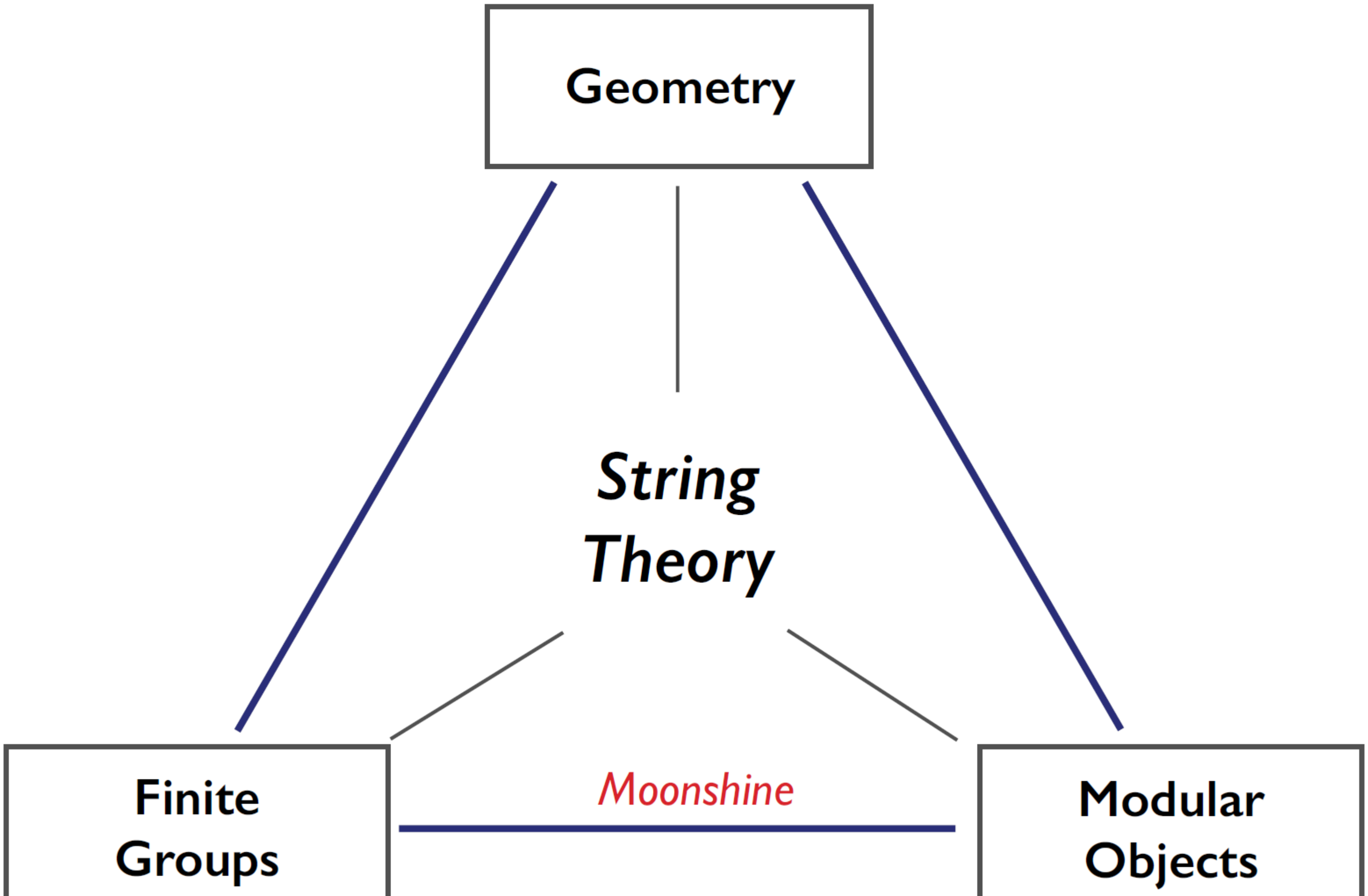
Moonshine

Miranda C. N. Cheng
University of Amsterdam

Universiteit van Amsterdam

European Research Council
Established by the European Commission

A Mysterious Story About



Outline

Moonshine

I. The Classic

II. The Modern

III. The Post-Modern

I. The Classic

- What it moonshine?
- A beautiful example: Monstrous Moonshine


Finite
Groups
Moonshine
Modular
Forms


Finite Groups
Moonshine
Modular Forms

Modular Forms

A modular form f(τ) transforms "covariantly" under a subgroup Γ of SL(2,R).

$$f : \mathbb{H} \to \mathbb{C}$$

(holomorphic)

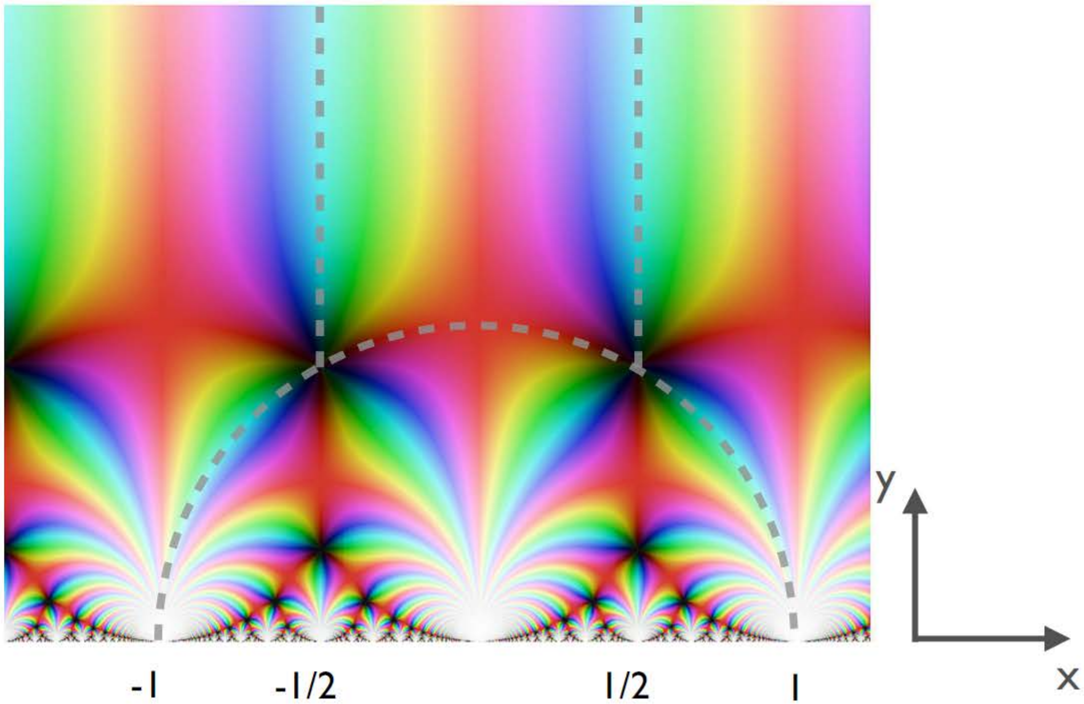
$$f(\tau) = (c\tau + d)^{-w} f(\frac{a\tau + b}{c\tau + d}) \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$\tau = x + iy$

