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Group Theoretic Approach to Theory of Fermion Production

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Based on
Min, SON, Suh 1808.00939

Particle Production

Preheating via parametric resonance or excitation in post-inflationary era vs instantaneous decay of inflaton

Kofman, Linde 97'

Axion-inflation via tachyonic gauge boson or fermion production

Anbor, Sorbo 10'

Adshead, Pearce, Peloso, Roberts, Sorbo 18'

Gravitational waves from preheating

Many literature

Greene, Kofman 99' 00'

Since our focus is on the reformulation of theory of fermion production, we will not get into any numerical simulation in this talk

Traditional Approach
To
Theory of Fermion Production

The model

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\bar{\psi} \left(i\, e^{\mu}{}_{a} \gamma^a D_\mu - m + g(\phi) \right) \psi + \frac{1}{2} \left(\partial_\mu \phi \right)^2 - V(\phi) \right]$$

On the metric:

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2 = a(t)^2 (d\tau^2 - d\mathbf{x}^2)$$

Under rescaling $\psi \to a^{-3/2} \psi$

$$\mathcal{L} = \bar{\psi} \left(i\, \gamma^\mu \partial_\mu - ma + \underline{g(\phi)} \right) \psi + \frac{1}{2} a^2 \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - a^4 V(\phi)$$

Interaction type commonly considered in literature

$$g(\phi) = \begin{cases} h\phi & : \text{Yukawa-type} \\ \dfrac{1}{f} \gamma^\mu \gamma^5 \partial_\mu \phi & : \text{Axion-type} \end{cases}$$

Benchmark model

1. Focus on axion-type interaction (derivative coupling)
2. Assume spatially homogenous scalar field : $\phi(\mathbf{x},\tau) = \phi(\tau)$

We will not distinguish t and τ

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - ma - \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

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A subtlety

$$\Pi_\psi = \frac{\delta\mathcal{L}}{\delta\dot{\psi}} = i\psi^+ \qquad \Pi_\phi = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = a^2\dot{\phi} - \frac{1}{f}\bar{\psi}\gamma^0\gamma^5\psi$$

$$\mathcal{H} = \Pi_\psi\dot{\psi} + \Pi_\phi\dot{\phi} - \mathcal{L}$$

$$= \bar{\psi}\left(-i\,\gamma^i\partial_i + ma + \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi - \frac{1}{2a^2}\frac{(\bar{\psi}\gamma^0\gamma^5\psi)^2}{f^2} + \frac{1}{2a^2}\Pi_\phi^2 + a^5V(\phi)$$

Definition of particle number is ambiguous

Massless limit is not manifest

Benchmark model

A way out

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - ma - \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

Adshead, Sfakianakis 15'

$\psi \to e^{-i\gamma^{5}\phi/f}\psi$

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - \underline{ma\cos\frac{2\phi}{f}} + i\,\underline{ma\sin\frac{2\phi}{f}}\gamma^{5}\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

Adshead, Pearce, Peloso, Roberts, Sorbo 18'

$= m_R$ $\quad$ $= m_I$

Benchmark model

A way out

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - ma - \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

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Adshead, Pearce, Peloso, Roberts, Sorbo 18'

$= m_R$ $= m_I$

Hamiltonian formalism

$$\Pi_{\psi} = \frac{\delta\mathcal{L}}{\delta\dot{\psi}} = i\psi^{+} \qquad \Pi_{\phi} = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = a^{2}\dot{\phi}$$

$$\mathcal{H} = \bar{\psi}\left(-i\,\gamma^{i}\partial_{i} + m_{R} - i\,m_{I}\gamma^{5}\right)\psi + \frac{1}{2a^{2}}\Pi_{\phi}^{2} + a^{4}V(\phi)$$

- ✓ No ψ - dependence in conjugate momentum Π_{ϕ}
- ✓ Entire fermion sector is quadratic in ψ
  : particle number is unambiguously defined
- ✓ Massless limit is manifest

To sum up

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - m_R + i\,m_I\gamma^5\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^4V(\phi)$$

$$\mathcal{H} = \bar{\psi}\left(-i\,\gamma^{i}\partial_{i} + m_R - i\,m_I\gamma^5\right)\psi + \frac{1}{2a^2}\Pi_{\phi}^2 + a^4V(\phi)$$

