The motivic Hodge conjecture and D-Brane Masses on the Quintic

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☆ The main conjecture

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❷ Periods polynomial for $f_k \in S(\Gamma_0(N))$

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☆ Remarks on number theory, physics and geometry

❶ Discrete symmetries and automorphic functions :

Often the connection between number theory, physics and geometry goes along the following lines:

- There is a *discrete symmetry* T- S- more generally U-duality acting on physical parameters often geometrised
- The *physical amplitudes* of the theory are then *automorphic functions or forms* under this U-duality group

- The *integral* Fourier expansion of *BPS saturated amplitudes* count the (net) *multiplicities* of these states
- In the geometry of string and gauge theory these multiplicities are related to geometric invariants of the target space M such as *Gromov-Witten, Donaldson-Thomas, Pandharipande-T. or Seiberg-W. invariants*.
- Holomorphicity of the local Fourier expansion is usually incompatible with the gobal duality group.
- Resolving this contradiction leads the *holomorphic anomaly*, *mock modularity* and wall crossing

❷ Counting over finite fields and modular manifolds:

- In this talk the *modularity* and integrality comes at a different stage. We discover it by counting points of M *over finite fields* $\mathbb{F}_{p^k}$ at values of the moduli $\in \mathbb{F}_{p^k}$ and use the L-function to calculate BPS masses, which control the *growth* of the symplectic invariants.

- There is a direct relation to the approach of (Candelas della Ossa Rodriguesz-Villegas '01 and '04) and (R-V '03) about modular Calabi-Yau spaces and to (Kontsevich, Schwarz, Vologodsky '06) unsing the P-adic B-model to establish integrality and divisibility properties of the genus zero PT invariants.

- This relation is furnished by the *arithmetic properties of the periods* of M.

- A principal of the analysis over finite fields formalised in the *motivic Hodge conjecture*, is that over finite fields the Hodge structures of lower dimensional manifolds embeds naturally (motivically) into the higher dimensional one.

Modular manifolds: simplest example modular curves :

E Elliptic curve with fixed moduli:

$$P_E = y^2 - (x^3 + ax^2 + bx + c) = 0$$

Let p be a prime: Count points on E over finite field $\mathbb{F}_p = \{0, \ldots, p-1\}$ of order p [Here also $a, b, c \in \mathbb{F}_p$.]

$$\varepsilon_p = p + 1 - \#\{P_E = 0 | x, y, z \in \mathbb{F}_p\} \ .$$

For modular curves (rational) E

$$f_2 = \sum_{n=1}^{\infty} \varepsilon_n q^n$$

is a weight $k = 2$ modular cusp Hecke eigenform of level N_E. We can associate an L function to f_2 or E

$$L(E, s) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^s} = \prod_{p \text{ prime}} \left(1 - \varepsilon_p p^{-s} + p^{1-2s}\right)^{-1} .$$

An aside: *Birch, Swinnerton-Dyer conjecture:*

$$\mathrm{ord}_{s=1}L(E,s) = \mathrm{rank}E(\mathbb{Q}) \ .$$

A *modular Calabi Yau* M is a Calabi for which cusp Hecke eigenforms of level N_M occur in a similar manner as we will see.

Both approaches ❶ and ❷ are related by the CY-periods of the B-model, which contain both informations:

- ❶: mirror symmetry calculates the symplectic invariants from the periods at the MUM point.

- ❷: B. Dwork recovers the point counting over finite field on M from an p-adic analysis of the periods. Due to the non-Archimedian property of the p-adic norm this is a non-local analysis.

Arithmetic of periods govern both BPS count and point count over finite fields

☆ A classical problem for differential equations

❶ Consider the class of 14 fourth order hypergeometric differential operators of the form ($\theta = z\frac{d}{dz}$)

$$\boxed{\mathcal{D} = \theta^4 - \mu z \prod_{k=1}^{4}(\theta + a_k)}$$

These 14 cases, specified by $\{a_k\} \in \mathbb{Q}$ (and a normalisation μ) have the associated Riemann symbol

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & \frac{1}{\mu} & \infty & \\ 0 & 0 & a_1 & \\ 0 & 1 & a_2 & z \\ 0 & 2 & a_3 & \\ 0 & 1 & a_4 & \end{array}\right\}$$

From the degeneration of the indicials shown in the columns one sees equivalently that at $z = 0$ one has a maximal unipotent monodromy point (m), at $z = \mu^{-1}$ a conifold point (c) and at $z = \infty$ for the quintic a simple orbifold point (o), since the a_i are all *different* rational numbers.

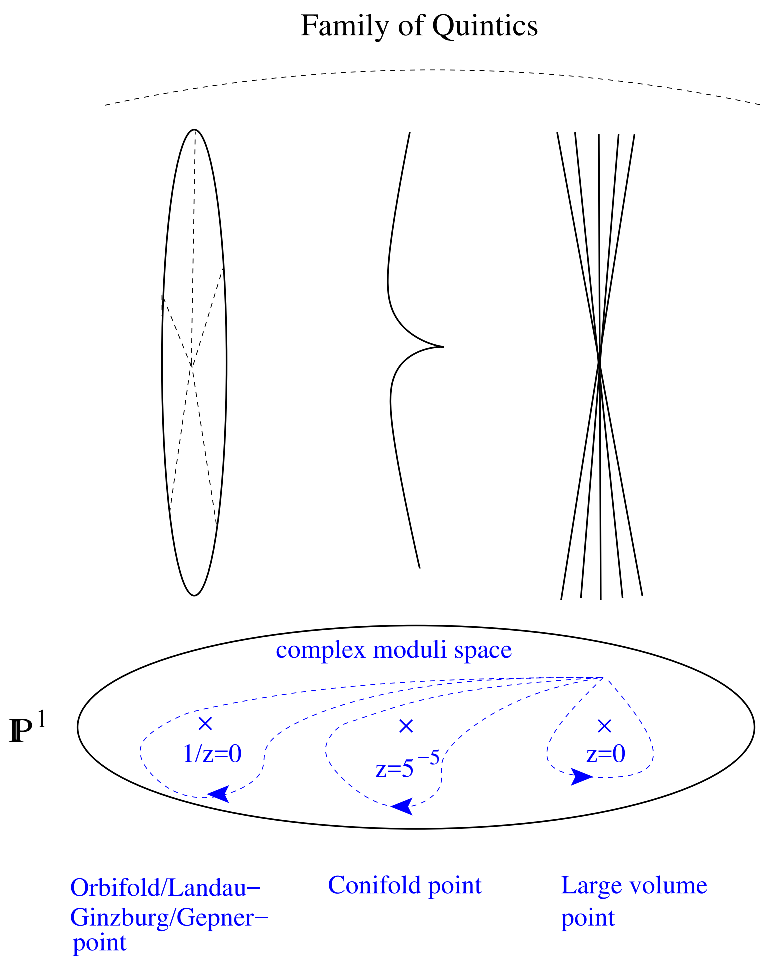

Figure 1: The family of quintics over the moduli space $\mathcal{M}_{cs} = \mathbb{P}^1 \setminus \{z = 0, z = \mu^{-1}, z = \infty\}$.