Example: the *J*-function

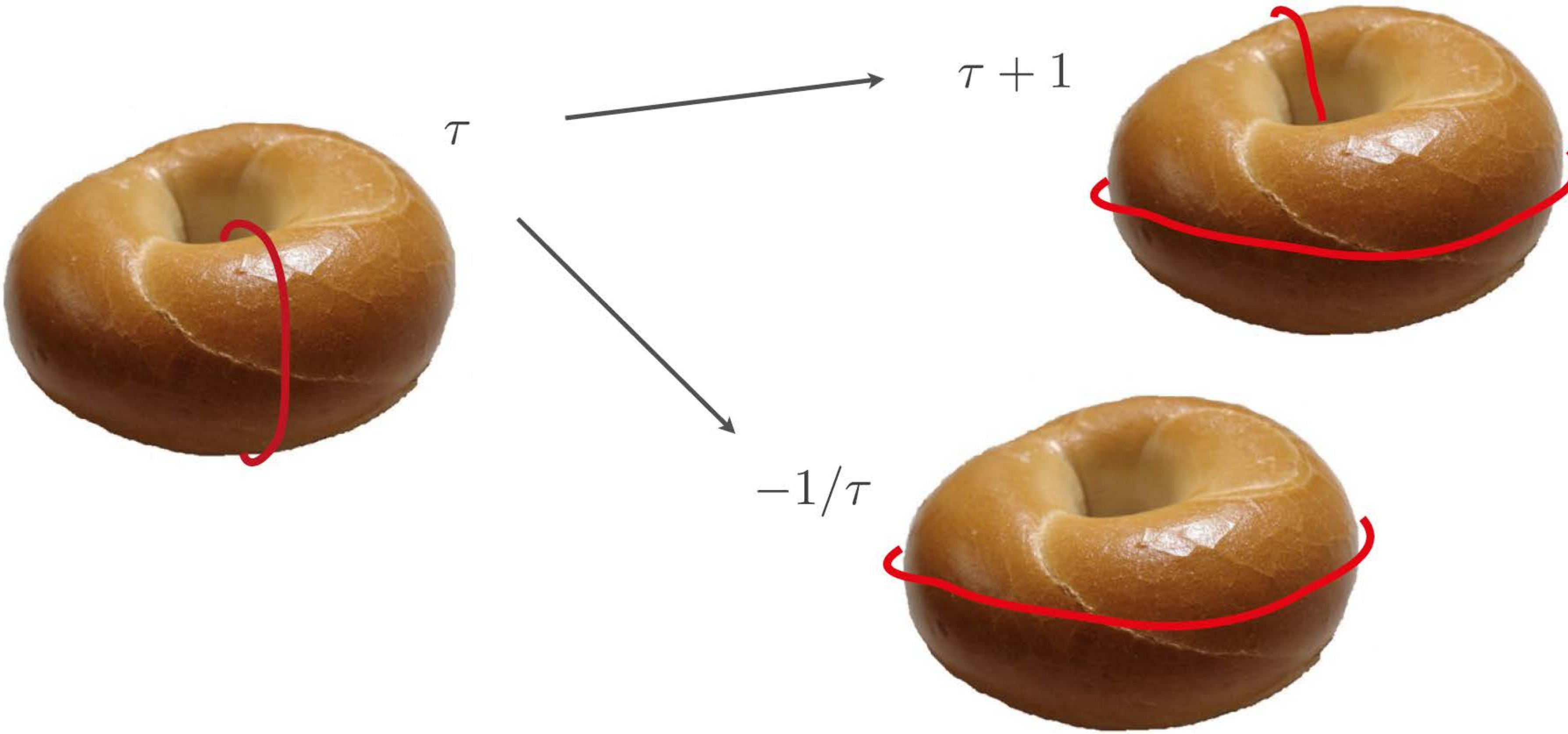
$$J(\tau) = J(\tau + 1) = J(-1/\tau)$$

$$\Gamma = SL_2(\mathbb{Z}), \ w = 0$$



Modular Forms

reflect the symmetry of a torus.



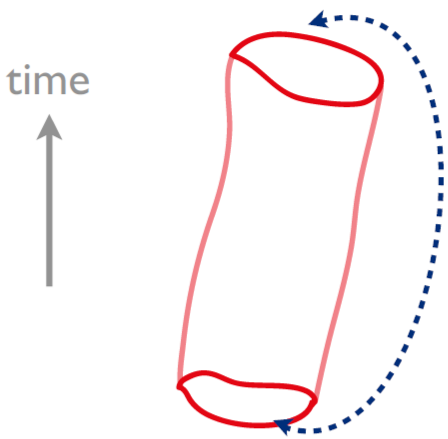
The modular action leaves the torus the same,
and only changes the choice of non-shrinkable circles on the torus.