Entire fermion sector is quadratic in ψ

: following the traditional technique of Bogoliubov coefficient, particle number is unambiguously defined in this basis

Adshead, Pearce, Peloso, Roberts, Sorbo 18'

Fermion Production

We follow notation and convention in Adshead, Pearce, Peloso, Roberts, Sorbo 18'

Quantize ψ

$$\psi = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{r=\pm}\left[U_r(\mathbf{k},t)a_r(\mathbf{k}) + V_r(-\mathbf{k},t)b_r^{+}(-\mathbf{k})\right]$$

$$U_r = \begin{pmatrix} u_r(\mathbf{k},t)\ \chi_r(\mathbf{k}) \\ r\ v_r(\mathbf{k},t)\,\chi_r(\mathbf{k})\end{pmatrix}, \quad V_r = C\,\bar{U}_r^T \qquad \text{with } C = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$$

$$\chi_r(\mathbf{k}) = \frac{k + r\,\vec{\sigma}\cdot\mathbf{k}}{\sqrt{2k(k+k_3)}}\bar{\chi}_r \qquad \text{where } \bar{\chi}_+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \bar{\chi}_- = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

$$B_r = \frac{r\, e^{ir\varphi_k}}{2}\left[2\, m_R u_r v_r - k(u_r^2 - v_r^2) - irm_I(u_r^2 + v_r^2)\right]$$

Diagonalization via $\begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix} \to \begin{pmatrix} \alpha_r^* & \beta_r^* \\ -\beta_r & \alpha_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$ with eigenvalues of $\pm\omega$

Fermion number for a particle with helicity r (similarly for a anti-particle)

$$N_r = \int \frac{dk^3}{(2\pi)^3} \langle 0|a^+(\mathbf{k})a(\mathbf{k})|0\rangle = \int dk^3\, n_{r,k}$$

$$n_{r,k} = |\beta_r|^2 = \langle 0|a^+(\mathbf{k})a(\mathbf{k})|0\rangle$$

$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

1. Initially one can diagonalize $\mathcal{H}_\psi$ such that a, a^+ are associated with one-particle states.
2. Time-dependent $\phi(t)$ induces off-diagonal elements, and a (a^+) do not annihilates (creates) one–particle states. Once hamiltonian is diagonalized, $a(t)^+a(t)$ includes $b(0)b(0)^+$ which gives non-zero particle number

looks too technical …
Any simplication?

$$n_{r,k} = |\beta_r|^2 = \langle 0|a(\mathbf{k})a(\mathbf{k})|0\rangle = \langle 0|b(\mathbf{k})b(\mathbf{k})|0\rangle$$

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Solving EOM of u_r, v_r is a tricky in current basis

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Solving EOM of u_r, v_r is a tricky in current basis

Recall a Fourier mode

$$\psi \sim U_r(\mathbf{k},t)\,a_r(\mathbf{k}) + V_r(-\mathbf{k},t)b_r^+(-\mathbf{k})$$

$$U_r = \begin{pmatrix} u_r(\mathbf{k},t)\ \chi_r(\mathbf{k}) \\ r\ v_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} u_r \\ r v_r \end{pmatrix} \otimes \chi_r \equiv \xi_r \otimes \chi_r$$

Then we realize that

$$\zeta_{r\,1} = \frac{1}{2} r(u_r^* v_r + u_r v_r^*) = r\ Re(u_r^* v_r)$$

$$\zeta_{r\,2} = -\frac{i}{2} r(u_r^* v_r - u_r v_r^*) = r\ Im(u_r^* v_r)$$

$$\zeta_{r\,3} = \frac{1}{2}(|u_r|^2 - |v_r|^2)$$

$\longrightarrow$ $\vec{\zeta}_r = \xi_r^+ \vec{\sigma}\ \xi_r$

collapses into one vector

$$n_{r,k} = |\beta_r|^2 = \langle 0|a(\mathbf{k})a(\mathbf{k})|0\rangle = \langle 0|b(\mathbf{k})b(\mathbf{k})|0\rangle$$

