Gauge Theory from Exceptional Geometry

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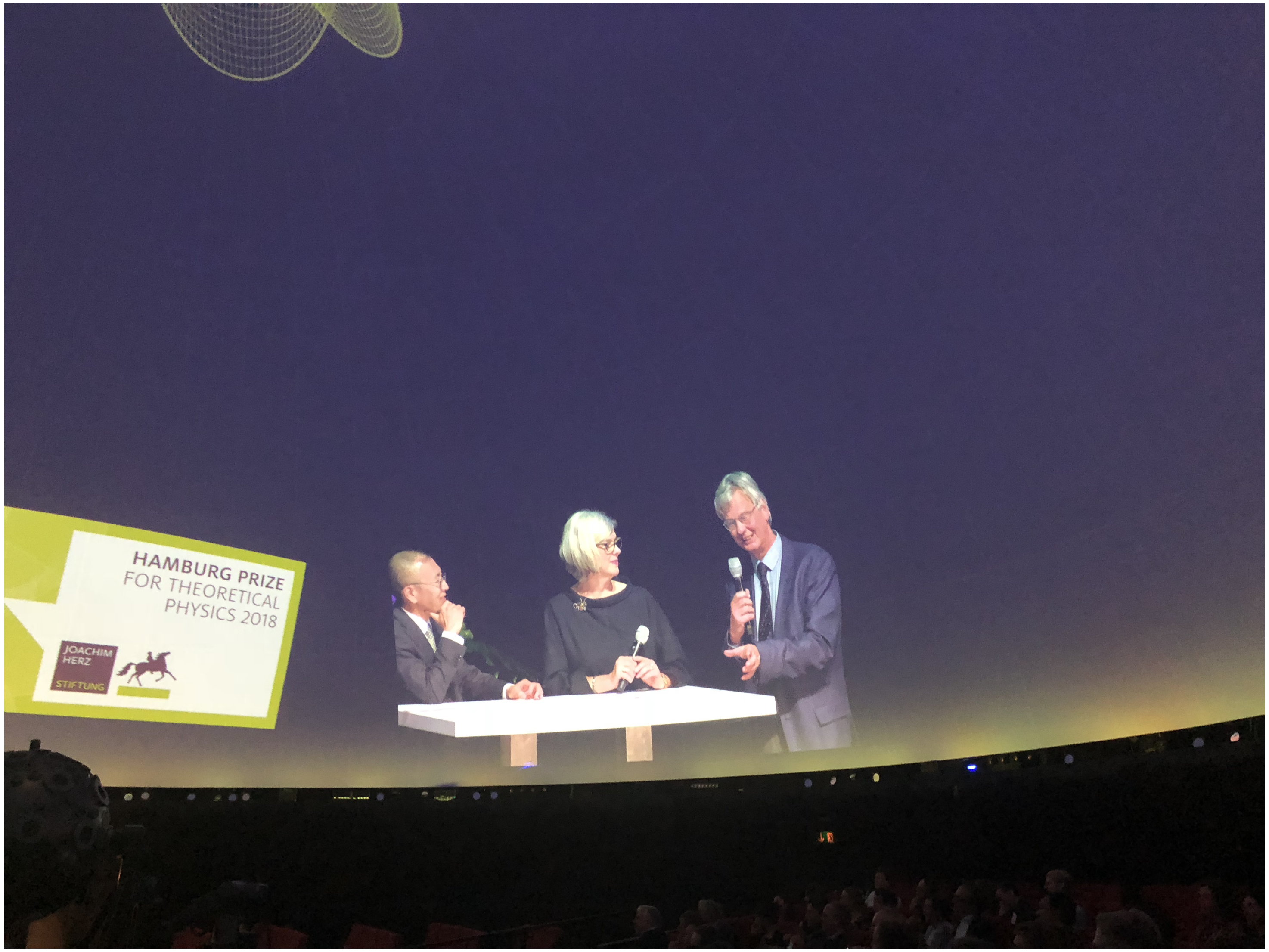


Many congratulations, Hirosi!

A more precise title of this talk is:

Gauge theories from Exceptional Holonomy
Compactifications of M-theory

Exceptional meaning: not Calabi-Yau, but G_2 or Spin(7) holonomy.

Motivation:

New constructions of gauge theories in 4d and 3d with minimal supersymmetry (i.e. hard to study)
Dualities relating 3d $\mathcal{N} = 1$ theories to TQFTs '3d 3d correspondence'.

References:

- with Andreas Braun (Oxford): `1708.07215`, construction of new G_2 manifolds
- with Andreas Braun (Oxford): `1803.10755`, construction of new Spin(7) manifolds
- with Andreas Braun , Sebastjan Cizel, Max Hubner (Oxford): `to appear` on gauge sector of G_2 manifolds
- with Braun, Del Zotto, Larfors, Halverson, Morrison: `1803.02343`
- with Julius Eckhard (Oxford), Jin-Mann Wong (KIPMU): `1804.02368` on N=1 version of 3d 3d correspondence
- with Julius Eckhard, Heeyeon Kim (Oxford): `in progress` on refinement of the N=1 3d 3d correspondence

Recap of some Basics

Clearly underlying any progress in string landscaping relies to some extent on the tools we have to study the relevant compactifications:

Low energy effective theory of the dimensional reduction depends on the geometry.

Kaluza-Klein theory:
gravity in 5d on a circle S^1: gravity + scalar + a $U(1)$ gauge field in 4d:
expanding g_{MN} along $\mu = 0, 1, 2, 3$, $x = 4$.

The topology/geometry determines what massless fields arise in the lower dimensions.

Enabled classification of 6d superconformal field theories using F-theory/geometry of elliptic Calabi-Yau manifolds.

What's also clear: this is just one small corner of the landscape.

Supersymmetry and Holonomy I

General Relativity: M_n with the Levi-Civita ∇. Parallel-transporting vectors they transforms by an element of $SO(n) \Rightarrow$ Holonomy

A manifold M_n is said to have reduced holonomy, if the parallel transport only acts with a strict subgroup $G \subset SO(n)$. All possible holonomy groups have been classified by Berger:

Holonomy	$\dim_{\mathbb{R}}$	Ricci	Type
$SO(n)$	n		Generic orientable manifold
$U(n)$	$2n$		Kähler
$SU(n)$	$2n$	0	Calabi-Yau
$Sp(n)$	$4n$	0	Hyper-Kähler
G_2	7	0	G_2 manifold
Spin(7)	8	0	Spin(7) manifold

G_2 and Spin(7) are so-called exceptional holonomy manifolds.

Supersymmetry and Holonomy II

Supersymmetric backgrounds are characterized by

$$\langle \delta\Psi \rangle \equiv \nabla \epsilon = 0 \,.$$

Covariant spinors exist only for reduced holonomy manifolds (no fluxes)

4d compactifications: $\mathbb{R}^{1,3} \times M$

1. 10d string theories:
   M is 6d has $SU(3)$ holonomy, i.e. Calabi-Yau, preserves $\mathcal{N} = 2$ (8 susies)
2. 11d M-theory:
   7+4 =11: G_2, preserves 4d $\mathcal{N} = 1$ (4 susies)
   8+3 = 11: Spin(7) preservies 3d $\mathcal{N} = 1$ (2 susies)

How does reduced holonomy induce spinors (supersymmetry) in 4d?

11d Lorentz group breaks in a compactification on a 7-manifold as

$$
\begin{aligned}
SO(1,10) &\rightarrow SO(1,3)_L \times SO(7) \\
\mathbf{32} &\rightarrow \mathbf{4} \otimes \mathbf{8}\,.
\end{aligned}
$$

To get spinors in 4d require these to be invariant under the local Lorentz symmetry of the compactification space. For $SO(7)$: No chance.
For G_2:

$$
\begin{aligned}
SO(7) &\rightarrow G_2 \\
\mathbf{8} &\rightarrow \mathbf{7} \oplus \mathbf{1}\,.
\end{aligned}
$$

$\Rightarrow$ M-theory on a 7-manifold of G_2 holonomy preserves 4d $\mathcal{N}=1$ supersymmetry.

Likewise: M-theory on Spin(7) results in 3d $\mathcal{N}=1$ theory.

Supersymmetry and Holonomy III: Submanifolds

Special holonomy: there are "volume-minimizing" cycles, which are calibrated by invariant forms.

Membranes in string/M-theory (D-branes or M-theory membranes) wrapped on these spaces preserves susy
$\Rightarrow$ Supersymmetric cycles

Examples:

- Calabi-Yau d-fold: Kähler submanifolds (calibrated by $\omega^{(1,1)}$) and d-real dimensional cycles (Lagrangians) (calibrated by $\Omega^{(d,0)}$)
- G_2: 3-cycles (associative) and 4-cycles (co-associatives) calibrated by Φ_3 and $*\Phi_3$

Wrapped branes give field theories, which depend on the geometry and embedding of susy cycles. Bonus: new dualities (AGT, 3d-3d).

