Chiral algebras of class $\mathcal{S}$ and symplectic varieties

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RIMS, Kyoto University

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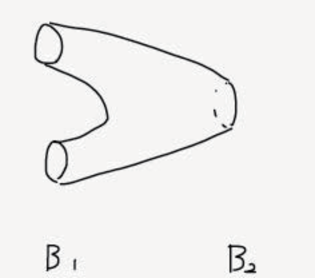

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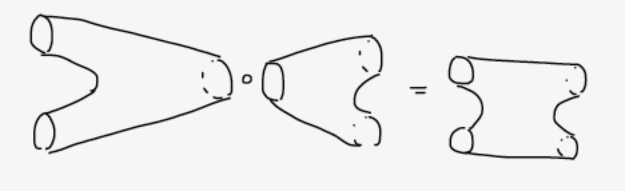

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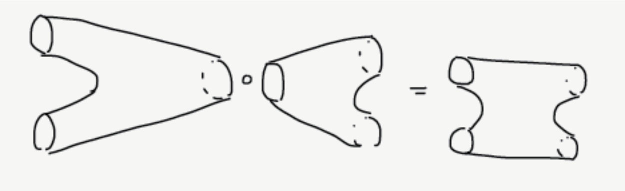

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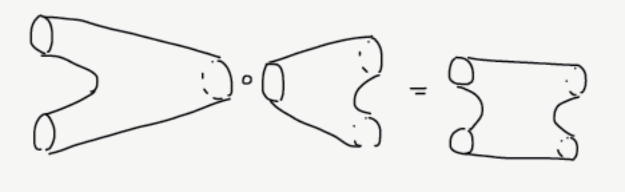

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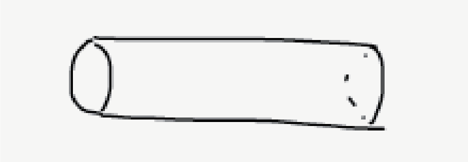

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$+$ some regularity conditions.

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where $\mu_X : X \mapsto \mathfrak{g}_2^*$, $\mu_Y : Y \mapsto \mathfrak{g}_2^*$ are moment maps for G_2-actions.

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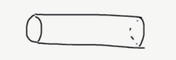

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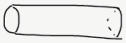

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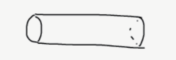

3. $\eta_G^{BFN}(cap) = G \times \mathcal{S}$, where $\mathcal{S}$ is a Kostant-Slodowy slice $e + \mathfrak{g}^f \subset \mathfrak{g} = \mathfrak{g}^*$ ($\{e, f, h\}$: $\mathfrak{sl}_2$-triple, e regular).

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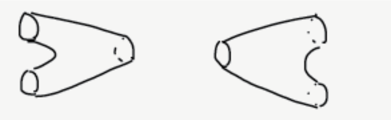

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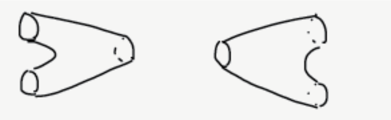

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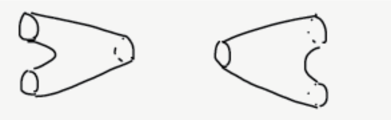

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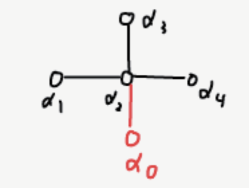

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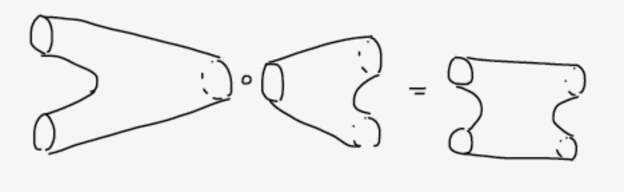

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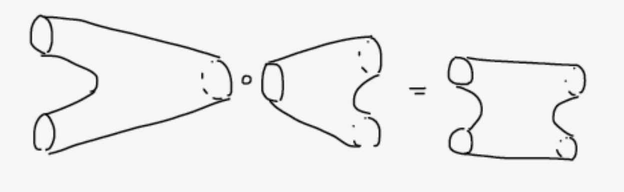

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which is known as the *ADHM construction* of $\overline{\mathbb{O}_{min}}$.

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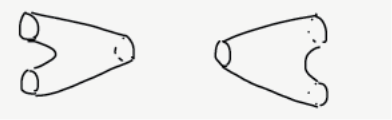
$\eta^{BFN}_{SL_2}(\text{pants}) = \overline{\mathbb{O}_{min}}$ in E_6.



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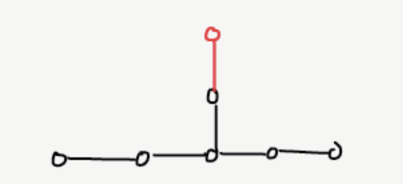

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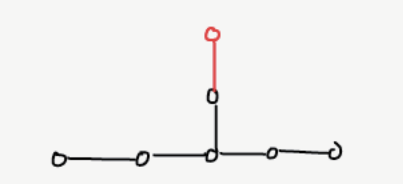

For other G there is no simple description of the varieties.