$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r\, m_I}{2\omega} Im(u_r^* v_r)$$

$$\mathbf{q} = rk\ \hat{x}_1 + m_I\ \hat{x}_2 + m_R\ \hat{x}_3$$

$$|\mathbf{q}| = \omega = \sqrt{k^2 + m^2}$$

We will see the origin of this vector later

$$\vec{\zeta}_r = \xi_r^+ \vec{\sigma}\, \xi_r \qquad \text{w/ } \xi_r \equiv \begin{pmatrix} u_r \\ r v_r \end{pmatrix}$$

$$\zeta_{r\,1} = \frac{1}{2} r(u_r^* v_r + u_r v_r^*) = r\ Re(u_r^* v_r)$$

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$$n_{r,k} = |\beta_r|^2 = \langle 0|a(\mathbf{k})a(\mathbf{k})|0\rangle = \langle 0|b(\mathbf{k})b(\mathbf{k})|0\rangle$$
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$$\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$$

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$$\zeta_{r\,3} = \frac{1}{2}(|u_r|^2 - |v_r|^2)$$

→ $$n_{r,k}(t) = \frac{1}{2}\left(1 - \frac{\mathbf{q}\cdot\vec{\zeta}_r}{|\mathbf{q}|}\right) = \frac{1}{2}(1 - \cos\theta)$$

$\vec{\zeta}_r$, $\mathbf{q}$ behave like vector reps of SO(3) !. What is this mysterious SO(3)?

Group Theoretic Approach

Lorentz Group

Weyl Representation

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\ \sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = -\sigma_3 \otimes I_2$$

Spinor rep. satisfying Lorentz algebra

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$$

$$J_i \equiv \frac{1}{2}\epsilon_{ijk} S^{jk} = \frac{1}{2} I_2 \otimes \sigma_i \ \text{(space rotation)}, \qquad K_i \equiv S^{i0} = \frac{i}{2}\sigma_3 \otimes \sigma_i \ \text{(boost)}$$

$$\psi \sim \xi_r \otimes \chi_r \ \rightarrow \ e^{-i\vec{\theta}\cdot\vec{J}}\psi = \xi \otimes e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} \chi_r$$

On the other hand

$$(J_{L,R})_i = \frac{J_i \mp i\ K_i}{\sqrt{2}} = \frac{1}{2}(I_2 \pm \sigma_3) \otimes \frac{\sigma_i}{2} \quad : \quad SU(2)_L \times SU(2)_R$$

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

: Rep. of $SU(2)_L \times SU(2)_R$ is constructed as a ‘tensor sum’

‘Reparametrization’ Group

While γ^μ is fixed (only ψ transforms) in the Lorentz group, there is a freedom in choosing a representation of the gamma matrices. This freedom is totally unphysical though.

Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}$$

$$\gamma^\mu \to U\gamma^\mu U^{-1} \quad : \quad \text{GL(4,C)}$$

Dirac Theory

We assign the transformation of ψ, $\psi \to U\psi$

$$\mathcal{L} = \psi^+\gamma^0(i\gamma^\mu\partial_\mu - m)\psi$$

$$\to \mathcal{L} = \psi^+\, U^+U\gamma^0U^{-1}(iU\gamma^\mu U^{-1}\partial_\mu - m)U\psi$$

$$U^+U = UU^+ = 1 \quad : \quad \text{U(4)}$$

This symmetry is not identified with the non-unitary Lorentz group
: note that both γ^μ and ψ transform under U unlike the Lorentz group

We consider the following subgroup of $U(4)$

$$SU(2)_1 \times SU(2)_2 \times U(1) \subset U(4)$$

The rep of subgroup is constructed as a 'tensor product' of two $SU(2)$'s and phase rotation, e.g.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes U_2 = \begin{pmatrix} a_{11}U_2 & a_{12}U_2 \\ a_{21}U_2 & a_{22}U_2 \end{pmatrix}$$

$$= U_1$$

Under $SU(2)_1 \otimes SU(2)_2$ transformation (we associate $U(1)$ with ξ_r)

$$\psi \sim \xi_r \otimes \chi_r \rightarrow (U_1 \otimes U_2)(\xi_r \otimes \chi_r) = (U_1 \xi_r) \otimes (U_2 \chi_r)$$