Special vs. Exceptional Holonomy

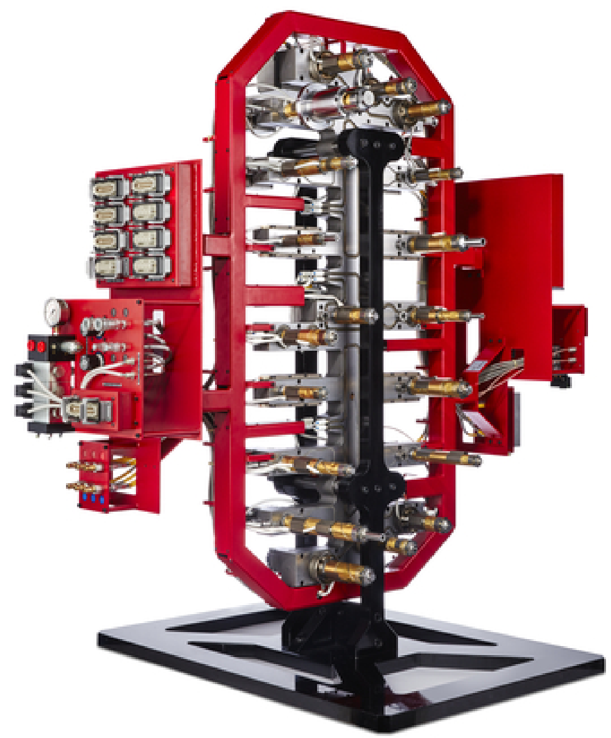
Lets recap from Wednesday evening:

Special vs. Exceptional Holonomy

As we all learned from Hirosi's movie: Calabi-Yau manifolds are friendly, squeaky, definitely 'kawai' creatures populating the extra 6 dimensions that string theory predicts.

To some extent, there's a mathematical truth to that, going back to Yau's proof of the Calabi conjecture: existence of a unique Ricci-flat metric on a Kähler manifold with trivial canonical class.

In this analogy, the way to think about exceptional holonomy G_2 manifolds (and likewise Spin(7)) is as miniature monsters populating 7d in an M-theory description of the real world. Google reveals: "G2 manifold"



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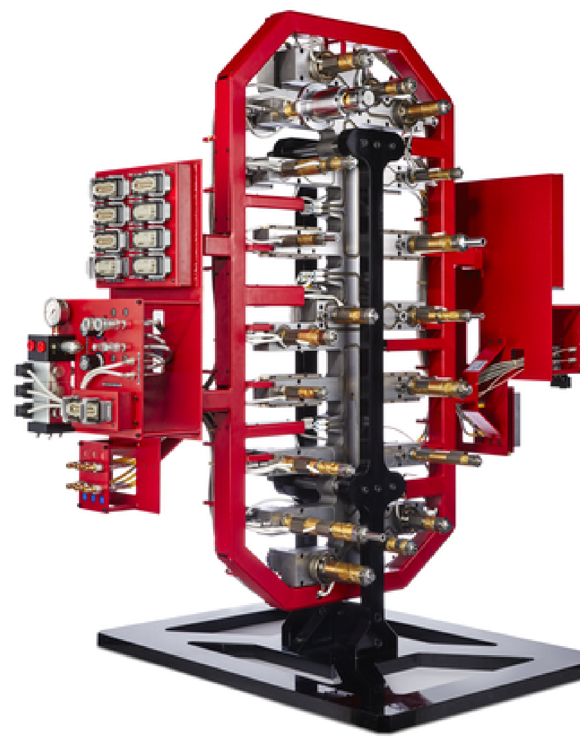


.....which is stragely close to what we call TCS-constructions of G_2.

ref: google images

Motivations for Exploring Exceptional Holonomy Manifolds

1. F-theory: Classification of 6d SCFTs ✓ What about 4d?
   ⇒F-theory on Calabi-Yau + fluxes
   ⇒ M-theory on G_2 manifolds (purely geometric problem)

   Question:
   Is there a classification of 4d $\mathcal{N} = 1$ SCFTs using F or M-theory?

2. Recent progress is 3d minimally (and also non-)supersymmetric field theories, which have interesting dualities and phase transitions.
   ⇒ Geometric engineering using M-theory on Spin(7)
   ⇒ Domain walls in 4d $\mathcal{N} = 1$ theories, i.e. M-theory on G_2 manifolds

   Question:
   Using geometric engineering in M-theory, can we construct 3d$\mathcal{N} = 1$ theories, and study the dualities and phases?

3. Recent progress in Mathematics: Finding explicit constructions of G_2 or Spin(7) holonomy manifolds is notoriously difficult. Recently an large class (order 10^3) of compact G_2 manifolds were constructed* in mathematics by Corti, Haskins, Nordstrom, Pascini, based on earlier work of Kovalev and of Donaldson, so-called Twisted Connected Sums (TCS).

   Questions:

   What are the properties of the 4d theories obtained from this new class of G_2 manifolds?

* This does not mean, they constructed the G_2 metric explicitly, but they proved that on these geometries there exists a Ricci-flat G_2 holonomy metric.

Setups to keep in mind:

1. M-theory on $\mathbb{R}^{1,3} \times M_7$ G_2 manifold.
   $\Rightarrow$ 4d $\mathcal{N} = 1$
2. M2 and M5-branes wrapping supersymmetric three-cycles $M_3 \subset M_7$, M_7= G_2 holonomy.
   - M2 instantons
   - M5-brane world-volume $\mathbb{R}^{1,2} \times M_3 \subset \mathbb{R}^{1,3} \times M_7$
     $\Rightarrow$ 3d $\mathcal{N} = 1$ domain wall theory in 4d $\mathcal{N} = 1$.
3. M-theory on $\mathbb{R}^{1,2} \times M_8$ with M_8 a Spin(7) manifold
   $\Rightarrow$ 3d $\mathcal{N} = 1$

G_2

- Lie group G_2 is defined as 14 dimensional subgroup of $GL_7\mathbb{R}$ that leaves in variant the three-form

$$\Phi_3 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}\,.$$

- G_2-holonomy manifolds are 7d admitting a Ricci-flat metric with holonomy G_2.

- Metric specified by a three-form, the G_2-form, Φ

$$\mathrm{d}\Phi = \mathrm{d}\star\Phi = 0\,.$$

- Calibrated submanifolds are 3d associatives M_3

$$\Phi|_{M_3} = \mathrm{vol}(M_3)\,.$$

  i.e. volume minimising in their homology class, or 4d co-associatives, which are calibrated by $\star\Phi$.

4d $\mathcal{N} = 1$ Gauge Theories from G_2 Holonomy

Gauge Sector of M-theory on G_2 Manifolds

- M-theory on a singular, non-compact K3, i.e. $\mathbb{C}^2/\Gamma_{ADE}$:
  C_{MNP} KK-reduction and M2-branes gives 7d SYM with G=ADE.

- Fiber ADE singularity over a three-manifold:

$$\mathbb{C}^2/\Gamma_{ADE} \to M_3$$

  This can be given a local G_2 structure.

- Adiabatic picture: 7d SYM on M_3.

$$SO(1,6)_L \times SU(2)_R \quad \rightarrow \quad SO(1,3)_L \times \underline{SO(3)_M \times SU(2)_R}$$

  M_3 has generic $SO(3)$ holonomy. To retain susy in 4d, we need to topologically twist $SO(3)_M$ with $SU(2)$ R-symmetry:
  $\Rightarrow SO(3)_{\text{twist}} = \text{diag}(SO(3)_M \times SU(2)_R)$.
  $\Rightarrow$ 4 supercharges in 4d.

Higgs bundle on M_3

The supersymmetric field configurations on M_3 are characterized by the BPS equations

$$\langle \delta\psi \rangle = 0$$

Background fields are one-forms **3** of $SO(3)_{\text{twist}}$:

- ϕ twisted scalars
- $\mathcal{A}$ gauge field components along M_3

$$0 = F_{\mathcal{A}} + i[\phi, \phi]\,, \qquad 0 = D_{\mathcal{A}}\phi$$
$$0 = D^{\dagger}_{\mathcal{A}}\phi\,.$$

For $[\phi, \phi] = 0$ and ϕ regular, non-trivial solutions only exist for $\pi_1(M_3) \neq 0$.

Matter field zero-modes

Zero-modes of 4d matter fields depend on background values of ϕ and $\mathcal{A}$:

$$\begin{aligned} \text{gauginos:} \quad & \chi_\alpha \in H^3_{\mathcal{D}}(M_3) \\ \text{Wilson-line-inos:} \quad & \psi_\alpha \in H^1_{\mathcal{D}}(M_3) \end{aligned} \qquad \text{where } \mathcal{D} = d - [(\phi + \mathcal{A}) \wedge \cdot]$$

Simplest class of solutions to BPS equations:

$$\mathcal{A} = 0 \quad \Rightarrow \quad d\phi = d^\dagger \phi = 0 \quad \exists f \text{ harmonic, with} \qquad \phi = df$$

For M_3 compact: no solutions.
M_3 with boundaries or alternatively, Poisson equation with source ρ.
$\Rightarrow$ Morse theory for critical loci points [Pantev, Wijnholt] or Morse-Bott theory for more general critical loci [Braun, Cizel, Hubner, SSN].

