

Conformal Symmetry and Feynman Integrals

Simone Zoia

zoia@uni-mainz.de

work in progress with **Dmitry Chicherin, Johannes Henn** and
Emery Sokatchev

Johannes Gutenberg Universität Mainz

DESY Theory Workshop, 26th September 2018



Conformal Symmetry and Feynman Integrals

Conformal symmetry breaking

Bootstrap approach

6D penta-box

Outlook



Conformal symmetry

- ▶ In **massless** theories the Poincaré group can be extended to the **conformal group** with the addition of
 - ▷ dilatations

$$x^\mu \rightarrow \lambda x^\mu$$

- ▷ conformal boosts

$$x^\mu \rightarrow \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2}$$

- ▶ **Euclidean spacetime**: transformations preserving **angles**
- ▶ **Minkowski spacetime**: transformations preserving **causality**

timelike points	→	timelike points
lightlike points	→	lightlike points
spacelike points	→	spacelike points

Conformal symmetry in momentum space

- ▶ Standard methods to compute correlation functions using conformal symmetry in position space date back to the '70s
- ▶ Goal: application to scattering amplitudes
 - ▷ momentum space
 - ▷ on-shell configuration $p_i^2 = 0$

Conformal symmetry in momentum space

- ▶ Standard methods to compute correlation functions using conformal symmetry in position space date back to the '70s
- ▶ Goal: application to scattering amplitudes
 - ▷ momentum space
 - ▷ on-shell configuration $p_i^2 = 0$
- ▶ The generator of conformal boosts becomes 2nd order

$$K_{\mu;\Delta} = \sum_{i=1}^n \left[-p_{i\mu} \square_{p_i} + 2p_i^\nu \frac{\partial}{\partial p_i^\nu} \frac{\partial}{\partial p_i^\mu} + 2(D - \Delta_i) \frac{\partial}{\partial p_i^\mu} \right]$$

- ▶ We cannot Fourier-transform from position to momentum space

Collinear anomaly

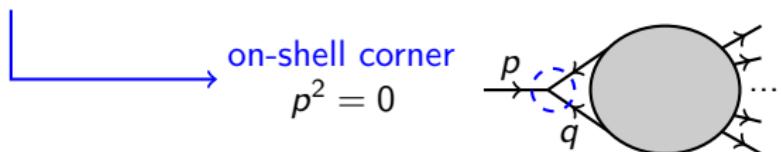
- ▶ Consider *naïvely conformal* Feynman integrals:
conformal symmetry may be broken in massless configurations

[Chicherin, Sokatchev 2017]

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- ▶ E.g. $D = 6$ scalar Φ^3 theory
 - ▷ Contact anomaly

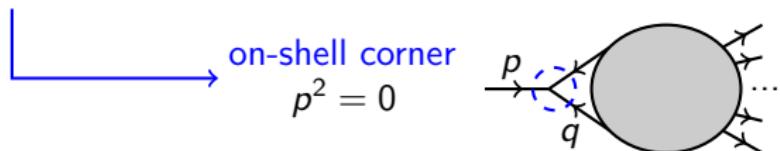
$$K_\mu \frac{1}{q^2(q+p)^2} = ?$$



Collinear anomaly

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conformal symmetry may be broken in massless configurations
[Chicherin, Sokatchev 2017]
- ▶ E.g. $D = 6$ scalar Φ^3 theory
 - ▷ Contact anomaly

$$K_\mu \frac{1}{q^2(q+p)^2} = 4i\pi^3 p_\mu \int_0^1 d\xi \xi(1-\xi) \delta^{(6)}(q + \xi p)$$



- ▷ Localized on the collinear configuration

$$q = -\xi p, \quad \xi \in [0, 1]$$

Powerful anomalous conformal Ward identities

- ▶ The contact anomaly localizes a loop-integration

$$K_\mu \int d^6 q \frac{p}{q} \text{I}(q, \dots) \propto p_\mu \int_0^1 d\xi \xi(1-\xi) \mathcal{I}(q = -\xi p, \dots)$$

- ▶ System of linear non-homogeneous 2nd-order DEs

$$K_\mu \delta^{(6)}(P) \mathcal{I}^{(\ell)} = \delta^{(6)}(P) \mathcal{A}_\mu^{(\ell-1)}$$

- ▷ $\mathcal{I}^{(\ell)}$ ℓ -loop Feynman integral
- ▷ $\mathcal{A}_\mu^{(\ell-1)}$ anomaly, 1-fold integration of $\ell - 1$ -loop integrals

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- ▶ Assuming we know $\mathcal{A}_\mu^{(\ell-1)}$, can we solve for $\mathcal{I}^{(\ell)}$?