This seems what we are looking for

In fact, we already have a well-known example for this symmetry

Looks similar to space rotation of Lorentz group. But it is not, and it does not play any role

Let us take a look at $SU(2)_1$ and $SU(2)_2$ to see what they are

An example of $SU(2)_1$

Weyl Representation $\psi_{\mathrm{Weyl}} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = -\sigma_3 \otimes I_2$$

Dirac Representation $\psi_{\mathrm{Dirac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \sigma_3 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2$$

Two representations are related via a similarity transformation

$$\gamma^\mu_{\mathrm{Weyl}} \;\to\; U_1 \gamma^\mu_{\mathrm{Weyl}} U_1^{-1} = \gamma^\mu_{\mathrm{Dirac}}$$

$$\psi_{\mathrm{Weyl}} \;\to U_1 \psi_{\mathrm{Weyl}} = \psi_{\mathrm{Dirac}}$$

w/ $U_1(\pi/2) = e^{i\frac{\pi}{2}\frac{\sigma_y}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$SU(2)_2$ can not be identified
with the space rotation subgroup of Lorentz group

** A mistake in 1808.00939. It will be corrected.

In terms of Gamma matrices

Rotation by $SU(2)_2$

$$U_2\gamma^\mu U_2^{-1} = \Lambda^\mu{}_\nu\gamma^\nu$$

$$(\gamma^\mu = U\gamma^\mu U^{-1}, \psi \to U\psi)$$

Space rotation of Lorentz group

$$\Lambda_{1/2}\gamma^\mu\Lambda_{1/2}^{-1} = \Lambda^\mu{}_\nu\gamma^\nu$$

$$(\gamma^\mu = \gamma^\mu, \psi \to \Lambda_{1/2}\psi)$$

In terms of transformation of ψ - bilinears

Under rotation by $SU(2)_2$

$$\bar{\psi}\gamma^\mu\psi \to \psi^+ U^+ U\gamma^0 U^{-1} U\gamma^\mu U^{-1} U\psi = \bar{\psi}\gamma^\mu\psi$$

Under space rotation of Lorentz group

$$\bar{\psi}\gamma^\mu\psi \to \bar{\psi}\,\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}\psi = \Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\mu\psi$$

Previously mysterious group
that we were looking for is $SU(2)_1 \times U(1)$

We will drop subscript from now on

This is what our group theoretic approach is based on

We will consider first fermion production
in `Inertial Frame'

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - ma\cos\frac{2\phi}{f} + i\,ma\sin\frac{2\phi}{f}\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

$$\mathcal{H} = \bar{\psi}\left(-i\,\gamma^{i}\partial_{i} + m_{R} - i\,m_{I}\gamma^{5}\right)\psi + \frac{1}{2a^{2}}\Pi_{\phi}^{2} + a^{4}V(\phi)$$

Group Theoretic Approach

Dirac equation in inertial frame

$$\left(i\,\gamma^{\mu}\partial_{\mu} - m_R + i\,m_I\gamma^5\right)\psi = 0$$

EOM in tensor form for a Fourier mode can be written as (using $(\vec{\sigma}\cdot\mathbf{k})\chi_r = rk\chi_r$)

$$[(i\,\sigma_3\partial_t - irk\sigma_2 - m_R I_2 + im_I\sigma_1)\otimes I_2](\xi_r\otimes\chi_r) = 0$$

Gives rise to EOM of fundamental rep.

$$\partial_t\xi_r \;=\; -i(\mathbf{q}\cdot\vec{\sigma})\xi_r$$

: it is called Weyl equation in condensed matter physics

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

SU(2) fundamental

SU(2) embedding of SO(3) vector $\mathbf{q}$

Group Theoretic Approach

Fundamental rep. of SU(2)

$$\xi_r \equiv \begin{pmatrix} u_r \\ r v_r \end{pmatrix}$$

EOM of fundamental rep.

$$\partial_t \xi_r \;=\; -i(\mathbf{q} \cdot \vec{\sigma})\xi_r$$

SU(2) embedding of
SO(3) vector

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

Group Theoretic Approach

Fundamental rep. of SU(2)

$$\xi_r \equiv \begin{pmatrix} u_r \\ r v_r \end{pmatrix}$$

EOM of fundamental rep.