Morse and Morse-Bott theory for Zero-Modes

Matter zero modes: $U(1)$-valued Higgs field then $G \to H \times U(1)$, then charge q states counted by cohomology of

$$\mathcal{D}_f = d + q df \wedge \cdot \,.$$

- Charge distribution: ρ support on $\Gamma \subset M_3$. Either + or - charge $\Gamma_\pm$, with total charge distribution 0.
- Boundary conditions: Excise tubular neighborhood of $\Gamma_\pm$ and impose Neumann or Dirichlet b.c.:

$$\text{Dirichlet}: \qquad \alpha_t = 0\,, \qquad \text{Neumann}: \qquad \star\alpha_n = 0\,.$$

Theorem:

$$H^*_{\mathcal{D}_f}(M_3) = H^*(M_3, \partial_- M_3)$$

Gives matter localized in codimension 7 (points) in the local G_2. Chiral index:

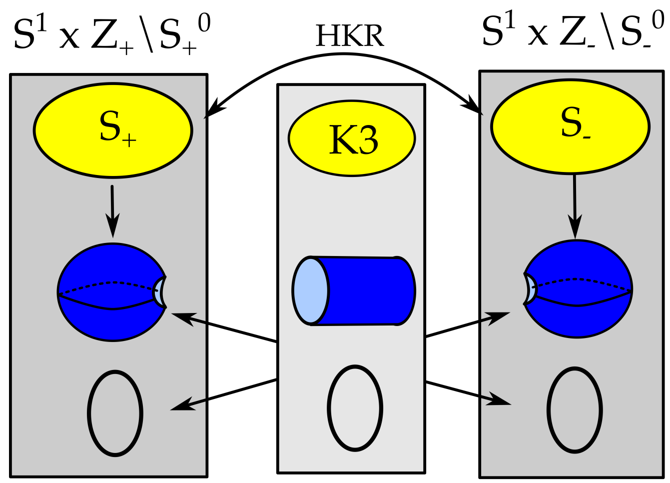
$$\chi(M_3, \partial_- M_3) = b^2(M_3, \partial_- M_3) - b^1(M_3, \partial_- M_3)\,.$$

All known compact G_2 manifolds

- First example: non-compact $(\mathbb{C}^2 \times S^3)/\Gamma_{ADE}$ [Bryant, Salamon (1989)]
- Compact: [Joyce (2000)] orbifolds T^7/Γ. Order 10 examples, but far from fully classified
- Compact: Calabi-Yau $\times S^1$ with antiholomorphic involution [Joyce, Karigiannis (2017), some earlier work]
- Compact: Twisted Connected sum: [Corti, Haskins, Nordström, Pacini (2015)]. Thousands of examples...
  ... but they are very special (see codim 6 singularities)

Except for non-compact constructions ([Acharya, Witten]) these do not have codimension 7 singularities, i.e. no chiral matter.

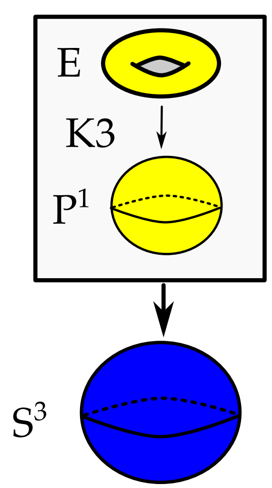
Twisted Connected Sums



Building blocks: Calabi-Yau three-folds that are fibered by K3s $S_\pm$ over $\mathbb{P}^1$. Remove a fiber ($S_0^\pm$), take a product with S^1 and glue $S_\pm$ with a hyper-Kähler rotation (HKR)

$$\omega_\pm \leftrightarrow \mathrm{Re}\,\Omega_\mp^{(2,0)}\,, \quad \mathrm{Im}\,\Omega^{(2,0)} \leftrightarrow -\,\mathrm{Im}\,\Omega^{(2,0)}$$

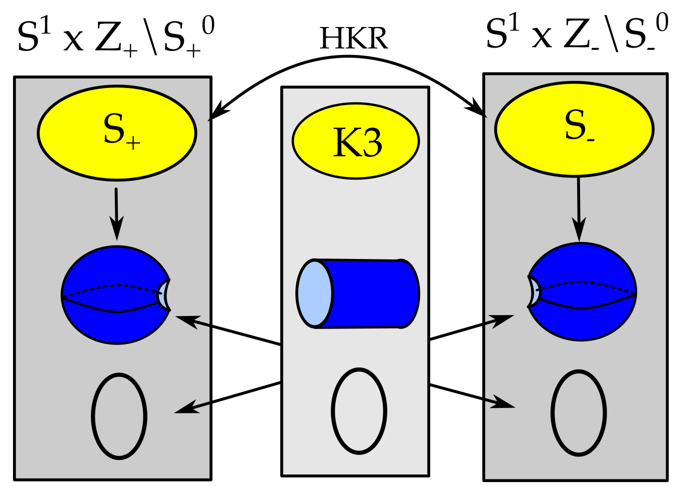
[Kovalev; Corti, Haskins, Nordström, Pacini]



Let $S_\pm$ be elliptically fibered K3 with sections, i.e. Weierstrass models over $\mathbb{P}^1$, and e.g.
S_+: smooth elliptic fibration
S_-: two II^* singular fibers
Singular K3-fibers result in non-abelian gauge groups, e.g. E_n

[Braun, SSN]

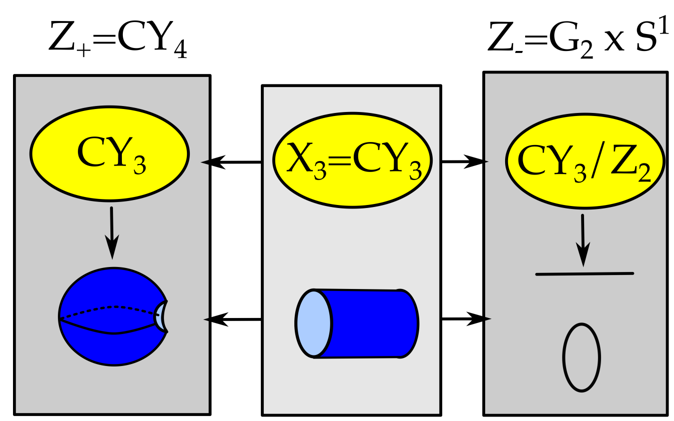
Field Theoretic Interpretation of TCS



- M-theory on Calabi-Yau $Z_\pm \times S^1$ preserves $\mathcal{N} = 2$ in 4d.
- Central region: $K3 \times T^2 \times$interval preserves $\mathcal{N} = 4$ in 4d.
- HyperKähler rotation and gluing retains only a common $\mathcal{N} = 1$ susy.
- Matter Spectrum: non-chiral, codimension 6 singularities (discriminant locus is circles in the base S^3)
  Can TCS be deformed to yield chiral 4d theories?
  No. [Braun, Cisel, Hubner, SSN][Chen]

Interlude: Compact Spin(7)

- [Joyce (2000)] orbifold T^8/Γ
- Calabi-Yau four-fold orientifold [Kovalev (2018?)]
- Generalized Connected Sum: [Braun, SSN (2018)]



Field theoretic construction: $Z_\pm$ preserves 3d $\mathcal{N}=2$. Central region preserves 3d $\mathcal{N}=4$, but gluing retains only common 3d $\mathcal{N}=1$.

For CY3 is elliptic, there is an F-theory dual with 4d '$\mathcal{N}=1/2$' [Vafa]. Recently used the Generalized Connected Sums construction to build F-theory dark matter model [Heckman, Lawrie, Lin, Zoccarato].

More on the Physics of TCS G_2

Computing non-perturbative corrections to G_2 is notoriously difficult [Harvey, Moore]. M2-brane instantons are hard, and even harder (mathematically!) to determine what the supersymmetric 3-cycles are in G_2!

In the TCS G_2 we can make a prediction for the existence of an infinite class of supersymmetric 3-cycles using string dualities.