- ▷ Open problem! Direct solution only achieved in simple cases
- ▷ Very powerful in a bootstrap approach

Bootstrap strategy

- ▶ Write down an ansatz ($s \equiv$ kinematic variables)

$$\mathcal{I}(s) = \sum_{i,j} c_{ij} r_i(s) f_j(s)$$

c_{ij} finite number of coefficients

⇒ to be fixed by imposing constraints
(anomalous conformal Ward identities)

$r_i(s)$ algebraic functions

⇒ leading singularities

[Cachazo 2008; Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010]

$f_j(s)$ special functions (polylogarithms)

⇒ symbol alphabet

Iterated integrals and symbols

- Polylogarithms can be written as Chen iterated integrals of logarithmic 1-forms

$$\int_{\gamma} d \log \alpha_1 \circ \dots \circ d \log \alpha_k = \int_0^1 \left(\int_0^t d \log \alpha_1 \circ \dots \circ d \log \alpha_{k-1} \right) d \log \alpha_k(t)$$

Weight: number of iterated integrations

- Letters** α_i : algebraic functions of the kinematic variables
- Symbol map** associates a “word” to each iterated integral

$$\mathcal{S} \left[\int_{\gamma} d \log \alpha_1 \circ \dots \circ d \log \alpha_k \right] = \alpha_1 \otimes \dots \otimes \alpha_k$$

- Alphabet:** set of all independent letters

$$\Omega = \{\alpha_1, \dots, \alpha_n\}$$

Massless 4-particle amplitudes @ 3-loop

- ▶ Alphabet

$$\Omega = \{x, 1+x\}$$

$$x = \frac{s}{t}$$

[Remiddi, Gehrmann, Henn, Smirnov, Mistlberger...]

- ⇒ Harmonic polylogarithms

$$H(\vec{m}_w; x) = \int_0^x dx' f(a; x') H(\vec{m}_{w-1}; x') \quad \vec{m}_w = (a, \vec{m}_{w-1})$$

$$f(0; x) = \frac{1}{x} \quad f(1; x) = \frac{1}{1-x} \quad f(-1; x) = \frac{1}{1+x}$$

- ▶ Symbol of the harmonic polylogarithms

$$H(0, -1; z) = \int_0^z \frac{dt}{t} H(-1, t) = \int_0^z d \log t \int_0^t d \log (1 + t')$$

$$\mathcal{S}\left[H(0, -1; z)\right] = (1+z) \otimes z$$

Massless 5-particle amplitudes @ 2-loop

- ▶ Five independent kinematic variables $v_i = 2p_i \cdot p_{i+1}$

Planar, 26 letters
[Gehrmann, Henn, Lo Presti 2015]

Non-planar, 31 letters
[Chicherin, Henn, Mitev 2017]

$$\left| \begin{array}{l} \alpha_i = v_i \\ \alpha_{5+i} = v_{i+2} + v_{i+3} \\ \alpha_{10+i} = v_i - v_{i+3} \\ \alpha_{15+i} = v_i + v_{i+1} - v_{i+3} \\ \alpha_{20+i} = v_{2+i} + v_{3+i} - v_i - v_{i+1} \\ \alpha_{25+i} = \frac{a_i - \sqrt{\Delta}}{a_i + \sqrt{\Delta}} \\ \alpha_{31} = \sqrt{\Delta} \end{array} \right.$$

$$a_i = v_i v_{i+1} - v_{i+1} v_{i+2} + v_{i+2} v_{i+3} - v_{i+3} v_{i+4} - v_{i+4} v_i$$

$$\Delta = \det(2p_i \cdot p_j)$$

Massless 5-particle amplitudes @ 2-loop

Given the pentagon alphabet \mathbb{A} , is any word made of letters $\alpha_i \in \mathbb{A}$ allowed?