Modular Forms

are natural products of string theory.

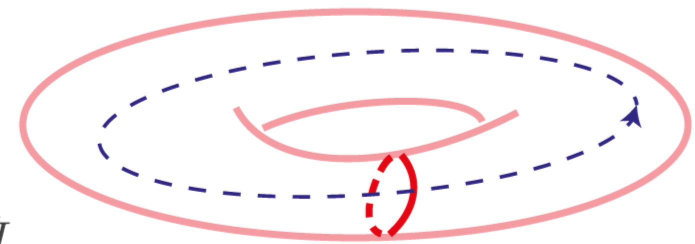
A string moving in time = a cylinder.

$t = \tau / i = 1/k_B T$



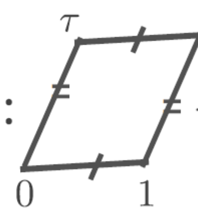
$t = 0$

partition function



$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}} e^{2\pi i \tau \hat{H}}$$

$$= \int \mathcal{D}[\phi : \text{(parallelogram)} \to X] e^{-S(\phi)}$$



The partition functions are computed by identifying the initial and final time. This turns the cylinder into a torus. As a result the CFT partition functions are modular forms!

String theory is good at producing functions with symmetries!

More generally, there can be space-time symmetries (such as E-M, S-, T-dualities) as well as world-sheet symmetries (such as SL(2,Z)).

All symmetries have to be reflected in suitable partition functions.

Finite Groups

Moonshine

Modular Forms

Finite Groups
Discrete Symmetries of Interesting Objects

Example. Close Packing Lattices: considering the most efficient way to stack up identical balls.

Face-Centered Cubic (fcc) Lattices
e.g. *Cu, Ag, Au*, *oranges, ...*

Some Interesting Examples

the "*Sporadic Groups*"

The only 26 groups that cannot be studied systematically.
They are usually symmetries of very interesting objects.

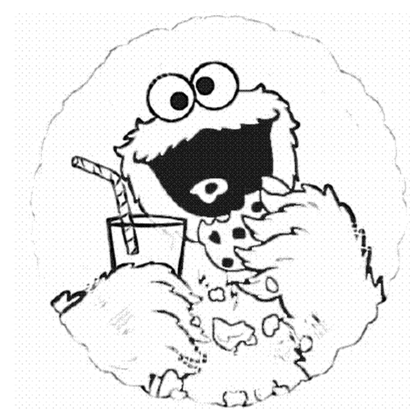
Example 1. M_{24} = the oldest sporadic *"Mathieu Groups"*. [Mathieu 1860]

$|M_{24}|$ = the number of elements in M_{24} = *244,823,040.*

Example 2. *"The **Monster**"* = the largest sporadic groups. [Fischer, Griess 1970-80]

$|M| \sim 10^{54}$ ~ the number of atoms in the solar system.

The smallest non-trivial representation has 196,883 dimensions.



Finite Groups

Moonshine

Modular Objects

Moonshine

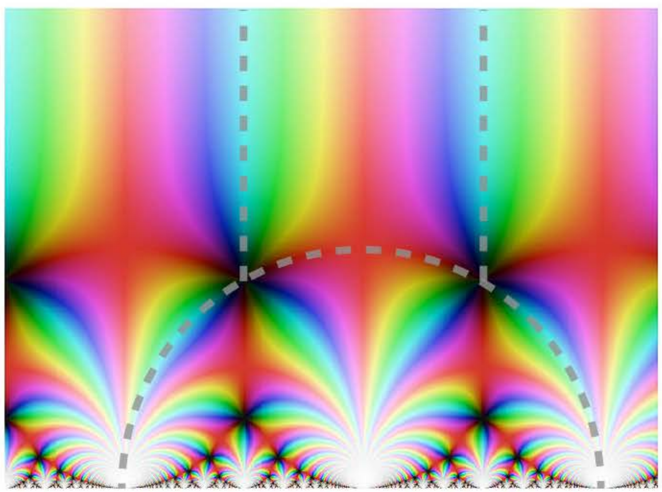
relates modular forms and finite groups.

$$J(\tau) = J(\tau+1) = J(-\tfrac{1}{\tau})$$
$$= q^{-1} + 196884\, q + 21493760\, q^2 + \ldots \qquad (q = e^{2\pi i \tau})$$

$$196884 = 1+196883 \qquad 21493760 = 1+196883+21296876$$

[McKay late 70's]

dim irreps of Monster

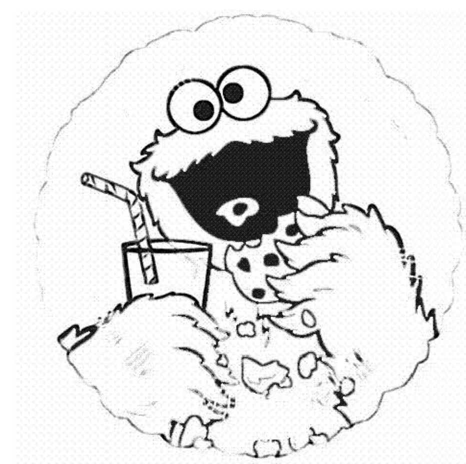


The Most Natural Modular Function

??

Monstrous Moonshine Conjecture

[Conway–Norton '79]

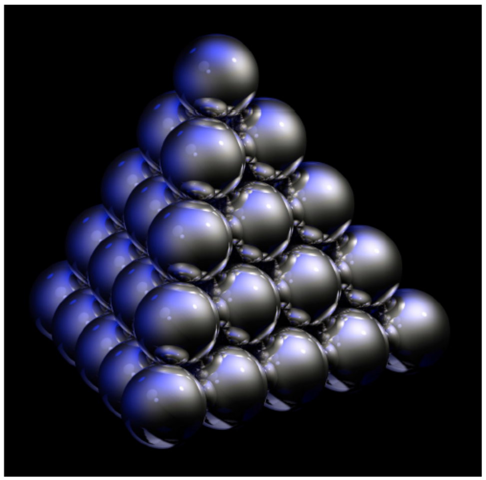


The Largest Sporadic Group

String Theory

explains Monstrous Moonshine.