M/K3 = Heterotic $/T^3$ applied fiberwise to TCS gives some surprising results [Braun, SSN]

WARNING: Non-string theorists: Take a 5 min break

M-theory/Heterotic String Duality for TCS

Moduli space for both theories: $\Gamma\backslash SO(3,19)/(SO(3)\times SO(19))\times\mathbb{R}^+$

M-theory on K3: moduli space of Einstein metrics on K3
Heterotic: Narain moduli space for T^3 compactification.
Specializing to elliptic K3s:
3 complex structures ω_i of the K3 are idenfied in the T^3 as follows:

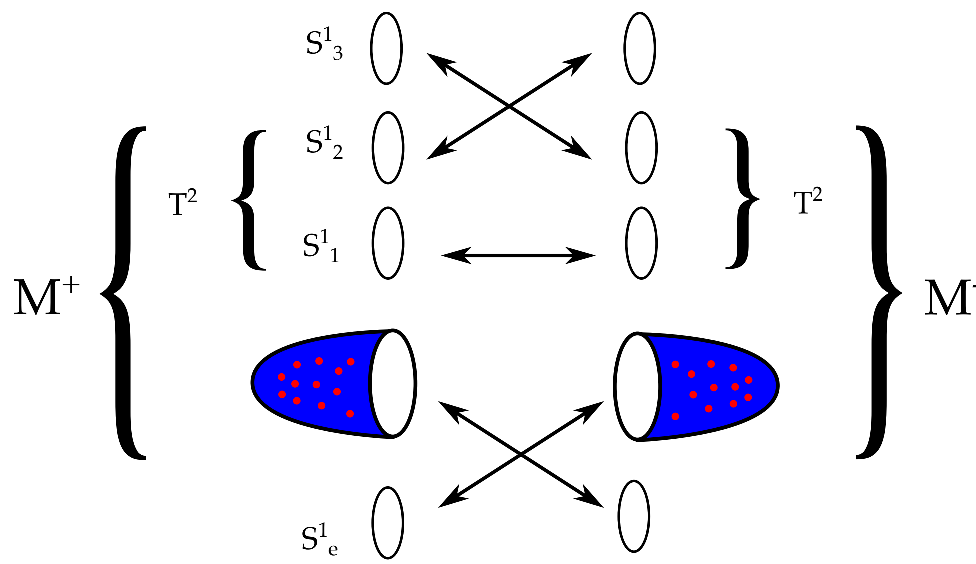
$$H^2(K3,\mathbb{Z}) = U_1 \oplus U_2 \oplus U_3 \oplus (-E_8)^{\oplus 2}$$

Periods of ω_i along U_i	$\leftrightarrow$	radii of the S^1_i
Periods of ω_i along $(-E_8)^2$	$\leftrightarrow$	Wilson lines along S^1_i

Fiber-wise duality for the TCS geometries with elliptic building blocks:
For an elliptic K3, additionally fibered over $\widehat{\mathbb{P}}^1$, only ω_1 and ω_2 vary.
By fiber-wise duality in heterotic only $T^2\subset T^3$ varies over the base $\widehat{\mathbb{P}}^1$, and the total space of the heterotic compactification is an elliptic K3$\times S^1_3$.

M-theory/Heterotic String Duality for TCS

[Braun, SSN, 2017]



Apply same gluing, i.e. HK rotation to these building blocks:

$$S^1_{2+} = S^1_{3-}\,, \quad S^1_{1+} = S^1_{1-}\,, \quad S^1_{3+} = S^1_{2-}\,.$$

We find: $h^{1,1}(X_{\text{het}}) = 19 = h^{1,2}(X_{\text{het}})$ for any such TCS!

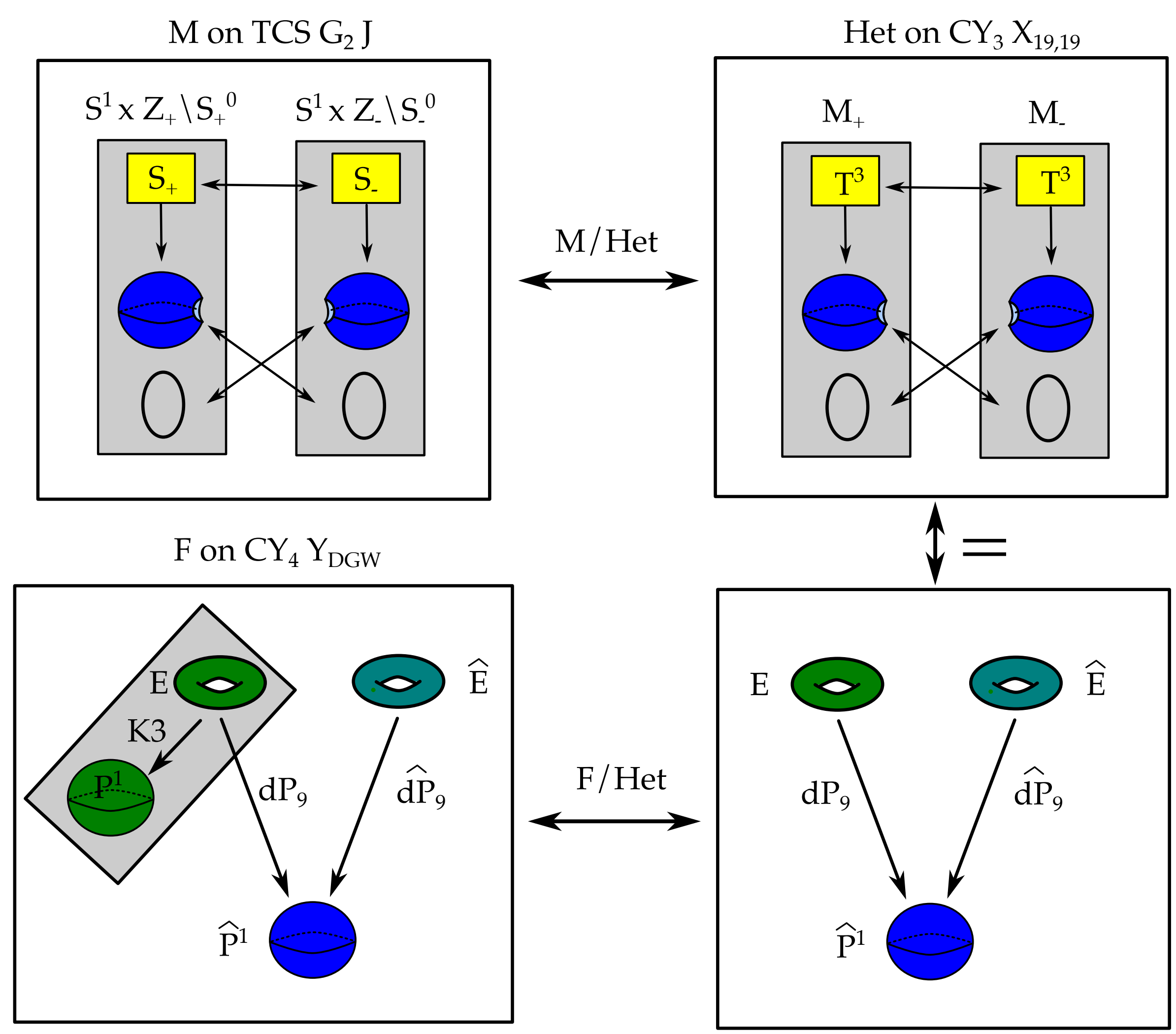
$\Rightarrow$ TCS-construction of SYZ-fibration of the Schoen CY3

$\Rightarrow$ All TCS with elliptic building blocks are dual to the Schoen CY3 with a choice of vector bundles.

Duality Chain for TCS G_2 Manifolds

[Braun, SSN (2017)]

Recap: M/K3 = Het/T^3 and Het/Elliptic CY3 = F-theory/K3-fibered CY4.



Instantons in the Duality Chain for TCS G_2 Manifolds

[BdZHLMS, 2018]

F-theory on $\mathbb{E} \hookrightarrow Y_{\text{DGW}} \to (\mathbb{P}^1 \times \widehat{dP_9})$ has inftinitely many D3-instantons [Donagi, Grassi, Witten], wrapping surfaces D which satisfy $\chi(D, \mathcal{O}_D) = 1$: $D_\gamma = \sigma_\gamma \times \mathbb{P}^1$, where σ_γ are sections of $\widehat{dP_9}$: choose in $H^2(dP_9, \mathbb{Z}) = U \oplus (-E_8)$

$$\sigma_\gamma = \sigma_0 + \gamma + n\hat{\mathbb{E}}$$

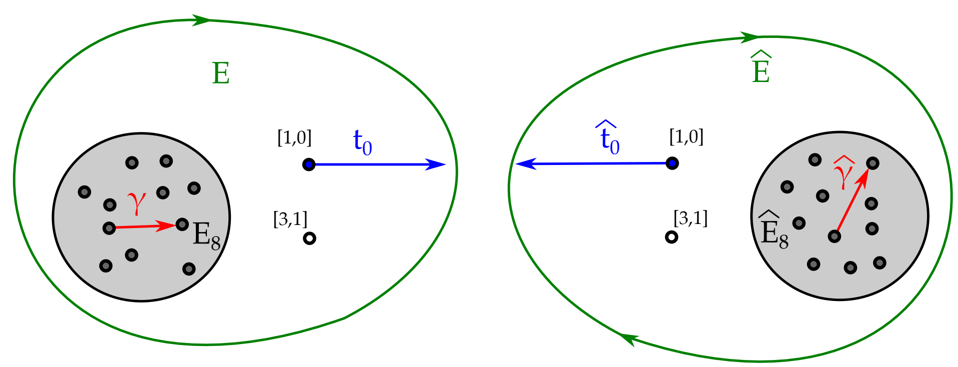
where $\sigma_0, \hat{\mathbb{E}} \in U$ are zero section and fiber class, $\gamma \in E_8$ with $\gamma^2 = -2n$. Then $\sigma_\gamma^2 = -1$ and $\sigma_\gamma \cdot \hat{\mathbb{E}} = 1$.