Massless 5-particle amplitudes @ 2-loop

Given the pentagon alphabet \mathbb{A} , is any word made of letters $\alpha_i \in \mathbb{A}$ allowed? **No!**

- ▶ Constraints on the symbols from mathematics
 - ▷ Integrability conditions

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Given the pentagon alphabet \mathbb{A} , is any word made of letters $\alpha_i \in \mathbb{A}$ allowed? **No!**

- ▶ Constraints on the symbols from mathematics
 - ▷ Integrability conditions
- ▶ and from physics
 - ▷ First entry condition
 - planar symbols $\rightarrow \{\alpha_i\}_{i=1}^5 = \{s_{12} \text{ and cyclic}\}$
 - non-planar symbols $\rightarrow \{\alpha_i\}_{i=1}^5 \cup \{\alpha_i\}_{i=16}^{20} = \{s_{ij}\}_{i < j=1}^5$
 - ▷ Conjectured second entry condition

[Chicherin, Henn, Mitev 2017]

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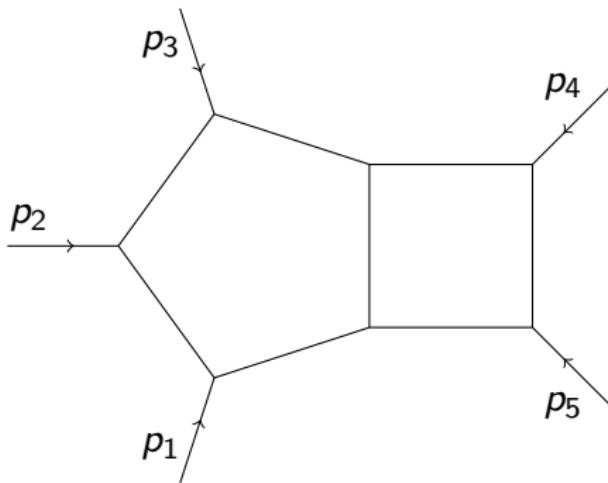
[Chicherin, Henn, Mitev 2017]

How many functions? E.g. for the **planar alphabet** (26 letters)

Weight	1	2	3	4	5	6
1 st entry cond.	5	25	126	651	3436	18426
2 nd entry cond.	5	20	81	346	1551	7201

[Chicherin, Henn, Mitev]

6D penta-box



Basic facts:

1. **naïvely conformal** \rightarrow 6D scalar ϕ^3 theory, finite
2. **planar alphabet** \mathbb{A}_P (26 letters, 1st entry condition)
3. **symmetric** under exchange $\{1 \leftrightarrow 3, 4 \leftrightarrow 5\}$
4. **parity even**
5. **leading singularity** $1/\sqrt{\Delta}, \Delta = \det(2p_i \cdot p_j)$

Last entry condition from conformal symmetry

- ▶ Conformal symmetry constrains the ansatz even before knowing the explicit expression of the anomaly

$$(q \cdot K) \left[\frac{\alpha_1 \otimes \dots \otimes \alpha_k}{\sqrt{\Delta}} \right] = (q \cdot K) \left[\frac{\log \alpha_k}{\sqrt{\Delta}} \right] (\alpha_1 \otimes \dots \otimes \alpha_{k-1}) + \text{weight}-(k-2)$$

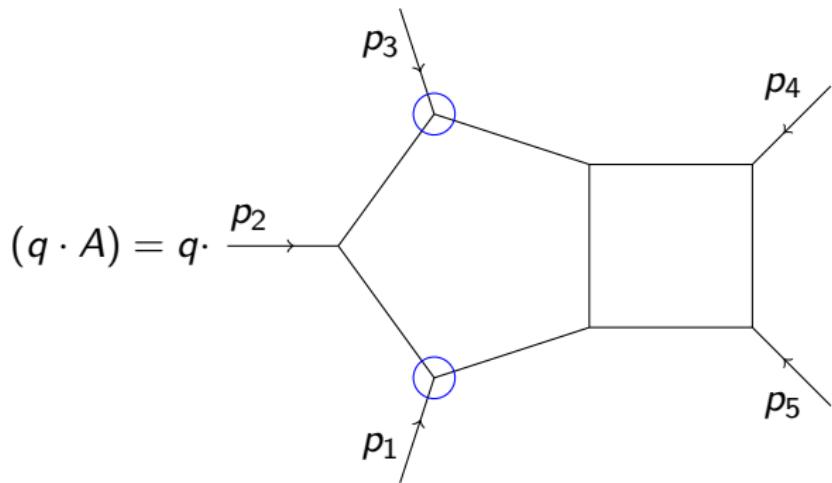
- ▶ If there exists a vector q^μ such that

$$(q \cdot K) \mathcal{I}_k = (q \cdot A) \equiv \text{weight}-(k-2)$$

⇒ Constraint on the **allowed last entries**

$$(q \cdot K) \left[\sum_i c_i \frac{\log \alpha_i}{\sqrt{\Delta}} \right] = 0$$

6D penta-box: anomaly



$$q \perp p_2, p_4, p_5$$

$$(q \cdot A) = \frac{(\text{weight-3})}{s_{12}s_{14}(s_{23} - s_{45})} + (1 \leftrightarrow 3, 4 \leftrightarrow 5)$$