Q: What's the most efficient way to stack up *24-dimensional* identical balls?



A: It's given by the *Leech lattice* Λ_{Leech}. [Leech 1967]

Λ_{Leech} has very interesting sporadic symmetries.

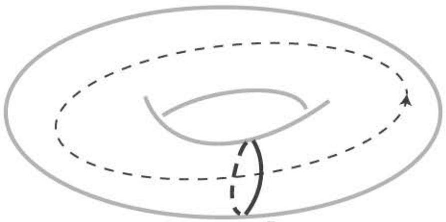
Q: But it's *24-dimensional!* What can we do with it?

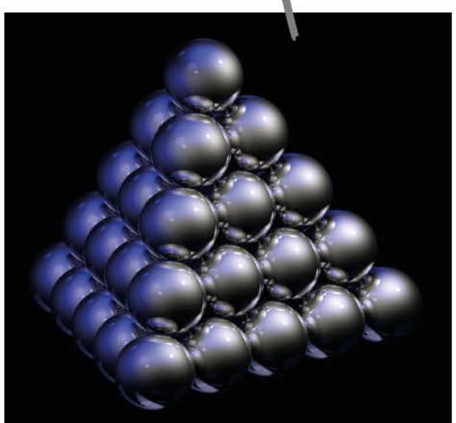
A: Just the right number of dimensions for string theory!

String Theory

explains Monstrous Moonshine.

Strings in the Leech lattice background:

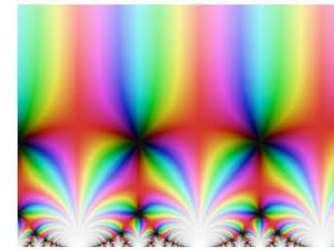




R^{24}/Λ_{Leech}

Modular Symmetry

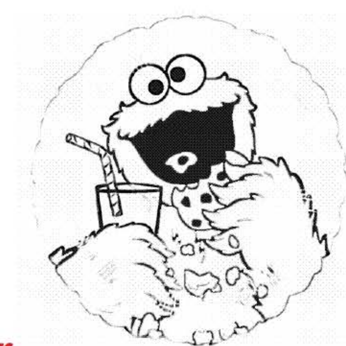
The Partition Function = $J(\tau)$



Monstrous Moonshine

Sporadic Symmetry

Λ_{Leech} ↻ $Co\,I$ $\xrightarrow{Z/2}$ ***Monster***



Monstrous Moonshine

A Meeting Place of Different Subjects

The proof of the Monstrous Moonshine Conjecture

- Used string theory
(orbifold conformal field theory, no-ghost theorem, ...)

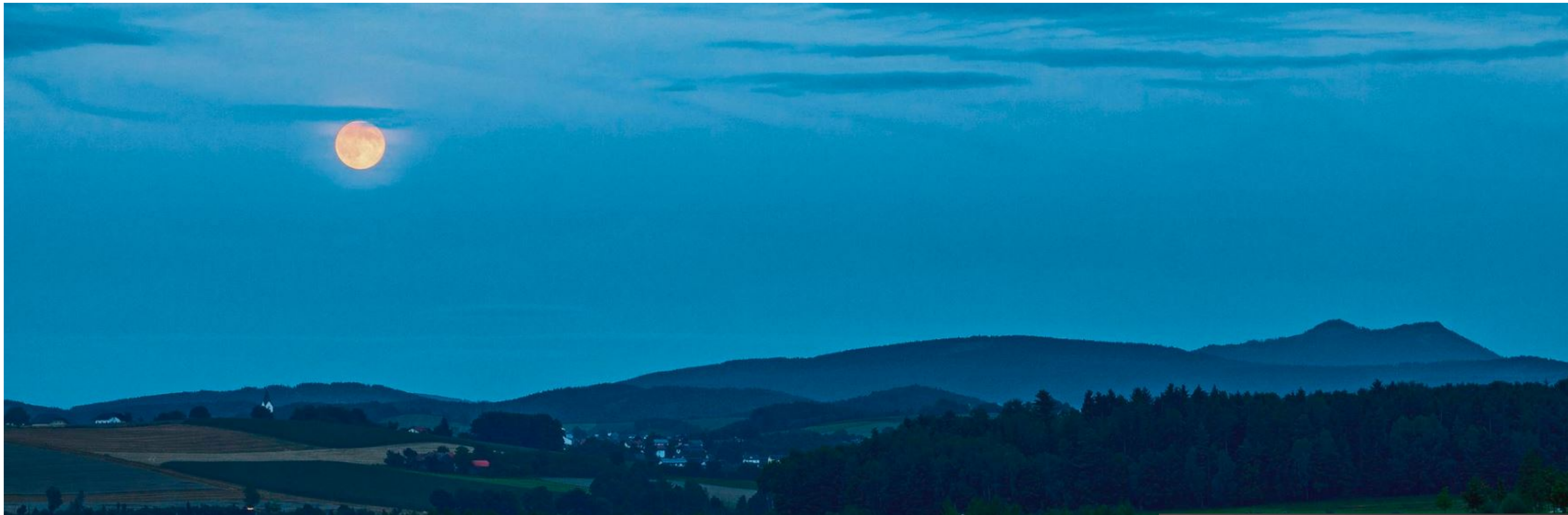
- Led to important developments in algebra and representation theory
(vertex operator algebra, Borcherds–Kac–Moody algebra, ...)