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SU(2) embedding of
SO(3) vector

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

In terms of SO(3) ~ SU(2) reps

Bilinear of ξ_r : $\xi_r^+ A\,\xi_r$ w/ A = arbitrary 2×2 complex matrix

$\xi^+\,\xi\;(=1)$: scalar

$\vec{\zeta}_r = \xi^+\vec{\sigma}\,\xi$: vector

the only non-trivial rep.

EOM of vectorial rep.

$$\partial_t \zeta_{r\,i} = \frac{1}{2}\xi_r^+[i\mathbf{q}\cdot\vec{\sigma},\sigma_i]\xi_r = 2\epsilon_{ijk}q_j\zeta_{r\,k}$$

$$\frac{1}{2}\partial_t\vec{\zeta}_r \;=\; \mathbf{q}\times\vec{\zeta}_r$$

Analog to classical precession motion

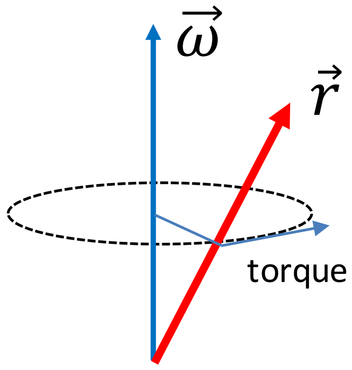
Quantum mechanical fermion production

$$\frac{1}{2}\frac{d\vec{\zeta}_r}{dt} = \mathbf{q}\times\vec{\zeta}_r$$

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

Classical precession of a vector $\vec{r}$ with angular velocity $\vec{\omega}$

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$



when $\vec{r} = \mathbf{M}$ (magnetization), $\vec{\omega} = \vec{\omega}_{\mathbf{M}} = -\gamma\mathbf{B}$

$$\frac{d\mathbf{M}}{dt} = \vec{\omega}_{\mathbf{M}}\times\mathbf{M}$$: called block eq.

$? = \boldsymbol{q}\cdot\vec{\zeta}_r$

$E = \vec{\omega}_{\mathbf{M}}\cdot\mathbf{M}$

Particle number density

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

$$B_r = \frac{r\, e^{ir\varphi_k}}{2}[2\, m_R u_r v_r - k(u_r^2 - v_r^2) - i r m_I (u_r^2 + v_r^2)]$$

Now it is clear that each matrix element should be a function of $\mathbf{q}$ and $\vec{\zeta}_r$ in our group theoretic approach

Particle number density

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

$$B_r = \frac{r\, e^{ir\varphi_k}}{2}[2\, m_R u_r v_r - k(u_r^2 - v_r^2) - irm_I(u_r^2 + v_r^2)]$$

Now it is clear that each matrix element should be a function of $\mathbf{q}$ and $\vec{\vec{\zeta}}_r$ in our group theoretic approach

Diagonal element

$$A_r = \mathbf{q} \cdot \vec{\vec{\zeta}}_r$$

$$= \omega \cos\theta$$

One can easily see why eigenvalues are $\pm\omega = \pm|\mathbf{q}|$

Off-diagonal element

$$|B_r|^2 = \left(\mathbf{q} \times \vec{\vec{\zeta}}_r\right)^2$$

$$|B_r| = \left|\mathbf{q} \times \vec{\vec{\zeta}}_r\right| = \omega \sin\theta$$

Particle number density

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \mathbf{q} \cdot \vec{\zeta}_r \,, \quad |B_r| = \left|\mathbf{q} \times \vec{\zeta}_r\right|$$

At time t, off-diagonal element becomes non-zero.
Diagonalization is done via

$$\begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix} \rightarrow \begin{pmatrix} \alpha_r^* & \beta_r^* \\ -\beta_r & \alpha_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$n_{r,k} = \langle 0 | a^+(\mathbf{k}) a(\mathbf{k}) | 0 \rangle = |\beta_r|^2 = f(\mathbf{q} \cdot \vec{\zeta}_r, |\mathbf{q}|)$$