❶ From the local Frobenius bases to the integral symplectic basis:

The local monodromy information gives a *somewhat* preferred *local basis* of the period vector $\tilde{\Pi}^* = \left(\int_{\tilde{B}_0}\Omega, \int_{\tilde{B}_1}\Omega, \int_{\tilde{A}^0}\Omega, \int_{\tilde{A}^1}\Omega\right)^T$ around each singular point $*$, the so called *Frobenius basis*, where in general $\tilde{A}^I, \tilde{B}_I \in H_3(M, \mathbb{C})$.

More fundamentally there is, up to Sp$(4, \mathbb{Z})$ transformations, an global *integral basis* Π over $A^I, B_I \in H_3(M, \mathbb{Z})$, *symplectic* w.r.t. the pairing $A^I \cap B_J = \delta^I_J$ rest zero. This *integral symplectic basis* was first determined for the quintic by COGP91 and gives in general the correct central charges for the B- and by mirror symmetry for the A-branes $Z_{D_{2n}}$, $n = 0, 1, 2, 3$ as we see next:

The simplest way to fix that basis is by a I_Γ-function defined in HKTY95 which encodes 4 solutions

$$I_\Gamma(z,\epsilon) = \sum_{k=0}^{\infty} \frac{\Gamma(5(k+\epsilon)+1)}{\Gamma(k+\epsilon+1)^5} z^{k+\epsilon} = \sum_{m=0}^{3} L_m(z)(2\pi i\epsilon)^m$$

at the mum point: From the analytic— L_0 to the triple logarithmic solution $L_3(z)$. These solutions form a $\mathbb{Q}$

basis from with the $\mathbb{Z}$ basis is given by

$$\Pi = \begin{pmatrix} F_0 \\ F_1 \\ X^0 \\ X^1 \end{pmatrix} = \begin{pmatrix} \int_{B_0} \Omega \\ \int_{B_1} \Omega \\ \int_{A^0} \Omega \\ \int_{A^1} \Omega \end{pmatrix} = \begin{pmatrix} \kappa L_3 + \frac{c_2 \cdot D}{12} L_1 \\ -\kappa L_2 + \sigma L_1 \\ L_0 \\ L_1 \end{pmatrix} .$$

Remarkably all constants are related to topological intersection numbers of M

- $\kappa = \int_M \omega^3 = D^3 \in \mathbb{N}$ is the triple intersection on M,
- $c_2 \cdot D = \int_M c_2(TM) \wedge \omega \in \mathbb{N}$,

- σ can chosen to be $\sigma = (\kappa \bmod 2)/2$.

These topological data also also determine the monodromy around $z = 0$ as

$$M_m = \begin{pmatrix} 1 & -1 & \frac{\kappa}{6} + \frac{c_2 \cdot D}{12} & \frac{\kappa}{2} + \sigma \\ 0 & 1 & \sigma - \frac{\kappa}{2} & -\kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

An interesting constant $\zeta(3)\chi(M)$ with $\chi(M)$ the Euler number of M appears due to the third derivative of the Γ function as coeff of L_0 in L_3. Generalisation the Γ-class (Kontsevich,....).

A volume grading given by mirror map at the point of

maximal monodromy of W
$t(z) = \frac{X^1}{X^0} = \frac{1}{2\pi i}\left(\log(z) + \varpi_1(z)/\varpi_0(z)\right)$, indentified with complexified area $t = \int_{\mathcal{C}} i\omega + b$ of the curve $\mathcal{C}$ defining the Mori cone of M yields the desired identification with the central charges of the A-branes:

$$\Pi = \begin{pmatrix} F_0 \\ F_1 \\ X^0 \\ X^1 \end{pmatrix} = X^0 \begin{pmatrix} 2\mathcal{F} - t\partial_t\mathcal{F} \\ \partial_t\mathcal{F} \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} Z_{D_6} \\ Z_{D_4} \\ Z_{D_0} \\ Z_{D_2} \end{pmatrix} \quad .$$

The second equality identifies the prepotential as

$$\mathcal{F} = -\frac{\kappa}{6}t^3 + \frac{\sigma}{2}t^2 + \frac{c_2 \cdot D_\alpha}{24}t + \frac{\zeta(3)\chi(M)}{2(2\pi i)^3} - \mathcal{F}_{inst}(Q) \; .$$

$$\mathcal{F}_{inst}(Q) = \frac{1}{(2\pi i)^3} \sum_{\substack{\beta \in H_2(M,\mathbb{Z}) \\ \beta \neq 0}} n_0^\beta \mathrm{Li}_3(Q^\beta) \, .$$

Here $Q_k = \exp(2\pi i t_k)$ and n_0^β the $g = 0$ BPS invs. Note that since $n_0^0 = \chi/2$, the degree zero instanton contribution (at $\beta = 0$) can be interpreted as the term in $\mathcal{F}$ involving the Euler number and the Zeta value $\zeta(3)$.