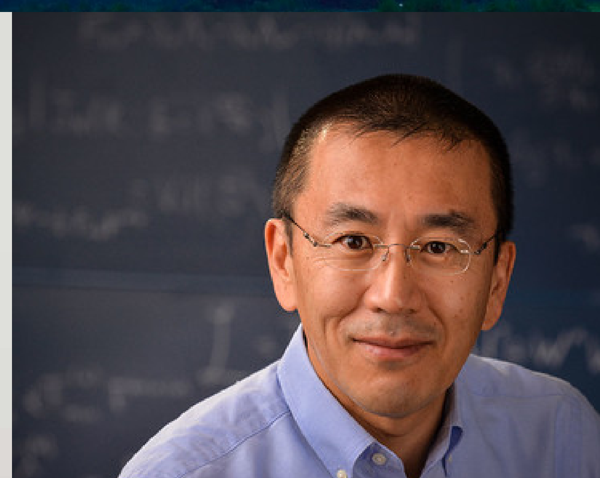
[Frenkel–Lepowsky–Meurman, 80's]

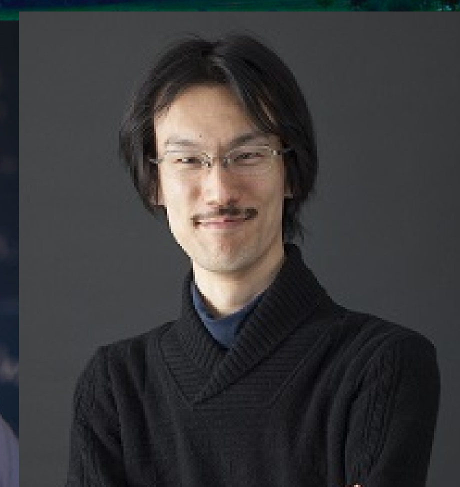
[Borcherds, 80-90's]

⋆ Recently, certain important properties of the moonshine functions have been explained using heterotic string theory. [Paquette–Persson–Volpato 16]

II. The Modern



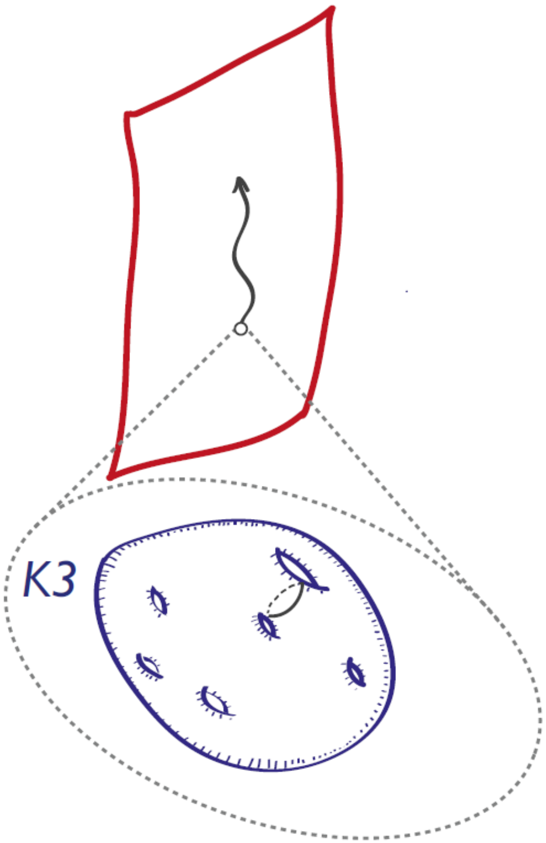




II. The Modern

- $K3$ and the Mathieu Group M_{24}
- Umbral Moonshine

K3 and the Mathieu Group M_{24}



⋆ $K3$ is an especially nice manifold of 4 dimensions (the unique non-trivial Calabi–Yau two-fold).

⋆ String theory accommodates extra dimensions and gives a useful tool to study its topological properties.

⋆ It gives rise to a superconformal field theory (SCFT), where extended symmetries (superconformal algebras) organise the spectrum into its representations ("multiplets").

Elliptic Genus

In a 2d *N>=(2,2)* SCFT, susy states are counted by the *elliptic genus*.

$$\mathbf{EG}(\tau, z; \mathrm{CFT}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\left((-1)^{J_0+\bar{J}_0} y^{J_0} q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24}\right) \quad (y = e^{2\pi i z})$$

It is modular (a Jacobi form):

$$\begin{aligned}\mathbf{EG}(\tau, z) &= \mathbf{EG}(\tau, z+1) = \exp(\tfrac{\pi i c}{3}(\tau + 2z))\,\mathbf{EG}(\tau, z+\tau)\\ &= \mathbf{EG}(\tau+1, z) = \exp(-\tfrac{\pi i c}{3}\tfrac{z^2}{\tau})\,\mathbf{EG}(-\frac{1}{\tau}, \frac{z}{\tau})\end{aligned}$$

K3 Sigma-Model

Recall: 2*d* sigma model on *K3* is a *N*=(4,4) SCFT.

⇒ The spectrum fall into irred. representations of the *N*=4 SCA.

$$\mathbf{EG}(\tau, z; K3) = 8 \sum_{i=2}^{4} \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)} \right)^2$$

[Eguchi–Ooguri–Taormina–Yang '88]

$$= \underline{24 \text{ short multiplets}} + \underline{\text{long multiplets with different } \Delta}$$

$$= \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \left(24\, \mu(\tau, z) + 2q^{-1/8} \left(-1 + 45\, q + 231\, q^2 + 770\, q^3 + \dots \right) \right)$$

number of long *N*=4 multiplets

(as recorded in Hirosi's PhD thesis)

K3 Sigma-Model

Recall: *2d* sigma model on *K3* is a *N=(4,4)* SCFT.

⇒ The spectrum fall into irred. representations of the *N=4* SCA.

$$\mathbf{EG}(\tau, z; K3) = 8 \sum_{i=2}^{4} \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)} \right)^2$$

[Eguchi–Ooguri–Taormina–Yang '88]

$$= \underline{24 \text{ short multiplets}} + \underline{\text{long multiplets with different } \Delta}$$

$$= \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \left(\underline{24\,\mu(\tau, z)} + \underline{2q^{-1/8}\left(-1 + 45\,q + 231\,q^2 + 770\,q^3 + \dots\right)} \right)$$

[Eguchi–Ooguri–Tachikawa '10]

number of long *N=4* multiplets

also dimensions of irreps of *M*$_{24}$!!!