Particle number density

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$$n_{r,k} = \langle 0 | a^+(\mathbf{k}) a(\mathbf{k}) | 0 \rangle = |\beta_r|^2 = f(\mathbf{q} \cdot \vec{\zeta}_r, |\mathbf{q}|)$$

It should be at most linear in $\vec{\zeta}_r$

$$n_{r,k} = A \pm B \frac{\mathbf{q} \cdot \vec{\zeta}_r}{|\mathbf{q}|}$$

which gives rise to inequality,

$$A - B \leq n_{r,k} \leq A + B$$

Using Pauli-blocking property:

$$0 \leq n_{r,k} \leq 1$$

→

$$n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q} \cdot \vec{\zeta}_r}{|\mathbf{q}|}\right)$$

$'-'$ sign chosen for the consistency with the form of A_r

(** agrees with our explicit computation)

Solution of EOM

Given the particle number density, closed form of solution is available

$$\frac{1}{2}\partial_t \vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r = (\mathbf{q}\cdot\mathbf{L})\vec{\zeta}_r \qquad n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q}\cdot\vec{\zeta}_r}{|\mathbf{q}|}\right)$$

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

Initial condition (↔ zero particle number) at $t = t_0$ is straightforward unlike other approach

$$\vec{\zeta}_r(t_0, t_0) = \frac{\mathbf{q}_0}{|\mathbf{q}_0|}$$

Just like solving Schrödinger eq. for the unitary op., EOM can be iteratively solved to be

$$\vec{\zeta}_r(t, t_0) = T\exp\left(\int_{t_0}^{t} dt'\ (\mathbf{q}\cdot\mathbf{L})(t')\right)\frac{\mathbf{q}_0}{|\mathbf{q}_0|}$$

Expanding involves commutator of series of $\mathbf{q}\cdot\mathbf{L}$

WKB solution might be the case with vanishing commutators

Switching to `Rotating Frame'

Via $\psi \to e^{+i\gamma^5\phi/f}\psi$

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^\mu\partial_\mu - ma - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

Equivalent to (in terms of $\vec{\zeta}_r$)

$$\vec{\zeta}_r \to R(t)\vec{\zeta}_r\,, \qquad R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi/f & -\sin 2\phi/f \\ 0 & \sin 2\phi/f & \cos 2\phi/f \end{pmatrix}$$

This rotating frame is non-inertial frame

Needs to supplement extra terms, e.g. Coriolis , centrifugal forces etc, to keep physics independent

EOM in `Rotating Frame'

Under $\vec{\zeta}_r \to R(t)\vec{\zeta}_r$, $$R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi/f & -\sin 2\phi/f \\ 0 & \sin 2\phi/f & \cos 2\phi/f \end{pmatrix}$$

Similarly to the classical mechanics, EOM transforms like

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r = (\mathbf{q}\cdot\mathbf{L})\vec{\zeta}_r \quad \to \quad \frac{1}{2}\partial_t(R\vec{\zeta}_r) = (\mathbf{q}\cdot\mathbf{L})(R\vec{\zeta}_r)$$

$$\frac{1}{2}\partial_t\vec{\zeta}_r = R^T(\mathbf{q}\cdot\mathbf{L})R\,\vec{\zeta}_r - \frac{1}{2}R^T\dot{R}\vec{\zeta}_r$$

w/ $(R^T\dot{R})_{ij} \equiv \epsilon_{ijk}\omega_{\zeta_r\,k}$

EOM can be brought back to the universal form

EOM in `Rotating Frame'

Under $\vec{\zeta}_r \to R(t)\vec{\zeta}_r$, $R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi/f & -\sin 2\phi/f \\ 0 & \sin 2\phi/f & \cos 2\phi/f \end{pmatrix}$

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w/ $\left(R^T\dot{R}\right)_{ij} \equiv \epsilon_{ijk}\omega_{\zeta_r k}$

EOM can be brought back to the universal form

$$\frac{1}{2}\partial_t\vec{\zeta}_r = R\mathbf{q}\times\vec{\zeta}_r + \frac{1}{2}\vec{\omega}_{\zeta_r}\times\vec{\zeta}_r = \left(R\mathbf{q} + \vec{\omega}_{\zeta_r}\right)\times\vec{\zeta}_r = \mathbf{q}'\times\vec{\zeta}_r$$

$$\mathbf{q}' = \left(rk + \frac{\dot{\phi}}{f}\right)\hat{x}_1 + ma\,\hat{x}_3$$