Heterotic string theory on the Schoen CY3 $X_{19,19}$: duality map allows to identify infinitely many world-sheet instantons.[Curio, Lüst]
These can be identified in the SYZ-description.

Define "string junction" $\mathfrak{t}_\gamma$ to each section σ_γ: for each building block of the TCS-description of the Schoen CY3 the T^2-fiber degenerates at 12 points: 10 realize the E_8 roots, and two with $[p,q]$ charges $[1,0]$ and $[3,1]$: the string junction (paths in the base)

$$\mathfrak{t}_\gamma = \gamma + \mathfrak{t}_0 + nE$$

with collapsing S^1 fibered above yield thimples (half-S^2).



To construct the sections σ_γ we glue the thimbles from each building block together.

<u>M-theory on the TCS J</u>: S^1-fibrations map to S^2-fibrations over junctions.
$\Rightarrow E_8 \oplus E_8$ worth of assocative homology three-spheres $\Sigma_{\gamma\hat{\gamma}}$.

Expanding $C_3 + i\Phi$ in terms of these $H^3(J,\mathbb{Z})$ cycles (coefficients given by ω_i) the superpotential correction by M2-instantons is then [BdZHLMS]

$$\begin{aligned}\Delta W^{\mathrm{M2}} &= \sum_{\Sigma_{\gamma\hat{\gamma}}} G(\gamma\hat{\gamma}) \exp\left[2\pi i \int_{\Sigma_{\gamma\hat{\gamma}}} C + i\Phi_3\right] \\ &= \sum_{m,\hat{m}\in\mathbb{Z}^8\times\mathbb{Z}^8} G(\gamma\hat{\gamma}) \exp 2\pi i \left[z + n\tau + \hat{n}\hat{\tau} + \sum_i m_i\varsigma_i + \hat{m}_i\hat{\varsigma}_i\right],\end{aligned}$$

For $G(\gamma\hat{\gamma}) = 1$ this just becomes a product of two E_8 θ-functions.

⇒ Using M/het/F duality applied to the TCS-construction with elliptic K3-building blocks as proposed in [Braun, SSN].

Conjecture:
For every element $(\gamma, \hat{\gamma}) \in E_8 \oplus E_8$ there is a pair of three-chains Σ^+_γ in Z_+ and Σ^-_γ in Z_-, with boundary a (-2) curve in the transcendental lattice of the asymptotic K3 S_0, which can be glued together to a $\Sigma_{\gamma\hat{\gamma}} \in H^3(J)$ We conjecture that the class of this three-cycle contains a unique associative representative that has the topology of a three-sphere.

Wrapped M2-branes give non-perturbative superpotential corrections.

In M-theory: there are also M5-branes.

What's the role of these? Domain wall theories in 4d.

Example:

$$M_7 = (\mathbb{C}^2 \times S^3)/\mathbb{Z}_k \,, \qquad M_3 = S_3/\mathbb{Z}_k$$

gives rise to 4d $\mathcal{N} = 1$ SQCD with gauge group $SU(k)$. M5-brane correspondes to domain wall theory. [Acharya, Vafa]

Goal is now to study what 3d domain wall theories there are, and we'll uncover some interesting connections to TQFTs via M5-brane correspondences.

3d $\mathcal{N} = 1$ and TQFTs

M5-branes are 6d membranes in M-theory. The effective theory is not a SYM theory (unlike D-branes) and most likely is non-Lagrangian, but is known to be the unique 6d $\mathcal{N} = (2, 0)$ superconformal field theory with gauge group ADE. Whatever can be learned about M5-branes should be, as they form one of the key missing pieces in our understanding of M-theory.

Recently a whole class of correspondences have been determined from M5-branes wrapped on supersymmetric cycles. The basic idea is:

- M5-branes on M_d yields a supersymmetric theory in $6-d$ dimensions: $T[M_d]$
- Observables such as partition functions on S^{6-d} or indices of $T[M_d]$ can be computed by considering a 'dual' theory obtained from M5-branes on S^{6-d}. This d dimensional theory is usually not supersymmetric, but a conformal or TQFT.
- Conjecture: TQFT partition function on M_d computes the supersymmetric partition function of $T[M_d]$.

M5-brane Correspondences: $\mathcal{N}=2$ SUSY

The sphere-partition functions for the $T[M_d]$ theories are computed by the following d-dimensional theories:

- d=2: AGT correspondence between 4d $\mathcal{N}=2$ theories and 2d Toda theories on M_2 [Alday, Gaiotto, Tachikawa]
  $\Rightarrow M_2$ is a curve in CY3
- d=3: 3d–3d correspondence between 3d $\mathcal{N}=2$ theories and complex Chern-Simons on M_3 [Gaiotto, Gukov, Dimofte]
  $\Rightarrow M_3$ is a Slag in a CY3
- d=4: 4d–2d correspondence between 2d $\mathcal{N}=(0,2)$ and topological sigma-model from M_4 into the Nahm moduli space [Assel, SSN, Wong]
  $\Rightarrow M_4$ is a Coassociative in G_2

M5-brane Correspondences $\mathcal{N} = 2$ SUSY

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M5-brane Correspondences: $\mathcal{N} = 1$ SUSY

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  $\Rightarrow M_3$ is a Slag in a CY3
- d=3: $\mathcal{N} = 1$ 3d–3d correspondence between 3d $\mathcal{N} = 1$ theories and Chern-Simons-Dirac on M_3 [Eckhard, SSN, Wong, 2018]
  $\Rightarrow M_3$ is an associative in G_2
- d=4: 4d–2d correspondence between 2d $\mathcal{N} = (0, 2)$ and topological sigma-model from M_4 into the Nahm moduli space [Assel, SSN, Wong]
  $\Rightarrow M_4$ is a Coassociative in G_2

In the 4d $\mathcal{N}=1$ theory from M/G_2 M5-branes on asssocatives M_3 correspond to domain walls. For SQCD this was studied in [Acharya, Vafa].

Complementary motivation to study such theories: partial topological twist results in 3d $\mathcal{N}=1$ theories: $T_{\mathcal{N}=1}[M_3]$ ($G=SU(N)$, but more generally can be any ADE). [Eckhard, SSN, Wong]

Questions:

How does the geometry of M_3 enter the 3d theory?

T^3 and S^3 partition functions for $T[M_3]$ via TQFTs and compute observables of the 3d theory from a dual topological theory

Recent progress in understanding of partition functions and generalized dualities in 3d $\mathcal{N}=1$ theories [Gaiotto, Gomis, Komargodski, Seiberg, Witten, Benini, Benvenuti,...]. What is the counterpart in the TQFT dual?

3d $\mathcal{N}=1$ Gauge Theories from M5-branes on Associative Three-Cycles

M5-branes

Nahm's classification of Superconformal theories implies that there is a unique up to choice of ADE-gauge group 6d $\mathcal{N} = (2,0)$ superconformal theory with superconformal algebra $OSp(6|4) \supset SO(6)_L \times Sp(4)_R$. For $G = A_N$ this is the effective theory on a stack of M5-branes. Single M5-brane has $G = U(1)$.

Dimensional reduction on a three-cycle:

$$
\begin{array}{lcll}
SO(1,5)_L & \to & SO(1,2)_L \times \underline{SO(3)_M} & \\
Sp(4)_R & \to & \left\{ \begin{array}{l} \underline{SU(2)_R} \times U(1)_R \\ \underline{SU(2)_r} \times SU(2)_\ell \end{array} \right. & \begin{array}{l} \text{3d } \mathcal{N} = 2;\ M_3 = \text{ sLag in CY}_3 \\ \text{3d } \mathcal{N} = 1;\ M_3 = \text{Associative in } G_2\,. \end{array}
\end{array}
$$

The main challenge is: we have absolutely no idea what the theory is for $G \neq U(1)$!

Local Geometry of Associatives in G_2-manifolds

Normal bundle of M_3 is the spin-bundle twisted with $SU(2)$-bundle

$$N_{M_3} = \mathbb{S} \otimes V$$

Linear deformations parametrised by twisted harmonic spinors satisfying

$$\not{D}_{\mathfrak{V}} \phi = 0$$

on M_3. Moduli space of solutions $\mathcal{H}_{\mathcal{D}}$ metric dependent!

VitualDim($\mathcal{H}_{\mathcal{D}}$) = 0 $\Rightarrow$ dim(Ker$\not{D}_{\mathfrak{V}}$) $=$ dim(Coker$\not{D}_{\mathfrak{V}}$).