We fixed the integral symplectic basis at the *maximal* *unipotent point*. The main objective of this talk is to study this basis at the conifold:

$$\delta = (1 - \mu z) \equiv 0 \ .$$

A Frobenius basis at the conifold $\tilde{\Pi}^c$ is given by

$$\tilde{\Pi}^c = \left(\begin{array}{l} 1 + \frac{2}{625}\delta^3 + O\left(\delta^4\right) \\ \nu \\ \delta^2 + \frac{37}{30}\delta^3 + O\left(\delta^4\right) \\ \nu \log(\delta) - \frac{23}{360}\delta^3 + O\left(\delta^4\right) \end{array} \right) ,$$

where $\nu = \delta + \frac{7}{10}\delta^2 + \frac{41}{75}\delta^3 + O\left(\delta^4\right)$ is the unique vanishing period, which can be identified from the local solutions of the Picard-Fuchs equation up to a constant as it is the unique period which multiplies the logarithm in the fourth solution. The existence of the vanishing cycle $\nu = \int_{B_0} \Omega$ implies a conifold monodromy in the integral symplectic basis

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

The key player will be the transition matrix T_{mc}

$$\Pi = \begin{pmatrix} 0 & -\frac{i\sqrt{\kappa}}{2\pi} & 0 & 0 \\ a + ig\sigma & b + ih\sigma & c + ir\sigma & 0 \\ d & e & f & \frac{\sqrt{\kappa}}{4\pi^2} \\ ig & ih & ir & 0 \end{pmatrix} \tilde{\Pi}^c = T_{mc}\tilde{\Pi}^c$$

defined by a path just above the real axis from $z = 0$ to $z = 1/\mu$. The zeros in this matrix can be inferred by compatibility of the monodromies M_c and M_c which determine the global mondromy $\Gamma \in \mathrm{Sp}(h_3(W), \mathbb{Z})$. e.g. $(M_m M_c)^5 = M_o^5 = 1$ for the quintic.

Note that $\nu \sim F_0 = |Z_{D_6}|$ and hence $m_{D_6} = e^{\frac{K}{2}}|Z_{D_6}|$, with $e^{-K} = -i\Pi^\dagger \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \Pi$, is zero at the conifold. $e^{\frac{K}{2}}(\delta = 0) = \frac{1}{\sqrt{2ag}}$ and the values of the other D-brane masses are finite

$$\begin{pmatrix} m^c_{D_6} \\ m^c_{D_4} \\ m^c_{D_0} \\ m^c_{D_2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{|a+ig\sigma|}{\sqrt{2ag}} \\ \frac{d}{\sqrt{2ag}} \\ \frac{g}{\sqrt{2ag}} \end{pmatrix} .$$

❸ Special geometry and Legendre relations

Lemma: For any of the 14 CY 3-fold hypergeometric systems let $\delta = 1 - \mu z$ the conifold variable and the Frobenius basis $\tilde{\Pi}^c$ basis as above. Then the transition matrix T_{mc} fullfils the following quadratic relation

$$\frac{1}{(2\pi i)^3}\begin{pmatrix} \mathbf{0} & \mathbf{t} \\ -\mathbf{t} & \mathbf{0} \end{pmatrix} = T_{mc}^T \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} T_{mc}$$

with $\mathbf{t} = \begin{pmatrix} \frac{\kappa}{2} & 0 \\ -\alpha\kappa & -\kappa \end{pmatrix}$, $\alpha = \frac{3}{4}\left(\sum_{i=1}^4 a_i - \sum_{i<j}^4 a_i a_j\right)$.

Proof: This follows from Griffiths transversality, i.e.

$\langle\alpha,\beta\rangle = \int_M \alpha\wedge\beta$ has the property that for $\alpha^{p,q}\in H^{p,q}$ one has $\langle\alpha^{p,q},\alpha^{r,s}\rangle = 0$ unless $p=s$ and $q=r$, the observation of Bryan and Griffiths BG83 that

$$\left\langle \Omega, \left(\frac{d}{dz}\right)^k \Omega \right\rangle = \tilde{\Pi}^T\tilde{\Sigma}\frac{d^k}{dz^k}\tilde{\Pi} = \begin{cases} 0 & \text{if } k<3 \\ C_{zzz}(z) & \text{if } k=3 \end{cases},$$

the fact that $\mathcal{D}\tilde{\Pi}=0$ and that

$$C_{zzz}(z) = \frac{\kappa}{(2\pi i)^3 z^3(1-\mu z)} \;.$$

is a global section $C_{zzz}\in\Gamma\left(\mathrm{Sym}^3(T^*\mathcal{M})\otimes\mathcal{L}^2\right)$, where $\mathcal{L}$ is the Kähler line bundle •

In physics the above geometrical package on $\mathcal{M}_{cs}$ is often called *special geometry*.

Due to the special form of T_{mc} the **Lemma** implies three quadratic relations among the nine unknowns $a, b, c, d, e, f, g, h, r$, which read for the Quintic

$$\frac{\kappa}{2} = 8\pi^3(ar - cg), \quad \sqrt{\kappa}d = 2\pi(bg - ah),$$
$$\alpha\frac{\kappa}{2} = 8\pi^3(\frac{\sqrt{\kappa}}{2\pi}f + ch - br) \ .$$

☆ The main conjecture (KSZ'18)

❶ The main conjecture:

Conjecture:

- The values a, g in the transition matrix T_{mc} are the up to rational numbers and trivial π factors the period values $\omega_\pm$ of the weight four holomorphic Hecke eigen form $f \in S_4(\Gamma_0(N))$, that has been associated to the degenerate $\zeta(M)$ function at the conifold by Schoen '80 and Rodriguez-Villegas '03 .

- The values c, r in T_{mc} are up rational numbers the quasi-period values $\eta_\pm$ of f, so that the quadratic relation

$$\boxed{\frac{\kappa}{2} = 8\pi^3(ar - cg)}$$

can be identified with the Legendre Relation on $\Gamma_0(N)$

Remarks and examples:

- For the quintic Schoen '80 determined $f \in S_4(\Gamma_0(25))$

to be

$$
\begin{aligned}
f &= \frac{\eta(5\tau)^{10}}{\eta(\tau)\eta(25\tau)} + 5\eta(\tau)^2\eta(5\tau)^4\eta(25\tau)^2 \\
&= \textstyle\sum_{n=1}^{\infty} \alpha_n q^n = q + q^2 + 7q^3 - 7q^4 + 7q^6 + \cdots \ .
\end{aligned}
$$

- The values a, g, which determine the masses of the $D2/D4$, can be alternatively described by L-function values of f with

$$
L_f(s) := \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s} = \prod_{p \ \text{prime}} \frac{1}{1 - \alpha_p p^{-s} + p^{k-1-2s}} \ .
$$

Relation to integrals over the modular form

$$\left(\frac{2\pi i}{\sqrt{N}}\right)^{-s}\Gamma(s)L_f(s) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} t^{s-1}e^{-2\pi n t}dt$$
$$= \int_0^{\infty} f\left(\frac{it}{\sqrt{N}}\right) t^{s-1}dt$$

- Example quintic

$$a = \frac{5^4 L_f(2)}{4(2\pi)^2} = \frac{5^2\omega_-}{2(2\pi)^4}, \qquad g = \frac{5L_f(1)}{2\pi} = \frac{5i\omega_+}{(2\pi)^4},$$
$$c = \frac{\eta_-}{2\cdot 5^5\pi^4}, \qquad r = \frac{i\eta_+}{5^6\pi^4}\ .$$

- This brings the values of unknowns in T_{mc} down to three as the Legendre relation between periods and quasi-periods of $S_4(\Gamma_0(25))$

$$\det \begin{pmatrix} \omega_+ & \omega_- \\ \eta_+ & \eta_- \end{pmatrix} = (2\pi i)^{k-1}\mathbb{Q}$$

  is already implied by the quadratic relations from special geometry.