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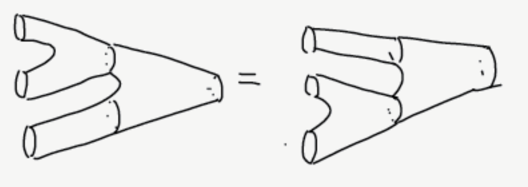

Some (physical) background and vertex operator algebras

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for a connected 2-bordism Σ, where C is a Riemann surface with the same genus as Σ, $\#\{\text{boundaries of } \Sigma\} = \#\{\text{marked pts on } C\}$.

Vertex operator algebras

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A *vertex operator algebra* (VOA) is a vector space V equipped with a linear map

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that satisfies certain properties such as locality:

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or equivalently, OPEs:

$$a(z)b(w) \sim \sum_{j \geq 1} \frac{1}{(z - w)^j} (a_{(j)} b)(w).$$

VOA = (chiral) 2d conformal field theory

Example

The VOA associated with an affine Kac-Moody algebra

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K$: ($\mathfrak{g}$: simple finite-dimensional)

$[xf, yg] = [x, y]fg + (x|y)\,\mathrm{Res}_{t=0}(gdf)K$, $[K, \widehat{\mathfrak{g}}] = 0$.

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The simple affine VOA $L_k(g)$ associated with $\mathfrak{g}$ at level k is the VOA generated by $x(z)$ ($x \in \mathfrak{g}$) with OPE

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Example

The VOA associated with an affine Kac-Moody algebra

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K$: ($\mathfrak{g}$: simple finite-dimensional)

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with relations coming from the maximal submodule ("null vectors").

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Exam@le

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Some of admissible representations do come from 4d SCFTs.

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$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$.

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Let $\mathfrak{g} \in DES$, $k = -h^\vee/6 - 1$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Then

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These are main examples of chiral algebras of class $\mathcal{S}$ considered by Rastelli et al. (coming with elliptic fibration via F-theory).

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2d TQFT whose targets are vertex algebras

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The composition of $V_1 \in \mathrm{Hom}(G_1, G_2)$ and $V_2 \in \mathrm{Hom}(G_2, G_3)$ is given by

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Under certain conditions

$$X_{V_1 \circ V_2} = X_{V_1} \circ X_{V_2} (:= X_{V_1} \times X_{V_2} /\!/\!/ G_2)$$

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$\mathbb{V}$ is a symmetric monodical category.

The following was conjectured by Beem, Peelaers, Rastelli and van Rees.

Main Theorem (A.)

For each semisimple group G there exists a unique monoidal functor

$$\eta_G^{VOA} : \mathbb{B}_2 \to \mathbb{V},$$

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Moreover, $X_{\eta_G^{VOA}(B)} \cong \eta_G^{BFN}(B)$.

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We have $X_{L_{-2}(D_4)} \cong \overline{\mathbb{O}_{min}}$ ([A.-Moreau'16]).

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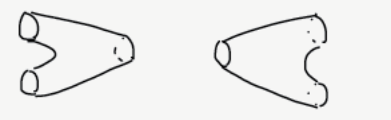

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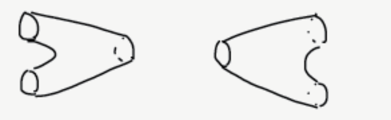

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$\mathcal{D}_G^{ch}$ is a simple VOA with central charge $2\dim G$, and

$$X_{\mathcal{D}_G^{ch}} \cong T^*G = \text{the cylinder symplectic variety.}$$

Construction

Theorem (Arkhipov and Gaitsgory)

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$\Rightarrow \mathcal{D}^{ch}_G$ is the cylinder VOA in $\mathbb{V}$.

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By the axiom,

$$\eta_G^{VOA}(cap) = \mathbf{W} := DS_{f_{prin}}(\mathcal{D}_G^{ch}) \quad \text{(equivariant affine } W\text{-algebra)}$$

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We have :

$$X_{\mathbf{W}} = G \times S_{reg} = \eta_G(cap)$$

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So we kill the difference of the two action of the center on $\mathbf{W} \otimes \mathbf{W}$, or more generally, on $\mathbf{W}^{\otimes r}$, using the BRST cohomology.

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$$\eta_G^{VOA}(\text{genus zero } \Sigma \text{ with } n\text{-boundaries})$$
$$= H^0_{BRST}(\mathbf{W}^{\otimes n} \otimes (\otimes_{i=1}^r (b_i, c_i))^{\otimes n-1}, Q_{(0)})$$

where

$$Q(z) = \sum_{i=1}^{n-1} Q_{i,i+1}(z),$$

$$Q_{i,i+1}(z) = \sum_{j=1}^{r} (\pi_i(p_j(z)) - \pi_{i+1}(p_j(z)))c_j^{(i)}(z).$$

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This correspondence extends to the functor with the required properties. $\square$.

Thank you!