So there can be obstructions. However, generically $d_{\not{D}} \equiv$ dim(Ker$\not{D}_{\mathfrak{V}}$) vanishes. [McLean]

Harmonic Spinors

When V is trivial i.e. $\mathfrak{V} = 0$ there are three distinct cases:

$$(\not{D})^2\psi = \nabla^*\nabla\psi + \frac{R}{4}\psi$$

- $R > 0$: $d_{\not{D}} = 0$ and the associative is rigid
- $R = 0$: $M_3 = T^3$ and harmonic spinors coincide with parallel spinors
- $R < 0$: Every closed spin manifold admits a metric with $d_{\not{D}} \geq 1$

Space of linear deformations depends on induced metric on M_3

Theory of a single M5-brane

\# Lorentz and R-symmetry:

$$SO(6)_L \times Sp(4)_R \subset OSp(6|4)$$

\# Tensor multiplet:

$$\begin{aligned} B_{\underline{ab}}: &\qquad (\mathbf{15},\mathbf{1}) \qquad \text{with selfduality } H = dB = *_6 H \\ \Phi^{\underline{\hat{m}\hat{n}}}: &\qquad (\mathbf{1},\mathbf{5}) \\ \varrho^{\underline{\alpha\hat{m}}}: &\qquad (\bar{\mathbf{4}},\mathbf{4}) \end{aligned}$$

\# EOMs:

$$H^- = dH = 0\,, \qquad \partial^2 \Phi^{\underline{\hat{m}\hat{n}}} = 0\,, \qquad \partial\!\!\!/ \rho = 0\,.$$

An M5-brane on an Associative

Recall: partial topological twist along M_3:

$$SU(2)_{\text{twist}} = \text{diag}(SU(2)_M, SU(2)_r)\,.$$

$$\begin{array}{rcccl} SO(6)_L \times Sp(4)_R & & \rightarrow & & SO(3)_L \times SU(2)_{\text{twist}} \times SU(2)_\ell \\ \Phi^{\underline{\hat{m}\hat{n}}}: & (\mathbf{1},\mathbf{5}) & \rightarrow & & (\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1}) \equiv (\phi^{\alpha\hat{\alpha}}, \varphi) \\ H_{\underline{abc}}: & (\mathbf{10},\mathbf{1}) & \rightarrow & & (\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{3},\mathbf{1}) \equiv (h, H_{axy}) \\ \varrho^{\underline{\alpha\hat{m}}}: & (\bar{\mathbf{4}},\mathbf{4}) & \rightarrow & & (\mathbf{2},\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{3},\mathbf{1}) \equiv (\rho^{\sigma\alpha\hat{\alpha}}, \lambda^\sigma, \xi^\sigma_a)\,. \end{array}$$

$SU(2)_\ell$ identified with the structure group of V, and ϕ a section of N_{M_3}.
The zero-mode spectrum depends on

$$H_1(M_3,\mathbb{Z}) \cong \mathbb{Z}^{b_1(M_3)} \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}$$

$$d_{\not{D}}(M_3, g) = \#\text{ of twisted harmonic spinors on } M_3 \text{ wrt metric } g$$

$T[M_3, U(1)]$

The theory $T[M_3, U(1)]$ enjoys $\mathcal{N} = 1$ supersymmetry and is a supersymmetric CS-theory coupled to scalar multiplets:

1. A single scalar multiplet $\mathcal{A}_\varphi \ni \{\varphi, \lambda^\sigma, h\}$. For $T_{\mathcal{N}=1}[M_3, U(1)]$ the domain wall in the 4d $\mathcal{N} = 1$, this is the center of mass multiplet.

2. $b_1(M_3)$ massless scalar multiplets $\mathcal{A}^I_\alpha \ni \{\alpha^I, \xi^{\sigma I}\}$ coming from the free part of the first homology group of M_3.

3. $d_{\not{D}}(M_3, g)$ massless scalar multiplets $\mathcal{A}^i_\phi \ni \{\phi^i, \rho^{\sigma i}\}$ which describe the deformations of the associative M_3 inside the G_2-holonomy manifold. These explicitly depend on the G_2-holonomy metric g restricted to the associative cycle M_3.

4. A set of r massive gauge multiplets $\mathcal{V}^m_A \ni \{A^m, \xi^{\sigma m}\}$ whose masses are generated by Chern-Simons terms at levels p_m. Each multiplet $\mathcal{V}^m_A$ is induced by a factor in the torsion part of $H^1(M_3, \mathbb{Z})$

Non-abelian Generalization

In general this is unknown. However we can use a key fact about the M5-brane theory:

6d (2,0) Theory on S^1 with gauge group G
= 5d Super-Yang Mills with gauge group G

In particular, if one wishes to compactify M5-brane on circle-fibration we can infer the non-abelian generalization by defining the 5d SYM theory in a suitable ”supergravity background”.

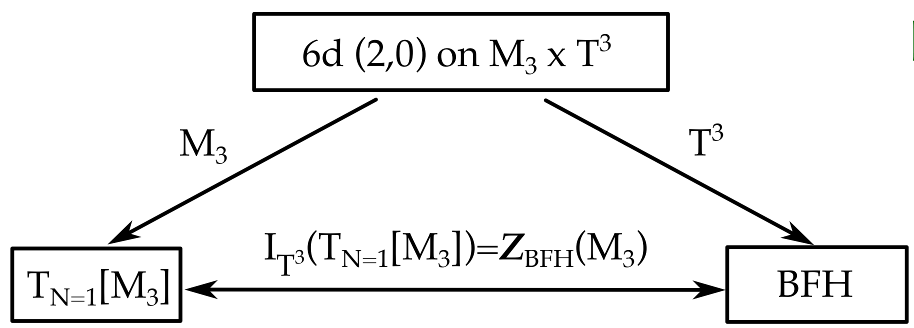
Examples:

- $M_3 = L(p, 1)$.
- S^3- or $L(p, 1)$ partition function, via 5d SYM on S^2 + graviphoton background that models the Hopf fibration.

A 3d–3d Correspondence:

TQFT Dual to 3d $\mathcal{N} = 1$

Witten-Index: 3d-3d Correspondence



[Eckhard, SSN, Wong]

BFH = BF-model coupled to a spinorial hypermultiplet. The Witten index $\mathrm{Tr}(-1)^F$ is

$$I_{T^3}\left(T_{\mathcal{N}=1}[M_3]\right) = Z_{\mathrm{BFH}}(M_3)\,.$$

BPS equations for $(\phi^{\alpha\hat{\alpha}}, A)$ fields of BFH on M_3 are generalized Seiberg Witten equations:

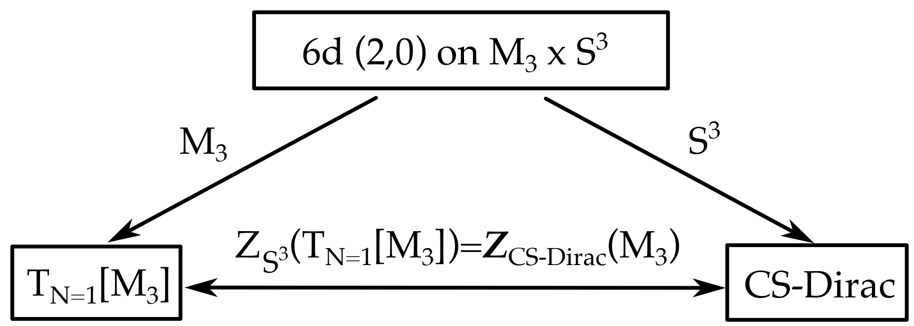
$$(\mathrm{gSW}_{M_3}):\qquad \begin{aligned} (\not{D}\phi)^{\alpha\hat{\alpha}} &= 0 \\ \varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^{\alpha}{}_{\beta}\phi^{\beta\hat{\alpha}}] &= 0\,. \end{aligned}$$

and

$$Z_{\mathrm{BFH}}(M_3) = \chi\left(\mathcal{M}_{\mathrm{gSW}_{M_3}}\right)$$

S^3-partition Function: 3d-3d Correspondence

[Eckhard, SSN, Wong]



CS-Dirac= level 1 CS coupled to a twisted harmonic spinor M_3, eom = gSW equations. S^3-partition function is computed by:

$$Z_{S^3}\left(T_{\mathcal{N}=1}[M_3,G]\right)=\mathcal{Z}_{\mathrm{CS}_1-\mathrm{Dirac},G}(M_3)$$