- The L function values a, g also determine the leading behaviour of the Weil-Petersson metric and its curvature at the conifold

$$g_{\delta\bar{\delta}} \sim -5\frac{\log|\delta|}{(2\pi)^3 ag}, \qquad R \sim -\frac{(2\pi)^3 ag}{10|\delta|^2 \log^3|\delta|}\,.$$

- The data for other cases are specified by a_i, $i = 1, \ldots, 4$, μ, the topological intersection numbers, level N and the Hecke eigenform $f \in S_4(\Gamma_0(N))$ and its dim d.

  These data are sumarized in the following table: $\Gamma \in \mathrm{SP}(4, \mathbb{Z})$ arith. (Enckefort, van Straten '04)

N	Name	a_1, a_2, a_3, a_4	μ	κ	$c_2.D$	χ	d	f
8	$X^*_{2,2,2,2}(1^8)$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	2^8	16	64	-128	1	$q - 4q^3 - 2q^5 \ldots$
9	$X_{4,3}(1^5 2)$	$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	$2^6 3^3$	6	48	-156	1	$q - 8q^4 + 20q^7 \ldots$
16	$X^*_{4,2}(1^6)$	$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	2^{10}	8	56	-176	1	$q + 4q^3 - 2q^5 - 24q^7 \ldots$
25	$X^*_5(1^5)$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	5^5	5	50	-200	3	$q + q^2 + 7q^3 - 7q^4 \ldots$
27	$X^*_{3,3}(1^6)$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	3^6	9	54	-144	4	$q - 3q^2 + q^4 - 15q^5 \ldots$
32	$X_{4,4}(1^4 2^2)$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	2^{12}	4	40	-144	3	$q - 8q^3 - 10q^5 - 16q^7 \ldots$
36	$X_{3,2,2}(1^7)$	$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	$2^4 3^3$	12	60	-144	1	$q + 18q^5 + 8q^7 \ldots$
72	$X^*_{6,2}(1^5 3)$	$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	$2^8 3^3$	4	52	-256	4	$q - 14q^5 - 24q^7 \ldots$
108	$X_6(1^4 2)$	$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	$2^4 3^6$	3	42	-204	4	$q - 9q^5 - q^7 - 63q^{11} \ldots$
128	$X^*_8(1^4 4)$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	2^{16}	2	44	-296	12	$q - 2q^3 + 6q^5 - 20q^7 \ldots$
144	$X_{6,4}(1^3 2^2 3)$	$\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}$	$2^{10} 3^3$	12	32	-156	7	$q + 16q^5 + 12q^7 - 64q^{11} \ldots$
200	$X_{10}(1^3 2, 5)$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$2^8 5^5$	1	34	-288	14	$q + q^3 - 6q^7 - 19q^{11} \ldots$
216	$X_{6,6}(1^2 2^2 3^2)$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$2^8 3^6$	1	22	-120	12	$q + q^5 - 9q^7 - 17q^{11} \ldots$
864	$X^*_{2,12}(1^4 4, 6)$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$2^{12} 3^6$	1	46	-484	48	$q - 19q^5 - 13q^7 - 65q^{11} \ldots$

- T_{mc} can be calculated numerically by analytic continuation

N	a	b	c	d	e	g
8	8.949091097	2.237272774	-0.3621599810	1.118636387	-0.03235356517	0.9027285783
9	6.237050975	1.129712764	-0.1566329435	1.076974686	-0.02632457013	1.200321020
16	7.221828627	1.400050550	-0.2068090005	1.087310326	-0.02804166236	1.118636387
25	6.195016277	1.016604717	-0.1408899794	1.070725868	-0.02470761380	1.293573985
27	7.226824010	1.497435282	-0.2210220703	1.092181914	-0.02911035146	1.065341522
32	5.179782801	0.8407582552	-0.1076083419	1.064285289	-0.02362785453	1.335647591
36	8.132044212	1.837251981	-0.2853273935	1.104571259	-0.03079199557	0.9839286744
72	6.134443907	0.8938516418	-0.1237752881	1.063732084	-0.02279636586	1.418734246
108	5.087245254	0.7102246789	-0.09063866981	1.056196517	-0.02123059234	1.500810815
128	4.951930090	0.5610465041	-0.07131091014	1.046427258	-0.01812826025	1.773813125
144	3.985991253	0.5172845846	-0.05929580447	1.046941792	-0.01889214034	1.636637656
200	3.757763918	0.3439738187	-0.03900080381	1.033797388	-0.01423540765	2.169822377
216	2.783509082	0.3070846753	-0.03005333986	1.034287017	-0.01495377663	1.938348345
864	4.733161845	0.3773526858	-0.04767792995	1.033377871	-0.01361692746	2.379289657

Thanks to Pari packages of Henri Cohen for calculating $\omega_\pm, \eta_\pm$ the conjecture supported by agreement to hundreds of digits.