: different basis amounts to choose different angular velocity

Particle number density in `Rotating (non-inertial) Frame'

Particle number density in rotating frame

$$n_{r,k} = \langle 0|a^{+}(\mathbf{k})a(\mathbf{k})|0\rangle = |\beta_r|^2 = f(\mathbf{q}'\cdot\vec{\zeta}_r, |\mathbf{q}'|)$$

It should be at most linear in $\vec{\zeta}_r$.
Higher order terms should vanish to match to the one in inertial frame in $\dot{\phi} \to 0$ limit

$$n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q}'\cdot\vec{\zeta}_r}{|\mathbf{q}'|}\right)$$

This particle number density matches to the quadratic terms in

See Adshead, Sfakianakis 15' for a related discussion

$$\mathcal{H}_\psi = \bar{\psi}\left(-i\gamma^i\partial_i + ma + \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi - \frac{1}{2a^2}\frac{(\bar{\psi}\gamma^0\gamma^5\psi)^2}{f^2}$$

1. It looks like particle numbers are different in two different frames.
2. Establishing the 'final' particle number as a basis-independent quantity seems very non-trivial, e.g. Inertial frame vs. Non-inertial frame

Summary

We proposed a new group theoretic approach to theory of fermion production

1. This approach is based on the reparametrization group of gramma matrcies

2. This approach applies to any fermion system

   Possible extension is gravitino production, fermion production from gravitational background, fermion production in extra-dim. spacetime

3. Needs an idea to simplify the solution for analytic understanding

It would be great if this approach can simplify some complicated computation or give a new insight

Extra Slides

Back reaction

$$\ddot{\phi} + 2\,\frac{\dot{a}}{a}\dot{\phi} + a^2 V(\phi) = \frac{2}{a^2 f}\langle\bar{\psi}(m_I + i m_R \gamma^5)\psi\rangle$$

$$= -\sum_{r=\pm}\int \frac{d^3k}{(2\pi)^3}\langle m_I \zeta_{r\,3} + m_R \zeta_{r\,2}\rangle$$

In the massless limit, $m \to 0$

$$\mathbf{q} = rk\,\hat{x}_1$$

Initally , $\vec{\zeta}_r$ should be parallel to $\mathbf{q}$, stay in $\hat{x}_1$-axis

$$\partial_t \vec{\zeta}_r = 2\,\mathbf{q} \times \vec{\zeta}_r = 0$$

Since $\mathbf{q}$ is constant, $\vec{\zeta}_r$ does not evolve. $\zeta_{r\,2} = \zeta_{r\,3} = 0$ for any time

$$n_{r,k} = |\beta_r|^2 = \langle 0|a(\mathbf{k})a(\mathbf{k})|0\rangle = \langle 0|b(\mathbf{k})b(\mathbf{k})|0\rangle$$

$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega}Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

Solve equations of motion for u_r, v_r

$$\left(i\gamma^\mu \partial_\mu - m_R + i m_I \gamma^5\right)\psi = 0$$

$$\left(i\gamma^\mu \partial_\mu - m - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi = 0$$

$$\psi \sim \tilde{U}_r(\mathbf{k},t)a_r(\mathbf{k}) + \tilde{V}_r(-\mathbf{k},t)b_r^+(-\mathbf{k})$$

$$\psi \sim \tilde{U}_r(\mathbf{k},t)a_r(\mathbf{k}) + \tilde{V}_r(-\mathbf{k},t)b_r^+(-\mathbf{k})$$

$$U_r = \begin{pmatrix} u_r(\mathbf{k},t)\,\chi_r(\mathbf{k}) \\ r\, v_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix}$$

$$\tilde{U}_r = \begin{pmatrix} \tilde{u}_r(\mathbf{k},t)\,\chi_r(\mathbf{k}) \\ r\, \tilde{v}_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix}$$