6D penta-box: ansatz

- ▶ Info from conformal symmetry:

6. at most weight 5

$$\underbrace{(q \cdot K)}_{-2 \text{ weight drop}} \quad \mathcal{I}_5 = \underbrace{q \cdot A}_{\text{weight 3}}$$

6D penta-box: ansatz

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7. last entry condition (11 allowed letters out of 26)

6D penta-box: ansatz

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6. at most weight 5

$$\underbrace{(q \cdot K)}_{-\text{2 weight drop}} \quad \mathcal{I}_5 = \underbrace{q \cdot A}_{\text{weight 3}}$$

7. last entry condition (11 allowed letters out of 26)

Constraints	weight-5 integrable symbols (\mathbb{A}_P)
1st entry condition	3436
parity	161
last entry condition	59
graph symmetry	33

- ▶ Ansatz

$$\mathcal{S}[\mathcal{I}_5] = \frac{1}{\sqrt{\Delta}} \sum_{i=1}^{33} c_i (\alpha_{i_1} \otimes \dots \otimes \alpha_{i_5}), \quad \alpha_i \in \mathbb{A}_P \quad \forall i$$

6D penta-box: symbol

- ▶ Plug the ansatz

$$\mathcal{S}[\mathcal{I}_5] = \frac{1}{\sqrt{\Delta}} \sum_{i=1}^{33} c_i (\alpha_{i_1} \otimes \dots \otimes \alpha_{i_5}), \quad \alpha_i \in \mathbb{A}_{\mathsf{P}} \quad \forall i$$

into the anomalous conformal Ward identity

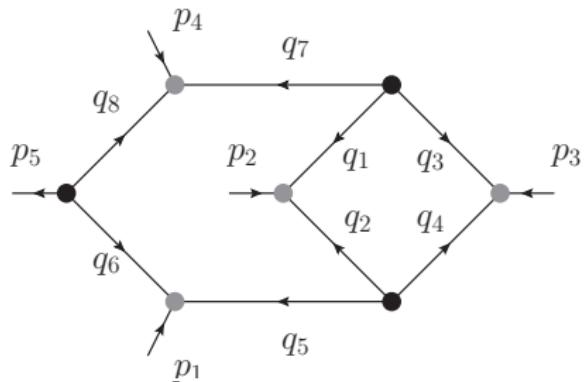
$$(q \cdot K) \mathcal{S}[\mathcal{I}_5] = q \cdot \mathcal{S}[A]$$

- ▷ All coefficients are fixed!
- ▷ Only one projection of the Ward identities is needed

$$\mathcal{S}[\mathcal{I}_5] \sim 10000 \text{ terms}$$

Outlook

- ▶ Solve the anomalous conformal Ward identities
- ▶ Upgrade bootstrap to function level (Goncharov polylogs)
- ▶ Study interplay with the β function
- ▶ Super-conformal symmetry [Chicherin, Henn, Sokatchev 2018]
 - ▷ Wess-Zumino model of $\mathcal{N} = 1$ massless supersymmetric matter
 - ▷ 1st order Ward identities \Rightarrow more powerful!



Backup slides

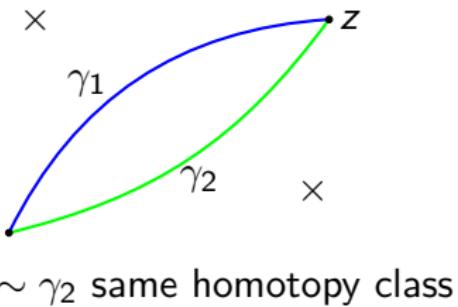
Integrable symbols

- Given an alphabet $\Omega = \{\alpha_1, \dots, \alpha_k\}$, is any word made of the letters $\alpha_i \in \Omega$ allowed? **No!**
- Local path independence**

$$\mathcal{S}(f) = \alpha_1 \otimes \dots \otimes \alpha_k$$

$$\omega = d \log \alpha_1 \circ \dots \circ d \log \alpha_k$$

$$f(z) = \int_{\gamma_1} \omega \stackrel{!}{=} \int_{\gamma_2} \omega$$



- Integrability conditions** for $\mathcal{S} = \sum_{i_1, \dots, i_n} c_{i_1 \dots i_n} (\alpha_{i_1} \otimes \dots \otimes \alpha_{i_n})$