- Local Calabi-Yau cases:

$$\mathcal{L} = \theta^3 - \mu z \prod_{i=1}^{3} (\theta + a_i) \ .$$

Now define the Dirichlets L-function

$$L_a(s) = \sum_{n=1}^{\infty} \frac{\left(\frac{a}{n}\right)}{n^s} \ .$$

with the Legendre symbol

$$\left(\frac{a}{n}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod n} \\ -1 & \text{if } a \text{ is a quadratic non} - \text{residue mod n} \\ 0 & \text{if } a \bmod n = 0 \,. \end{cases} \tag{1}$$

Now the transition matrix

$$\Pi_l = \begin{pmatrix} 0 & \frac{\kappa}{2h\pi} & 0 \\ 1 & 0 & 0 \\ \frac{1}{2}v + ig & i\frac{h\sqrt{\kappa}(\log|\mu|+1)}{(2\pi)^2} & -i\frac{h\sqrt{\kappa}}{4\pi^2} \end{pmatrix} \tilde{\Pi}^c_l = T^l_{mc}\tilde{\Pi}^c \,,$$

is completly known

Base S	a_1, a_2, a_3	μ	κ	σ	$c_2 \cdot J$	v	g	h	$n_1^{(0)}$	$n_2^{(0)}$
$\mathbb{F}_0$	$0, \frac{1}{2}, \frac{1}{2}$	-2^4	1	0	-2	0	$\frac{16 L_{-4}(2)}{(2\pi)^2}$	2	-4	-4
$\mathbb{P}^2$	$0, \frac{1}{3}, \frac{2}{3}$	-3^3	$\frac{1}{3}$	$\frac{1}{6}$	-2	1	$\frac{27\sqrt{3} L_{-3}(2)}{2(2\pi)^2}$	3	3	-6
B_5 $[D_5]$	$0, \frac{1}{2}, \frac{1}{2}$	2^4	4	2	-20	1	$\frac{16 L_{-4}(2)}{(2\pi)^2}$	1	16	-20
B_6 $[E_6]$	$0, \frac{1}{3}, \frac{2}{3}$	3^3	3	$\frac{3}{2}$	-18	1	$\frac{27\sqrt{3} L_{-3}(2)}{2(2\pi)^2}$	1	27	-5
B_7 $[E_7]$	$0, \frac{1}{4}, \frac{3}{4}$	2^6	2	1	-16	1	$\frac{16\sqrt{2} i L_{-8}(2)}{(2\pi)^2}$	1	56	-272
B_8 $[E_8]$	$0, \frac{1}{6}, \frac{5}{6}$	$2^4 3^3$	1	$\frac{1}{2}$	-14	1	$\frac{40 i L_{-4}(2)}{(2\pi)^2}$	1	252	-9252

and all constants including the occurrence of the Dirichlet L-functions can be proven (KSZ '18). Relation to BPS count $d = 1$

$$n_0^\beta \underset{\beta \to \infty}{\sim} \left(\frac{2\pi d^2}{g}\right)^2 \frac{e^{2\pi\beta(g/d)}}{\beta^3 \log^2 \beta} \, .$$

❷ Point counting over finite fields $\mathbb{F}_{p^k}$ and the Hasse Weil ζ function

$\mathbb{F}_p$ is the finite cyclic field of order p given by the integers modulo the prime p.

$\mathbb{F}_{p^k}$ for $k \in \mathbb{N}$ is the group of order p^k is an extension of the finite field $\mathbb{F}_p$

$$\mathbb{F}_{p^k} = \left\{ a_0 + a_1\rho + \ldots + a_{k-1}\rho^{k-1} \middle| \, a_j \in \mathbb{F}_p \right\}$$

using a root ρ of a suitable irreducible monic polynomial of order k, for example of $x^k + 1$

Consider now the one parameter family of quintics:

$$P_\psi = \sum_{i=1}^{5} x_i^5 - (5\psi)\prod_{i=1}^{5} x_i = \sum_{i=1}^{5} x_i^5 - z^{-\frac{1}{5}}\prod_{i=1}^{5} x_i = 0$$

To see the relation to form of (Chad Schoen '80) and the L-function (CdOR-V '01) let $\psi \in \mathbb{F}_{p^k}$ and

$$N_k = \#\{x \in \left(\mathbb{P}\mathbb{F}_{p^k}\right)^5 | P_\psi(x) = 0\},$$

the number of projective solutions of the equation (**??**)

over $\mathbb{PF}_{p^k}$ and the ζ-function of the quintic

$$\zeta_M(T) = \exp\left(\sum_{k=1}^{\infty} N_k \frac{T^k}{k}\right) \tag{2}$$

There are various facts known

- $\zeta_M(T)$ is rational as proven by Dwork:
- Functionality $\zeta\left(\frac{1}{p^d}T\right) = \pm p^{d\frac{\chi}{2}} T^{\chi} \zeta(T)$, with χ the Euler number and d the dimension of M (Grothendieck) .

- Riemann Hypothesis proven by Deligne:

$$\zeta_M(T) = \frac{P_1(T)P_3(T)\dots P_{2d-1}(T)}{P_0(T)P_2(T)\dots P_{2d}(T)}\ , \tag{3}$$

  where the $P_i(T)$ have coefficients in $\mathbb{Z}$ and degree given by the Betti number $b_i(M)$ and $P_i(0) = 1$, $P_0 = 1 - T$, $P_{2d}(T) = (1 - p^d T)$ and $P_i(T) = \prod_{j=1}^{b_i}(1 - \alpha_{ij}T)$ with $|\alpha_{ij}| = p^{\frac{i}{2}}$.

For example for elliptic curves one has

$$\zeta_E(T) = \frac{1 - \alpha_p T + pT^2}{(1 - T)(1 - pT)}$$

and for the quintic M with $h_{11} = 1$ and $h_{21} = 101$ for $p \not\equiv 1 (mod\ 5)$

$$\zeta_M(T,\psi) = \frac{R_1(T) R_A^{20}(T) R_B^{30}(T)}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}$$

with

$$R_1 = 1 + a(z)T + b(z)T^2 + a(z)p^3 + p^6T^4$$

and R_A and R_B are similar quartics. Hence the degree of the numerator is indeed 204 while exponents in the the denominator correspond to the vertical Hodge numbers $b_{2k} = h_{kk} = 1$, for $k = 0, \ldots, 3$. Similary for the mirror quintic one has for $h_{11}(W) = 101$ and $h_{21}(W) = 1$

$$\zeta_W(T,\psi) = \frac{R_0(T)}{(1-T)(1-pT)^{101}(1-p^2T)^{101}(1-p^3T)}$$

with the same $R_1(T)$. In particular for $z = 5^{-5}$, $R_1(T)$ becomes cubic and factorises as

$$R_1(T) = \left(1 + \left(\frac{p}{5}\right) pT\right)(1 - \alpha_p T + p^3 T^2) \; .$$

where the α_p are the coefficients of the modular form f of Chad Schoen and $\left(\frac{p}{5}\right)$ is the Legendre symbol.