$$s_r = \frac{u_r + v_r}{\sqrt{2}}\ , \quad d_r = \frac{u_r - v_r}{\sqrt{2}}$$

$$\tilde{s}_r = \frac{\tilde{u}_r + \tilde{v}_r}{\sqrt{2}}\ , \quad \tilde{d}_r = \frac{\tilde{u}_r - \tilde{v}_r}{\sqrt{2}}$$

$$u_r = \frac{1}{\sqrt{2}}\left(e^{i\,r\,\phi/f}\ \tilde{s}_r + e^{-i\,r\,\phi/f}\tilde{d}_r\right)$$

$$v_r = \frac{1}{\sqrt{2}}\left(e^{i\,r\,\phi/f}\ \tilde{s}_r - e^{-i\,r\,\phi/f}\tilde{d}_r\right)$$

In inflationary era, EOM in the original basis has an analytic solution

$$\left(i\gamma^\mu \partial_\mu - m - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi = 0$$

$$\psi = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{r=\pm}\left[\tilde{U}_r(\mathbf{k},t)a_r(\mathbf{k}) + \tilde{V}_r(-\mathbf{k},t)b_r^+(-\mathbf{k})\right]$$

$$\tilde{U}_r = \begin{pmatrix} \tilde{u}_r(\mathbf{k},t)\,\chi_r(\mathbf{k}) \\ r\,\tilde{v}_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix}$$

In an inflationary era

Adshead, Sfakianakis 15’

$\dot{\phi} \sim$ const. $\quad \frac{ma}{k} = -\frac{m}{H\tau k} = \frac{\mu}{x}, \quad \frac{a\dot{\phi}}{fk} = \frac{2\xi}{x}$ and $\tilde{s}_r, \tilde{d}_r = \frac{s_r, d_r}{\sqrt{x}}$

$$\partial_t^2 s_r + \left[-\frac{1}{4} + \frac{1}{x}\left(\frac{1}{2} + i\,2\xi r\right) + \frac{1}{x^2}\left(\frac{1}{4} + \mu^2 + 4\xi^2\right)\right] s_r = 0$$

$$\partial_t^2 d_r + \left[-\frac{1}{4} + \frac{1}{x}\left(-\frac{1}{2} + i\,2\xi r\right) + \frac{1}{x^2}\left(\frac{1}{4} + \mu^2 + 4\xi^2\right)\right] s_r = 0$$

Whittaker Equation

$$\frac{d^2w}{dz^2} + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2}\right] w = 0$$

The formula in a static Universe has a close similarity to the case with Yukawa-type interaction

$$\left(i\gamma^\mu \partial_\mu - m - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi = 0$$

$$\psi = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{r=\pm}\left[\tilde{U}_r(\mathbf{k},t)a_r(\mathbf{k}) + \tilde{V}_r(-\mathbf{k},t)b_r^+(-\mathbf{k})\right]$$

$$\tilde{U}_r = \begin{pmatrix} \tilde{u}_r(\mathbf{k},t)\,\chi_r(\mathbf{k}) \\ r\,\tilde{v}_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix}$$

In a static Universe

Adshead, Sfakianakis 15'

$$n_{r,k} = \frac{1}{2} - \frac{\tilde{k}}{4\tilde{\omega}}\left(|\tilde{s}_r|^2 - \left|\tilde{d}_r\right|^2\right) - \frac{m}{2\tilde{\omega}} Im(\tilde{s}_r^*\tilde{d}_r)$$

$$= \frac{1}{4\tilde{\omega}(\tilde{k}+\tilde{\omega})}\left[\left|\dot{\tilde{d}}_r\right|^2 + \tilde{\omega}^2\left|\tilde{d}_r\right|^2 - 2\tilde{\omega}\, Im(\tilde{d}_r\dot{\tilde{d}}_r^*)\right]$$

$$\partial_t^2\tilde{s}_r + \left[m^2 + \left(k + r\,\frac{\dot{\phi}}{f}\right)^2 + i\,\partial_t\tilde{k}\right]\tilde{s}_r = 0$$

$$\partial_t^2\tilde{d}_r + \left[m^2 + \left(k + r\,\frac{\dot{\phi}}{f}\right)^2 - i\,\partial_t\tilde{k}\right]\tilde{d}_r = 0$$