$$\sum_{i_1, \dots, i_n} c_{i_1 \dots i_n} (d \log \alpha_{i_a} \wedge d \log \alpha_{i_{a+1}}) \alpha_{i_1} \otimes \dots \otimes \hat{\alpha}_{i_a} \otimes \hat{\alpha}_{i_{a+1}} \otimes \dots \otimes \alpha_{i_n} = 0$$
$$\forall a = 1, \dots, n-1$$

Symbols vs functions

Integrand \longrightarrow Symbol \longrightarrow Function (polylogs...)

- ▶ Symbols fix the **leading functional transcendentality** piece

$$\text{Li}_2(1-x) + \log(x) \log(1-x) = -\text{Li}_2(x) + \frac{\pi^2}{6}$$
$$-x \otimes (1-x) + x \otimes (1-x) + (1-x) \otimes x + (1-x) \otimes x = (1-x) \otimes x$$

\Rightarrow they capture the most complicated part of the function

- ▶ The lower functional transcendentality pieces can be fixed as well \Rightarrow Bootstrap strategy: accommodate them in the ansatz
- ▶ Additional work required to fully upgrade a symbol to function (Goncharov polylogarithms)

Iterated integrals and symbols

- ▶ Differential equations in the canonical form

$$d\vec{f}(s; \epsilon) = \epsilon \, d \left(\sum_k A_k d \log \alpha_k(s) \right) \vec{f}(s; \epsilon) \quad [\text{Henn}]$$

$\alpha_k(s)$ rational functions of the kinematic variables $s \equiv \text{letters}$
 A_k constant matrices

- ▶ **Alphabet** $\Omega = \{\alpha_1, \dots, \alpha_n\}$ set of all possible letters
→ specifies which class of functions is required
- ▶ General solution

$$\vec{f}(s; \epsilon) = \mathcal{P} \exp \left[\epsilon \int_{\gamma} d \left(\sum_k A_k d \log \alpha_k(s) \right) \right] \vec{f}_0(\epsilon)$$

Iterated integrals and symbols: a few examples

- ▶ Classical polylogarithms

$$\text{Li}_s(z) = \int_0^z d \log(t) \text{Li}_{s-1}(t), \quad \text{Li}_1(z) = -\log(1-z)$$

$$\text{Li}_s(z) = - \int_0^z d \log(t_1) \int_0^{t_1} d \log(t_2) \dots \int_0^{t_{s-1}} d \log(1-t_s)$$

$$\mathcal{S}[\text{Li}_s(z)] = -(1-z) \otimes \overbrace{z \otimes \dots \otimes z}^{s-1 \text{ times}} \Rightarrow \Omega = \{z, 1-z\}$$

From symbols to functions

- Differentiation acts on the last entry of the symbol

$$d(\alpha_1 \otimes \dots \otimes \alpha_n) = d \ln \alpha_n (\alpha_1 \otimes \dots \otimes \alpha_{n-1})$$

- The differential of a weight- n symbol is written in terms of weight- $(n - 1)$ symbols

$$d\mathcal{S}[f^{(n)}] = \sum_i c_i d \ln \alpha_i \mathcal{S}[f_i^{(n-1)}]$$

- Differential equation with a natural grading

$$d \begin{pmatrix} f^{(n)} \\ \vec{f}^{(n-1)} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & A & & & 0 \\ 0 & B & & & \\ & 0 & \ddots & & \\ 0 & & \ddots & C & \\ & & & 0 & \end{pmatrix} \begin{pmatrix} f^{(n)} \\ \vec{f}^{(n-1)} \\ \vdots \\ 1 \end{pmatrix}$$

From symbols to functions

- ▶ Algorithm: given a symbol \mathcal{S}^n of weight- n
 - by differentiating from weight- n down to weight-0 we determine the **block triangular** matrix M in

$$d\vec{f} = dM \vec{f} \quad \vec{f} = \left(f^{(n)}, \vec{f}^{(n-1)}, \dots, 1 \right)^T$$

- we integrate iteratively from weight-0 up to weight- n

✓ Function whose symbol matches \mathcal{S}^n

✗ The lower functional transcendentality pieces are still missing!

⇒ Bootstrap strategy: accommodate them into the ansatz