Remarks on the reduction of $R_1(T)$ In the mirror W of one moduli examples $R_1(T)$ is of degree 4 because $\dim(H_3(W,\mathbb{Z}))=4$. When does it reduce so that the coefficients a_p give rise to a modular form? In the case of *rigid Calabi-Yau* with middle cohomology

$$h_{3,0}=1,\quad h_{2,1}=0,\quad h_{1,2}=0,\quad h_{0,3}=1$$

then R_1 is of the form

$$R_1=1-a_pT+p^3T^2$$

again with $f=\sum_{k=1}a_kq^k\in S_4(\Gamma_0(N))$. These are known modular Calabi-Yau with ca 50 examples studied by Hulek, Verill, Van Straten, Cynk, Noriko Yui.

Attractor mechnism Kallosh, Ferrara, Strominger, Gibbons : Fix an element $\gamma \in H^3(\mathbb{W}, \mathbb{Z})$. This fixes a flux and determines an attractor locus in the complex moduli space $\mathcal{M}_{cs}(W)$. The latter is characterised by

$$\gamma = \Pi_z^{3,0}(\gamma) + \Pi_z^{0,3}(\gamma)$$

The latter have two interpretations: 1.) The moduli values (scalars in the vector multiplets) are the ones of an $N = 2$ dyonic black holes of charge γ at its horizon.

2.) At these points the flux superpotential $W_\gamma = \gamma_i P(z)$ allows for a vacuum $D_\alpha W_\gamma = 0$.

Conjecture KSZ, Candelas, Van Straten : These are preferred values, where the the Calabi-Yau becomes modular and $R_1(T)$ factorizes

$$R_1(T) = (1 - \alpha_p T + pT^2)(1 - a_p T + p^3 T^2)$$

where $g = \sum_{k=1} \alpha_k q^k \in S_2(\Gamma_0(M))$ and $f = \sum_{k=1} a_k q^k \in S_4(\Gamma_0(M))$. The L function values determine the value of the $U(1)$ gauge coupling $\tau = \frac{4\pi}{e^2} + \frac{\theta}{2\pi}$ with $\theta \in \mathbb{Q}$ Cecotti and Vafa.

Relation to periods over the p-adic numbers after Dwork

A natural completion of $\mathbb{F}_{p^k}$ are the p-adic numbers. Consider now x_i in such a bigger field K and consider the Frobenius map (not to be confused to the Frobenius method above) $\psi \in \mathbb{F}_p$

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1^p, x_2^p, x_3^p, x_4^p, x_5^p)$$

This is an *automorphism* on M as

$$P(x,\psi) = 0 \bmod p \qquad \leftrightarrow 0 = P(x,\psi)^p = P(x^p,\psi) \bmod p\,.$$

By Fermat's little theorem $a^p = a \bmod p$ the fix points are the $x_i \in \mathbb{F}_p$ that solve $P(x,\psi)=0$. The Lefshetz fix point theorem suggest that these numbers are given by the trace of the the Frobenius action on the cohomology of M. It turns out that this trace can be calculated by p-adic periods. More genrally one expects

$$L(f,s)=\prod_p \frac{1}{\det(1-F_pH(*))}\ .$$

In particular (R-V '03) showed for good primes p the so called super-congruence relation to the fundamental period L_0

$$^{(p-1)}L_0 = \sum_{n=0}^{p-1} \frac{(5n)!}{(n!)^5} \frac{1}{5^{5n}} = \alpha_p \bmod p^3 \ . \qquad (4)$$

Here the modulus for the rationals number on the left is taken by the p-adic expansion. For example for $p = 3$ one gets for the quintic

$$\sum_{n=0}^{2} \frac{(5n)!}{(n!)^5} \frac{1}{5^{5n}} = \frac{31 \cdot 101 \cdot 131}{5^8} = 1+2\cdot 3+3^4+2\cdot 3^6+O(3^7) \ .$$

Hence $\frac{31\cdot 101\cdot 131}{5^8} \bmod 3^3 = 7$. Since the space of

$S_4(\Gamma_0(N))$ is finite it is possible to fix the form by evaluating the super-congruence relation at a few good primes.

☆ Periods and quasi periods

❶ Useful definitions for $f \in S(\Gamma_0(N))$

- For $f : \mathbb{H} \to \mathbb{C}$ differentiable, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{R})$ and $k \in \mathbb{Z}$ we define

$$\left(f|_k\right)(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

- If $\Gamma \in \mathrm{Sl}_2(\mathbb{R})$ is commensurable with $\mathrm{Sl}_2(\mathbb{Z})$ we define $M_k^{mero} = \{f : \overline{\mathbb{H}} \to \mathbb{C} | f \text{ meromorphic}, \ f|_k\gamma = f, \forall \gamma \in \Gamma\}$

$$M_k^! = \{f \in M_k^{mero} | \text{poles only at cusps}\}$$
$$M_k^\infty = \left\{f \in M_k^! | \text{poles only at } i\infty\right\}$$
$$M_k = \{f \in M_k^\infty | \text{holomorphic}\}$$

- Let $f = \sum_{n\in\mathbb{Z}} a_n q^n$ Fourier expansion near a cusp
$$S_k^{mero} = \left\{f \in M_k^! | \text{all residue vanish}\right\}$$
$$S_k^! = \left\{f \in M_k^! | a_0 = 0 \text{ at all cusps}\right\}$$
$$S_k^\infty = \{f \in M_k^\infty | a_0 = 0 \text{ at } i\infty\}$$
$$S_k = \{f \in M | a_0 = 0\}$$

- $M_k^\#, S_k^\#$ for $\# = \text{mero}, !, \infty$

❷ Periods polynomial for $f \in S(\Gamma_0(N))$

We define a map

$$r : S_k(\Gamma) \to V_k \ ,$$

where V_k is the space $\mathrm{Sym}^{k-2}(\mathbb{C}^2)$, by the Eichler integral

$$r_f(\gamma)[x] = \int_{\tau_0}^{\gamma^{-1}(\tau_0)} f(\tau)(\tau - x)^{k-2}\mathrm{d}\tau \ .$$

This choice seems to depend strongly on γ and τ_0, but one really needs to understand this as a representative in a homology theory.