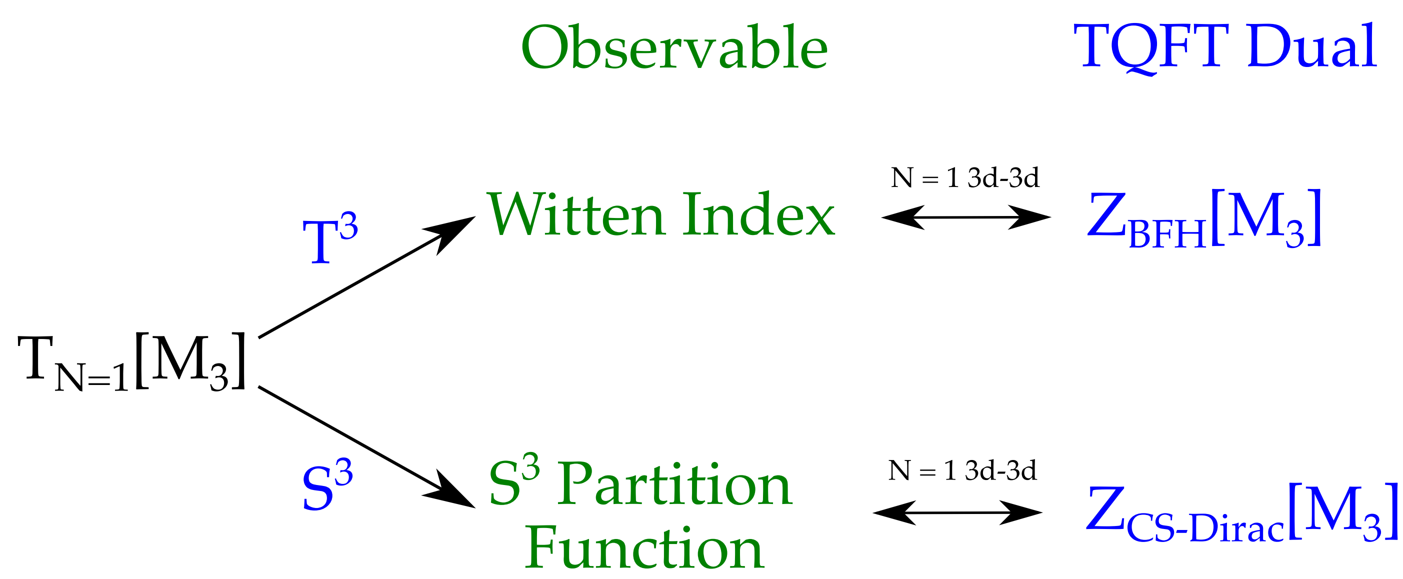
No twisted harmonic spinors for a given metric g induced from the G_2:

$$d_{\not{D}}(M_3,g)=0: \qquad Z_{S^3}\left(T_{\mathcal{N}=1}[M_3,G]\right)=\mathrm{WRT}(\mathrm{M}_3)$$

Generalization: $L(p,1)$ reduction instead of S^3:

$$Z_{L(p,1)}\left(T_{\mathcal{N}=1}[M_3,G]\right)=\mathcal{Z}_{\mathrm{CS}_p-\mathrm{Dirac},G}(M_3)$$

Summary of the $\mathcal{N} = 1$ 3d–3d Correspondence



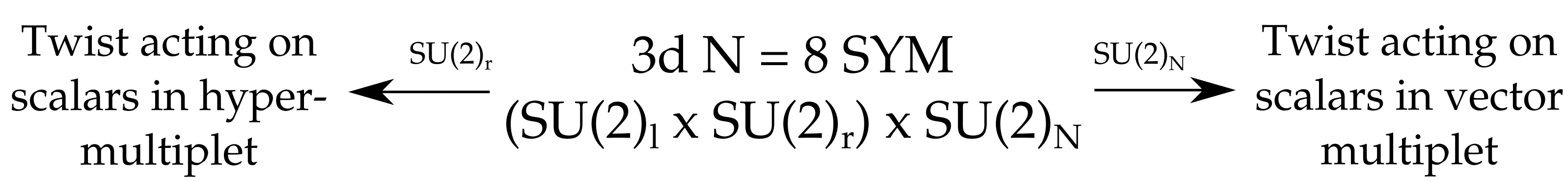
BFH: supersymmetric BF model coupled to spinorial hypermultiplet

CS-Dirac: Chern-Simons-Dirac theory

Witten Index: Derivation

M5-branes compactified on $T^3 \Rightarrow$ 3d $\mathcal{N} = 8$ SYM

Two topological twists of 3d $\mathcal{N} = 8$ SYM, both preserving two topological supercharges



$SU(2)_r$ twist: scalars $\phi^{\alpha\hat{\alpha}}$ in $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ twisted into 'bispinors' under twisted Lorentz group and $SU(2)_\ell$

$\Rightarrow$ sections of N_{M_3}, where $SU(2)_\ell$ identified with structure group of V

BFH-Model

[Eckhard, SSN, Wong]

BF-model coupled to spinorial Hypermultiplet preserving two topological supercharges

$$\mathcal{L}_{\text{BFH}} = B^a(B_a - \varepsilon_{abc}F^{bc} + \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^\alpha{}_\beta \phi^{\beta\hat{\alpha}}]) + \frac{1}{2}W_{\alpha\hat{\alpha}}(W^{\alpha\hat{\alpha}} - 2i\not{D}^\alpha_\beta \phi^{\beta\hat{\alpha}}) + \cdots$$

where B_a, $W_{\alpha\hat{\alpha}}$ are auxiliary fields, whose eoms are

$$\begin{aligned} B_a &= \frac{1}{2}\left(\varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, \phi^{\beta\hat{\alpha}}](\sigma_a)^\alpha{}_\beta\right) \\ W^{\alpha\hat{\alpha}} &= i\not{D}^\alpha{}_\beta \phi^{\beta\hat{\alpha}}, \end{aligned}$$

The action can be written as

$$S_{\text{BFH}} = \varepsilon_{\sigma\tau}Q^\sigma Q^\tau V_{\text{BFH}}$$

and the energy-momentum tensor is Q-exact, however partition function depends on ambient G_2-metric, due to the dependence of the bispinors on g.

BFH Partition Function $Z_{\text{BFH}}(M_3)$

[Eckhard, SSN, Wong]

BPS equations given by generalised Seiberg-Witten equations

$$(\text{gSW}_{M_3}): \qquad \begin{aligned} (\not{D}\phi)^{\alpha\hat{\alpha}} &= 0 \\ \varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^{\alpha}{}_{\beta}\phi^{\beta\hat{\alpha}}] &= 0 \end{aligned}$$

Partition function of $N_T = 2$ TQFTs computes $\chi(\mathcal{M}_{\text{BPS}})$ [Blau, Thompson][Dijkgraaf, Moore]. Applied to this theory, we expect:

$$Z_{\text{BFH}}(M_3) = \chi(\mathcal{M}_{\text{gSW}_{M_3}})$$

Checks: Abelian Theory

Abelian spectrum depends on first integral homology group

$$H_1(M_3, \mathbb{Z}) \cong \mathbb{Z}^{b_1(M_3)} \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}$$

Reduction of topologically twisted 6d EoMs yielded:

- Centre of mass scalar multiplet
- $b_1(M_3)$ scalar multiplets
- $d_{\not{D}}(M_3, g)$ scalar multiplets
- r vector multiplets with Chern-Simons interactions at level p_m

Checks: Abelian Theory

Witten index: $I = \mathrm{Tr}(-1)^F$

Multiplet	Contribution to $I(T_{\mathcal{N}=1}[M_3, U(1)])$
Chern-Simons at level k	k
Free scalar multiplet	0

Combining with spectrum of the abelian theory:

$$I(T_{\mathcal{N}=1}[M_3, U(1)]) = \begin{cases} \prod_{m=1}^{r} p_m & b_1 = d_{\not{D}} = 0 \\ 0 & \text{else} \end{cases}$$

Checks: Abelian Theory

$\mathcal{N} = 1$ 3d–3d correspondence implies

$$I(T_{\mathcal{N}=1}[M_3, U(1)]) = \chi(\mathcal{M}_{U(1)\text{-Flat}})\chi(\mathcal{H}_{\not{D}})$$

$U(1)$-flat connections: $\mathrm{Hom}(\pi_1(M_3), U(1))$

Topologically, $\mathcal{M}_{F=0} = T^{b_1} \times (\prod_{m=1}^{r} p_m)$pts so for generic embeddings of M_3

$$d_{\not{D}} = 0: \quad Z_{\text{BFH},U(1)}(M_3) = \begin{cases} \prod_{m=1}^{r} p_m & b_1 = 0 \\ 0 & \text{else} \end{cases}$$

Matches abelian Witten index when associative is obstructed.

Jump in Witten Index

Conjecture

$$d_{\not{D}} \neq 0 : \quad I(T_{\mathcal{N}=1}[M_3, U(1)]) \Rightarrow \chi(\mathcal{H}_{\not{D}}) = 0$$

Consider deforming metric on M_3 such that $d_{\not{D}} \neq 0$

$T_{\mathcal{N}=1}[M_3, U(1)]$ now has $d_{\not{D}}$ additional scalar multiplets

$$\Rightarrow I(T_{\mathcal{N}=1}[M_3, U(1)]) = 0$$

$\Rightarrow$ Witten index for abelian theory is not a metric independent quantity, but jumps when M_3 admits twisted harmonic spinors.