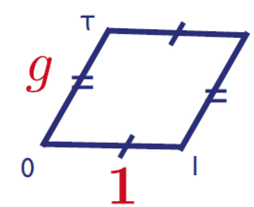
Interlude: Twisting the Symmetries

Hilbert space $\mathcal{H}$

$\circlearrowleft$ $g \in G$

The *twined* partition function $Z_g = \mathrm{Tr}_{\mathcal{H}}(g \ldots)$
gives finer information about $\mathcal{H}$ than just its graded dimensions.

e.g. 2d CFT $\qquad Z_g(\tau) = \mathrm{Tr}_{\mathcal{H}}(g\, q^{\hat{H}} \cdots)$



boundary condition: ~~$SL(2,Z)$~~ $\rightarrow \Gamma_g$

One expects the function to be at *level* $\mathrm{ord}(g)$, namely the group Γ_g is restricted to $\mathrm{ord}(g)|c$.

Questions :

⋆ What is this infinite q-series?

$$+\,2q^{-1/8}\left(-1+45q+231q^2+770q^3+\dots\right)$$

number of long N=4 multiplets

also dimensions of irreps of M_{24}!!!

⋆ How about the group characters? Given a group representation V ($\rho : G \to End(V)$)

$$Tr_V g := Tr\rho(g).$$

Need to consider

$$2q^{-1/8}\left(-1+(Tr_{V_{45}}g)q+(Tr_{V_{231}}g)q^2+\dots\right)$$

"Niemeier"
Finite
Groups

Umbral Moonshine

"Mock"
Modular
Forms

Mock Modular Form

: a variant of modular form that comes with a
non-holomorphic correction, given by the shadow (or umbra) of the mmf.

$$f(\tau) + \int_{-\bar{\tau}}^{\infty} (\tau + \tau')^{-w} \overline{s(-\bar{\tau}')} = \hat{f}(\tau, \bar{\tau})$$

[Ramanujan '20]
....
[Zwegers '02]

$f(\tau)$ → holomorphic, mock mod.

s → shadow

$\hat{f}$ → modular

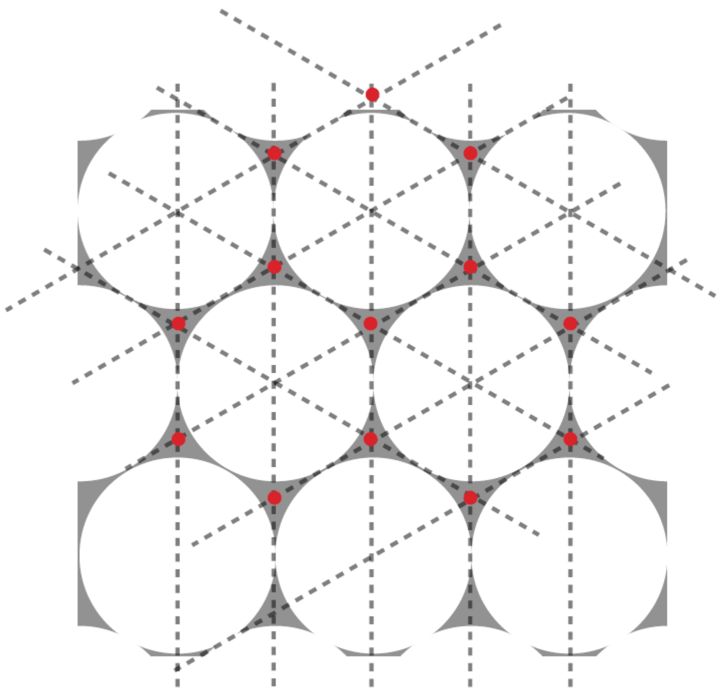
Mock modular forms appear in physics!

[Vafa–Witten '94, Dabholkar–Murthy–Zagier '12, Troost, Aschok, Maschot,....]

Holomorphic Anomaly ↔ Non-Compactness of the Moduli Space

eg. wall-crossing, the cigar theory, ...

Niemeier Groups
More Symmetries From Packing Balls



- :“*deep holes*” of the sphere packing = the points where the distance to any centre of a ball is maximum

The deep holes define a lattice themselves.

In the 24-dimensional sphere packing given by the Leech lattice Λ_{Leech}, there are 23 such deep hole lattices, $N^{(1)}, N^{(2)}, ..., N^{(23)}$, called the ***Niemeier Lattices***.

Theorem (Niemeier 1973)
In 24 dimensions, there are exactly 24 interesting (even, self-dual, negative-definite) lattices.
They are Λ_{Leech} and the 23 Niemeier lattices $N^{(1)}, N^{(2)}, ..., N^{(23)}$.

Niemeier Groups

More Symmetries From Packing Balls

Consider the symmetries of these 23 Niemeier lattices, we obtain the 23 finite groups, which we call the **Niemeier Groups**.

Def: $G^X := Aut(N^X)/W^X$

X = the roots $(\langle x,x\rangle=-2)$ of the Niemeier lattice, used to label the lattice N^X.