- Let $\tilde{f}$ sufficiently differentiable and an Eichler integral for f, i.e. $D^{k-1}\tilde{f} = f$. Then Bol's identity states

$$D^{k-1}\left(\tilde{f}|_{2-k}(\gamma - 1)\right) = f|_k(\gamma - 1) = 0 \ .$$

  Note that $D^{k-1}(M_{2-k}^{\#}) \subset S_k^{\#}$.

- For any $f \in S_k^{\#}$ the Eichler integral is determined by a meromorphic function $\tilde{f} : \mathbb{H} \to \mathbb{C}$ such that $D^{k-1}(\tilde{f}) = f$. $\tilde{f}$ is unique up to $p \in V_k = span\langle 1, \ldots, \tau^{k-2}\rangle$

- For any $\gamma \in \Gamma$ the period polynomial is defined as $r_f(\gamma) = \tilde{f}_{2-k}\gamma - \tilde{f}$ and by Bol's identity $r_f(\gamma) \in V_k$

Theorem (Eichler '57) 1. $r_f \in H^1_{par}(\Gamma, V_k)$, i.e. $r_{f,\gamma\gamma'} = r_{f,\gamma}|\gamma' + r_{f,\gamma'}$ fullfils the co-cycle condition and $r_f(\gamma) \in V_k|_{2-k}(\gamma - 1)$ for γ parabolic
2. $H^1_{par}(\gamma) = S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$

$$r_f(W_N)[x] = \sum_{n=0}^{k-2} \frac{L_f(n+1)}{(k-2-n)!}(2\pi i x)^{k-2-m} N^{k/2},$$

Here $W_N = \sqrt{N}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

Let $f|_k T_p = t_p f$ the Hecke operation on f and

$\mathbb{Q}_f = \mathbb{Q}(t_1, t_2, \ldots)$

Theorem (Eichler '57, Shimura '59, Manin '73) There exits $\omega_+ \in \mathbb{R}$, $\omega_- \in i\mathbb{R}$ such that

$$r_f(\gamma) \in \mathbb{Q}_f\omega_+ \oplus \mathbb{Q}_f\omega_- \ .$$

$\omega_\pm$ are *the periods* of f.

Proposition KSZ'18 Let Γ be $\Gamma_0(N)$ and $r(\gamma)$ be co-cycles for $\gamma \in \Gamma$. Then the action of the Hecke operator $\mathbb{T}_k$ is given by

$$\left(r|_{2-k}\mathbb{T}_n\right)(\gamma) = \sum_{i=1}^{R} r(\gamma_i)|_{2-k}M_{\pi_\gamma(i)} \ . \tag{5}$$

Here M_i with $i = 1, \ldots, R$ are representatives of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ with

$$\mathcal{M}_{n,N} = \left\{ m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det(m) = n,\ c \cong 0 \bmod N \right\} \tag{6}$$

and the cardinality R is given in terms of the divisor function as $R = \sigma_1(n)$. The $\gamma_i \in \Gamma$ are determined from the identity

$$M_i\gamma = \gamma_i M_{\pi_\gamma(i)}\ . \tag{7}$$

$\pi_\gamma(i)$ denotes a permutation of the indices $i = 1, \ldots, R$, whose γ dependence on γ is uniquely determined.

In particular if we know the generators of $\Gamma_0^+(N)$ we can calculate the operation of the Hecke algebra on the period polynomials and simultaneously diagonalize w.r.t. to the Hecke operators and the $\mathbb{Z}_2$, which corresponds to complex conjugation $r_f(\gamma)(\tau) = \overline{r_f(\gamma^\epsilon)(-\bar\tau)}$, where $\gamma^\epsilon = \epsilon\gamma\epsilon$ is conjugation by $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