Checks: Lens-Space Theories

Consider G_2-manifolds $X_7 = (S^3 \times \mathbb{R}^4)/\mathbb{Z}_p$, where action on S^3 is free. Associative is a Lens spaces $L(p,1)$, and is embedded with $\mathfrak{V} = 0$

$$T_{\mathcal{N}=1}[L(p,1), U(N)] = \begin{cases} \text{3d } \mathcal{N}=1 \text{ Chern-Simons-Yang-Mills at level} \\ p \text{ coupled to adjoint scalar multiplet} \end{cases}$$

Witten index computed by considering

$$U(N) = \frac{U(1) \times SU(N)}{\mathbb{Z}_N}$$

and discarding fermion zero mode from centre of mass $U(1)$ factor

[Acharya, Vafa]

$$I(T_{\mathcal{N}=1}[L(p,1), U(N)]) = p \times \frac{(p-1)!}{(N-1)!(p-N)!} \times \frac{1}{N} = \binom{p}{N}$$

Check via 3d-3d Correspondence

Metric on $L(p,1)$ does not admit harmonic spinors

$$d_{\not{D}} = 0 : \quad Z_{\text{BFH},U(N)} = \chi(\mathcal{M}_{U(N)\text{-Flat}})$$

Flat connections correspond to $\text{Hom}(\pi_1(M_3), U(N))$

Moduli space consists of N-dimensional representations of $\mathbb{Z}_p$

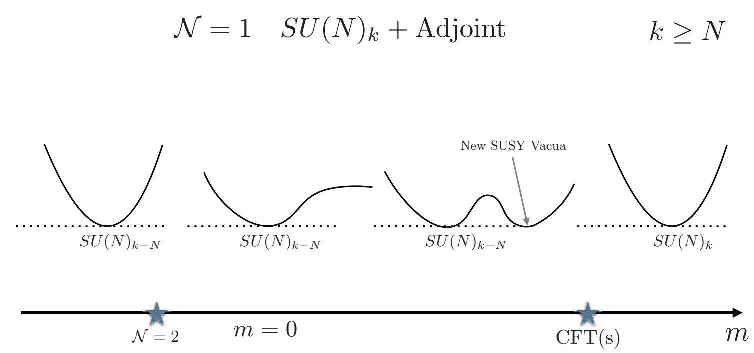
Abelian flat connections $\Leftrightarrow$ Irreducible representations of $\mathbb{Z}_p$

$$\Rightarrow \chi(\mathcal{M}_{U(N)}) = \binom{p}{N}$$

Extension

[work in progress: Julius Eckhard, Heeyeon Kim, SSN]

[Bashmakov, Gomis, Komargodski, Sharon, 2018]: Witten index for 3d $\mathcal{N}=1$ $SU(N)_k$+ adjoint multiplet of mass M, has undergoes phase transitions, as a function of the mass M. Consider: $k > N$.



We can incorporate mass deformation into supergravity background:

$$T_{\mathcal{N}=1}[L(k,1), U(N), M] = U(N)_k + \text{adjoint scalar multiplet of mass } M$$

For $|M| >> 0$ we can integrate out both the gaugino (which has negative) mass and the massive adjoint fermion. $\Rightarrow$ shifts $SU(N)$ level by $\text{sign}(m)\frac{N}{2}$, while the $U(1)$ level is unchanged.

Thus, the theory admits a single vacuum TQFT for parametrically large mass M:

$$M \gg 0: \qquad U(N)_{k-N,k} \quad \Rightarrow I_+ = \begin{pmatrix} k \\ N \end{pmatrix}$$

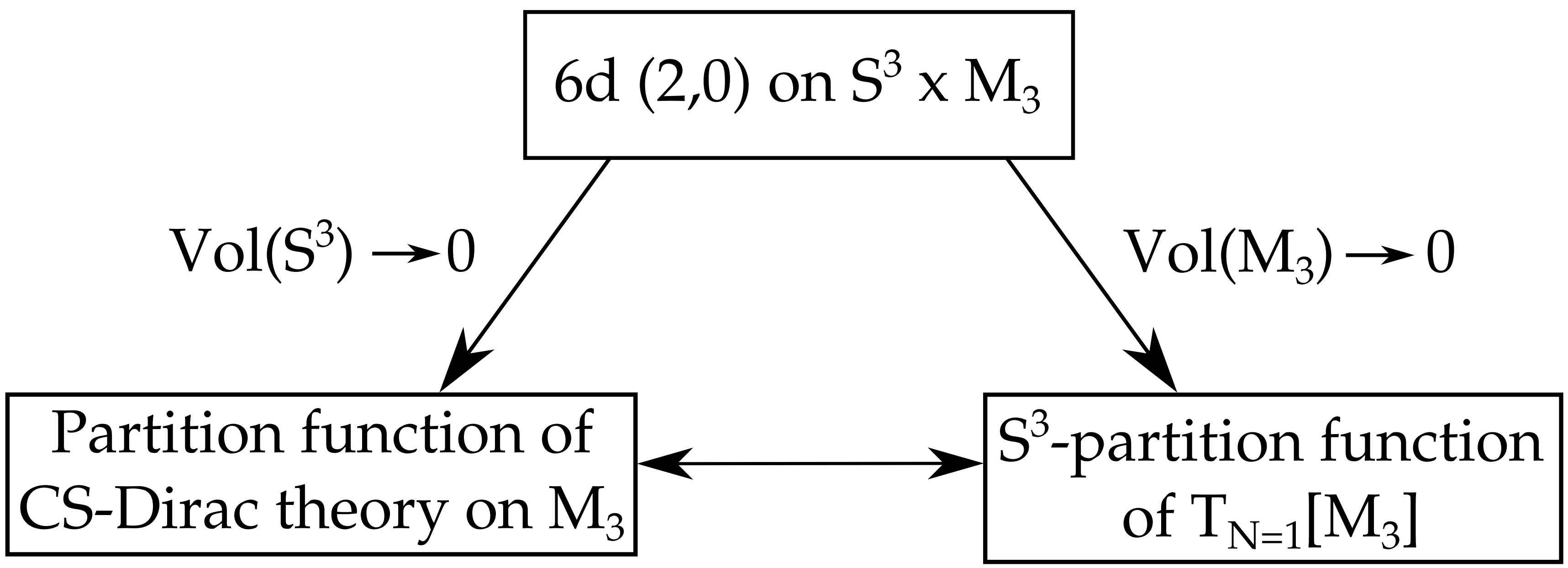
$$M \ll 0: \qquad U(N)_{k,k} \quad \Rightarrow I_- = \begin{pmatrix} k+N-1 \\ N \end{pmatrix} .$$

Note: $U(N)_{k,q} = \frac{SU(N)_k \times U(1)_{Nq}}{\mathbb{Z}_N}$ has $I = \begin{pmatrix} k+N-1 \\ N-1 \end{pmatrix} \cdot \frac{q}{N}$.

Note that the index for $M \gg 0$ agrees with the index of $\mathcal{N}=2\; U(N)_k$. The reason for this is that at $M = \frac{kg^2}{4\pi}$ supersymmetry enhances to $\mathcal{N}=2$. g-independence then implies that the index will only depend on the sign of M.

Work in progress: show this phase transition as a function of M from the dual M-deformed TQFT.

S^3 Partition Function



Chern-Simons-Dirac Theory

Computation of S^3 reduction by coupling to off-shell conformal sugra [Kugo][Cordova, Jafferis], via 5d on S^2. Captures metric dependence expected from S^3-partition function

$$\mathcal{L} = \frac{r}{8\pi}\left(F \wedge \star F - \frac{1}{2}\phi_{\alpha\hat{\alpha}}(\not{D}^2\phi)^{\alpha\hat{\alpha}}\right) + \frac{i}{4\pi}\left(\mathrm{CS}(A) + \frac{i}{2}\phi_{\alpha\hat{\alpha}}(\not{D}\phi)^{\alpha\hat{\alpha}}\right)$$

In the limit $r = S^2$-radius $\to 0$ we obtain CS coupled to 'bispinor' $\phi^{\alpha\hat{\alpha}}$ i.e. Chern-Simons-Dirac theory.

EoMs given by the gSW equations on M_3

$$(\mathrm{gSW}_{M_3}) : \quad \begin{aligned} (\not{D}\phi)^{\alpha\hat{\alpha}} &= 0 \\ \varepsilon_{abc}F^{bc} - \frac{i}{2}[\phi_{\alpha\hat{\alpha}}, (\sigma_a)^{\alpha}{}_{\beta}\phi^{\beta\hat{\alpha}}] &= 0 \end{aligned}$$

Mathematics question: what are the properties of the moduli space of these gSW? Coincidentally, in 10/2018 they have appeared independently in Doan and Walpuski's work on counting associative three-manifolds.

Summary and Outlook

Geometric structures underlie constructions of supersymmetric gauge theories in string and M-theory.

String theory embeddings rely heavily on a comprehensive understanding of the underlying geometries, and there is exciting new progress in the construction of exceptional holonomy manifolds, G_2 and Spin(7).

String theoretic embedding can realize field theory dualities, uncover new dualities (see 3d–3d correspondence), and can give full classifications of theories, such as superconformal theories.