W^X= Weyl reflection w.r.t. X

Example:

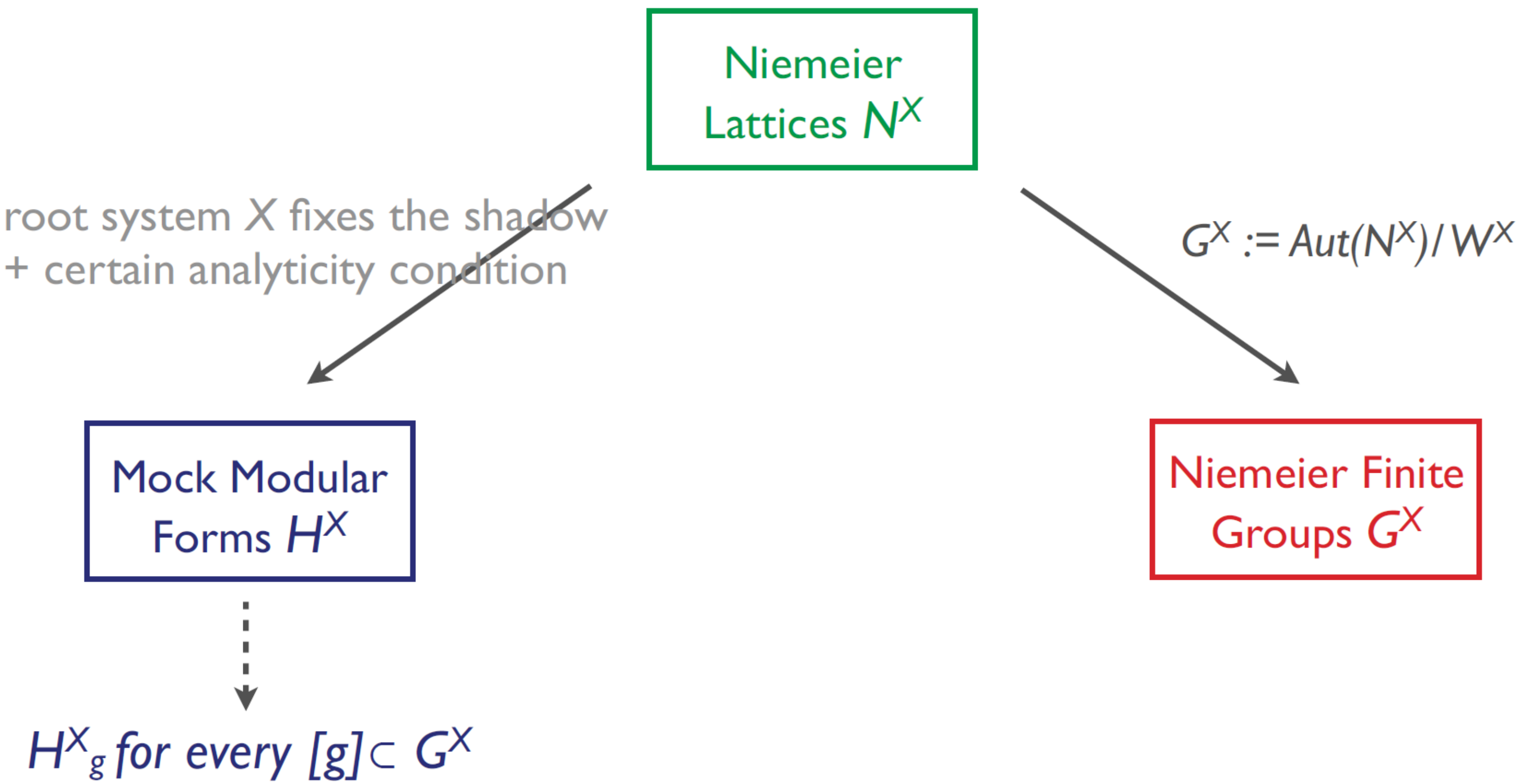
Consider the "simplest" Niemeier lattice with $X=24A_1$.

Its symmetry gives $G^X=M_{24}$!

Niemeier Mock Forms

For each of the 23 Niemeier lattice N^X, we associate a (unique) mock modular form $H^X{}_g$ for every element g of G^X via a procedure related to the A-D-E classification of modular invariants.

[Inspired by Cappelli–Itzykson–Zuber '87, Dabholkar–Murthy–Zagier '12]



Umbral Moonshine

Relating Mock Modular Forms and Niemeier Symmetries

Mock Modular Forms H^X

Umbral Moonshine ⟷

MC–Duncan–Harvey 2013

Niemeier Finite Groups G^X

Example: Corresponding to the "simplest" Niemeier lattice N^X, $X=24\,A_1$

$$H^X = -2q^{-1/8}\Big(-1+24\sum_{k=1}^{\infty}\frac{k}{1-q^k}\left(q^k+(-1)^k q^{k(k+1)/2}\right)\Big)\prod_{j=1}^{\infty}(1-q^j)^{-3}$$

$$= 2q^{-1/8}(-1+\underline{45}\,q+\underline{231}\,q^2+\underline{770}\,q^3+\underline{2277}\,q^4+\underline{5796}\,q^5\dots)$$

They are dimensions of irreps of the largest Mathieu sporadic group $G^X=M_{24}$!

Umbral moonshine conjectures: [MC–Duncan–Harvey 2013]
For every one of the 23 Niemeier lattices N^X, there exists a naturally defined infinite-dimensional G^X-module $K^X = \oplus_\alpha K^X_\alpha$ (the *umbral module*) such that

$$H^X_g(\tau) = \sum_\alpha q^\alpha \left(\mathrm{Tr}_{K^X_\alpha} g\right).$$

Status:
The module K^X have been constructed for 8 of the 23 cases. There has been steady progress. But many important cases, including the M_{24} case originally discovered by Eguchi–Ooguri–Tachikawa, is still mysterious.

III. The Post-Modern

what is moonshine, really?

The relation between the two does not necessarily have to be exotic, exceptional, or unique.

In fact, there seems to be a **ubiquitous** relation between these two mathematical structures.

A More General Connection

Connections between modular objects and finite groups seem much more common if we allow for 2 new features:

1. Group rep's come with **signs** and one has the **supertrace**:

$V = \oplus_n \left(V_{0,n} \oplus V_{1,n}\right), \qquad f_g(\tau) := \sum_n q^n \sum_{i\in\{0,1\}} (-1)^i \left(Tr_{V_{i,n}} g\right)$

2. The functions are no longer determined ~~**uniquely**~~ by modularity and pole structure; there are **cusp forms** (modular forms that vanish at the boundary) to add.

A much more flexible game!