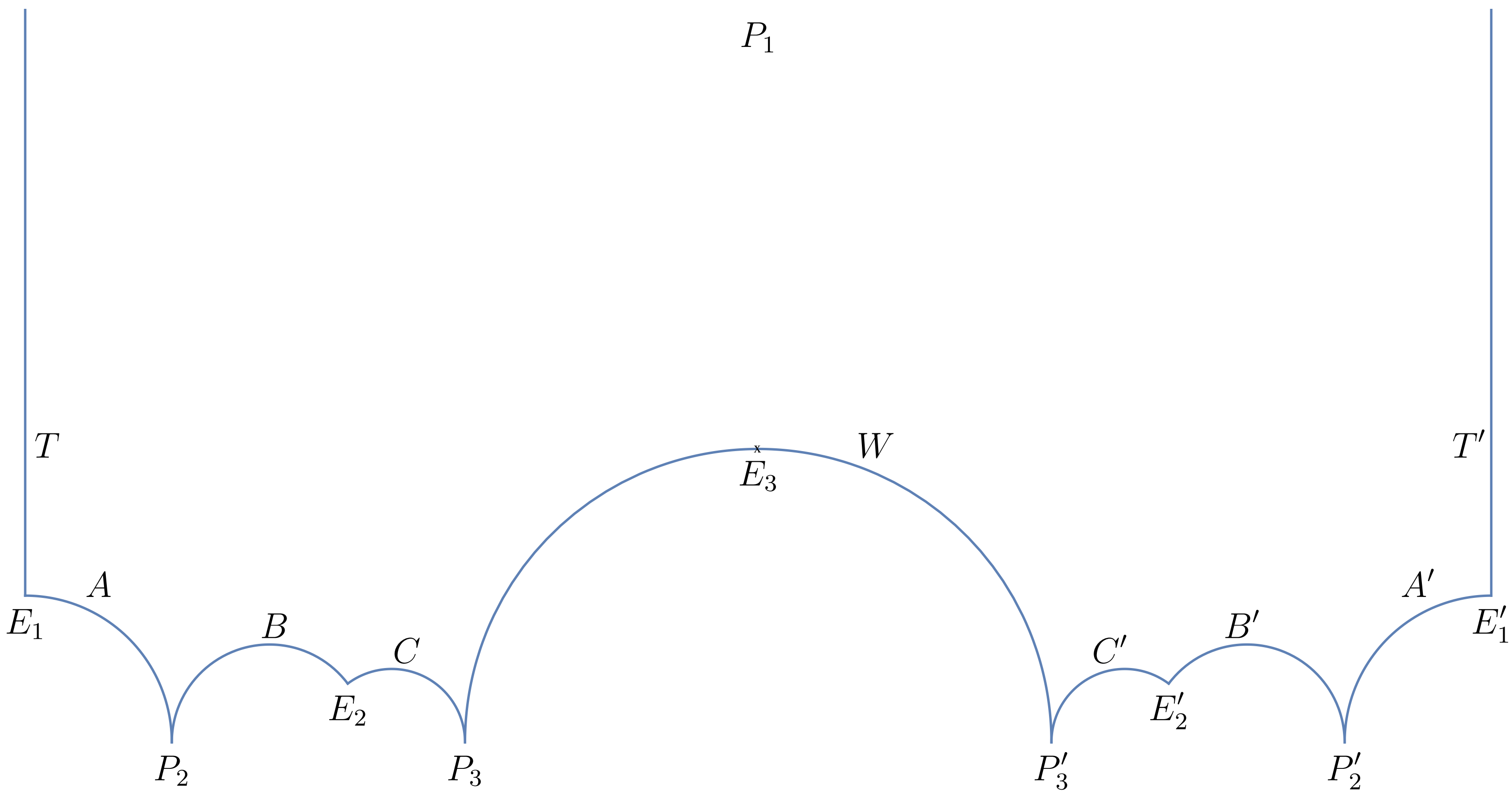

Figure 2: The fundamental region of $\Gamma_0^*(25)$ with three parabolic P_1, P_2, P_2 and three elliptic points E_1, E_2, E_3 of order two, yields all generators and its relations.

$r^{\pm}_{\lambda(2)}$	T	W	A	B	C
$r^{+}{}_{1}$	0	0	$x^2+x+\frac{6}{25}$	$\frac{21x^2}{16}+\frac{7x}{8}+\frac{7}{50}$	$\frac{5x^2}{2}+\frac{5x}{4}+\frac{3}{20}$
$r^{-}{}_{1}$	0	$-\frac{2x}{3}$	$x+\frac{1}{2}$	$\frac{125x^2}{12}+\frac{41x}{6}+\frac{7}{6}$	$\frac{125x^2}{6}+11x+\frac{3}{2}$
$r^{+}{}_{4}$	0	0	$x^2+x+\frac{6}{25}$	$3x^2+2x+\frac{8}{25}$	$4x^2+2x+\frac{6}{25}$
$r^{-}{}_{4}$	0	$-\frac{x}{8}$	0	$\frac{5x^2}{4}+x+\frac{1}{5}$	$\frac{35x^2}{8}+\frac{9x}{4}+\frac{3}{10}$
$r^{+}{}_{-4}$	0	0	$x^2+x+\frac{6}{25}$	$x^2+\frac{2x}{3}+\frac{8}{75}$	0
$r^{-}{}_{-4}$	0	$-\frac{x}{4}$	$x+\frac{1}{2}$	$\frac{25x^2}{2}+\frac{17x}{2}+\frac{3}{2}$	$\frac{75x^2}{4}+\frac{17x}{2}+1$

Table 1: The eigen basis of parabolic cocycles w.r.t. $\mathbb{T}_2$ and $r \mapsto r^*$.

❸ Quasi-periods for $f \in S(\Gamma_0(N))$

Proposition (KSZ '18) 1. $\mathbb{S}_k = S_k^{\infty}/D^{k-1}(M^{\infty}) = S^{!}/D^{k-1}(M_k^{!}) = S_k^{mero}/D^{k-1}(M_k^{mero})$

2. $\mathbb{S}_k \cong H^1_{par}(\Gamma, V_k)$
3. $0 \to S_k \to \mathbb{S}_k \to S^V \to 0$ The third arrow is defined by the $(-1)^{k-1}$ symmetric pairing $\{,\} : S_k^{mero} \times M_k^{mero} : (f, g) \mapsto 2\pi i \sum_{\tau \in \Gamma \backslash \mathbb{H}} \mathrm{Res}_\tau \tilde{f} g d\tau$

S_k admits a basis of simultaneous eigenfunctions w.r.t. to the Hecke operation $(T_m)_{m \geq 1}$, i.e. if $f \in S_k$, then $f_k T_m = t_m f$. The $(T_m)_{m \geq 1}$ act also on $M_k^{\#}, S_k^{\#}$ hence on $\mathbb{S}_k$.

Proposition (KSZ '18) Let $f \in S_k$ with $f_k T_p = t_p f$.

Then $\exists f^V \in S_k^V$ with

$$f^V|_k T_p = t_p f^V + D^{k-2} g$$

for some $g \in M_{k-2}^{\#}$. For the quintic the dual to holomorphic form of Chad Schön f with Hecke eigenvalue $\lambda^{(2)} = 1$ is

$$f^V \ = \frac{1}{q^4} - \frac{27}{32q^3} - \frac{1}{8q^2} - \frac{1}{8q} + \frac{q}{240} - \frac{7q^2}{120} + (q^3)$$

Theorem (KSZ '18)
1. Let $f^V \in S_k^V$ be dual to f, then there exits $\eta_+ \in \mathbb{R}$

and $\eta_- \in i\mathbb{R}$ such that

$$\boxed{r_{fV}(\gamma) = \mathbb{Q}_f\omega_+ \oplus \mathbb{Q}_f\omega_- \oplus \mathbb{Q}_f\eta_+ \oplus \mathbb{Q}_f\eta_-}$$

2. The following Legendre relation holds

$$\boxed{\det \begin{pmatrix} \omega_+ & \omega_- \\ \eta_+ & \eta_- \end{pmatrix} = (2\pi i)^{k-1}\mathbb{Q}}$$

☆ Conclusions, Outlook and and Questions

- There is new connection between counting points over finite fields $\mathbb{F}_{p^k}$ on the compactification geometry M and physical quantities in the effective theory.

- In particular the physical D-brane masses at attractor points (D2/D4 at conifold) are given by *Hecke L*-function values associated to the modular functions counting these points and are the *first example* of this connection.

- Physically evaluating the L-function is like regulating the trace of an operator, the Frobenius operator. Given that the trace has a physical meaning. What is the physical meaning of the individual eigenvalues?

- More generally, if the $\zeta_T(M,\psi)$-function factorise for ψ, $\psi \in \mathbb{F}_p^k$ seems always to be an attractor point (Candelas et al.). What is about the converse ?

- This factorisation allows at this to deconstruct the period problem of M into lower dimensional period problems in the sense of motifs.

- For CY its leads in particular to the period– and quasi period theory of Eichler, Shimura, Manin and Zagier to which we added some insights for $\Gamma_0^+(N)$.