Examples:

⋆ Weight 3/2 modular forms are connected to “Pariah” sporadic groups. [Duncan–Mertens–Ono 2017]

⋆ Class number mock modular forms are connected to cyclic group $\mathbb{Z}/p$ for **all** prime p. [Cheng–Duncan–Mertens, to appear]

Class number $H(D)$ counts quadratic forms $Q(x,y) = Ax^2 + Bxy + Cy^2$ with discriminant $D := B^2 - 4AC$ up to equivalence under $SL_2(\mathbb{Z})$:

$$H(D) := \sum_{Q \in \mathcal{Q}(D)/SL_2(\mathbb{Z})} \frac{1}{\# SL_2(\mathbb{Z})_Q}$$

The generating function

$$H(\tau) = -\frac{1}{12} + \sum_{D<0} H(D)\, q^{-D} = -\frac{1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + q^7 + q^8 + \dots$$

is a weight $3/2$ mock modular form, which also appears as the Vafa–Witten partition function on $\mathbb{CP}^2$. There is a straightforward generalisation $H^{(N)}(\tau)$ for all $N \in \mathbb{Z}_+$. For p prime, define

$$h_p(\tau) = \frac{p}{(p+1)(p-1)} H^{(p)}(\tau) - \frac{1}{p+1} H(\tau).$$

Theorem

For all prime p, there exists virtual modules for $G = \mathbb{Z}/p$ whose supertraces for the trivial and non-trivial elements are given by $m_p(12H(\tau))$ and $m_p h_p(\tau) = -m_p + O(q)$ respectively, where

$$m_p = \mathrm{num}(\frac{p-1}{12})\,\mathrm{num}(\frac{p+1}{6}).$$

e.g. $m_{11} = 10$.

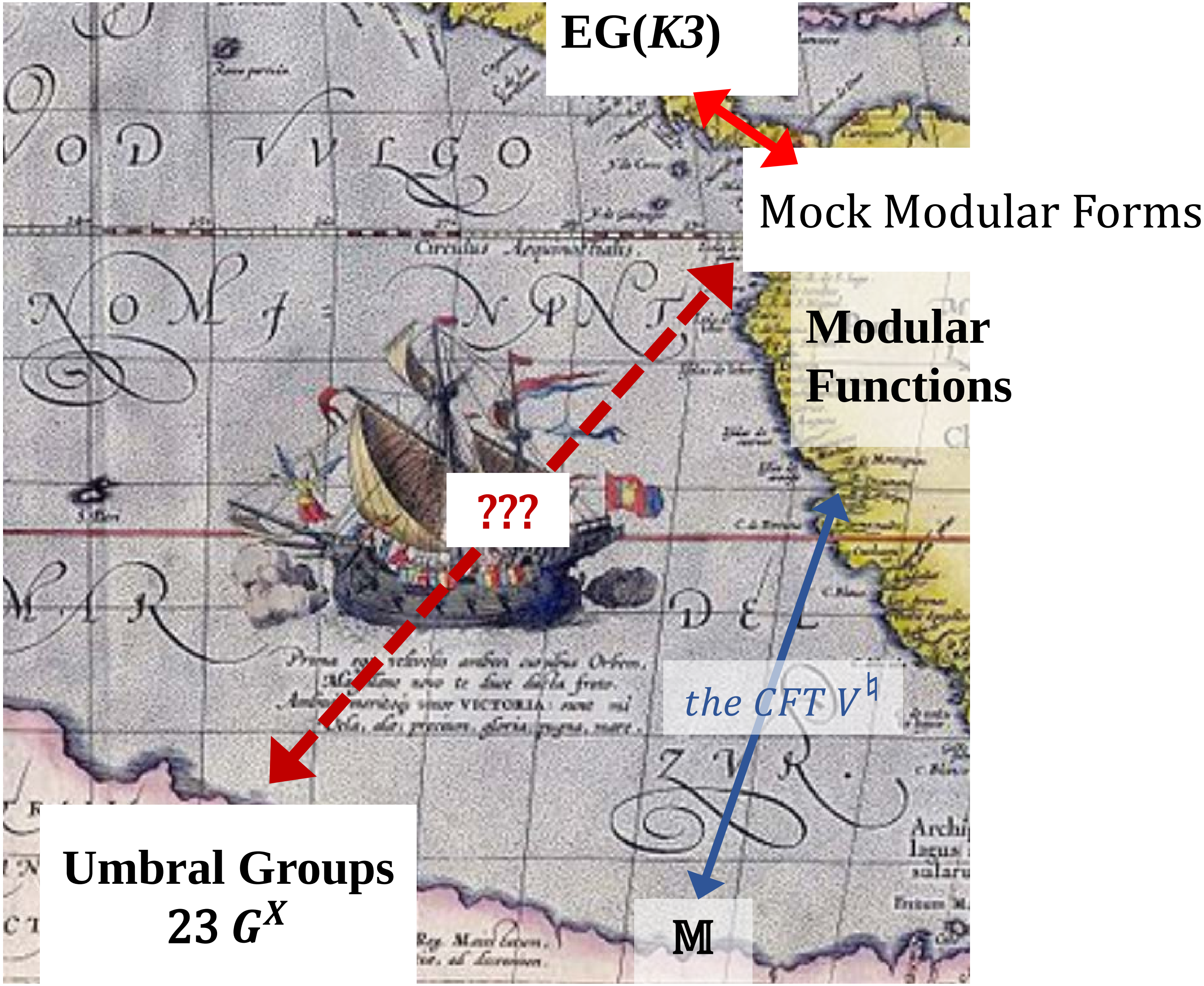
When p is such that there are cusp forms at level p, one can use the cusp forms to change the supertrace function into something with smaller (in magnitude) constant terms, and the ratio is given in terms of concrete arithmetic geometric data.

e.g. $m_{11} = 10$. $\mathrm{Jac}(X_0(11)) \cong \mathbb{Z}/5$. $n_{11} = 5$.
There exists virtual modules for $G = \mathbb{Z}/11$ whose supertrace is given by

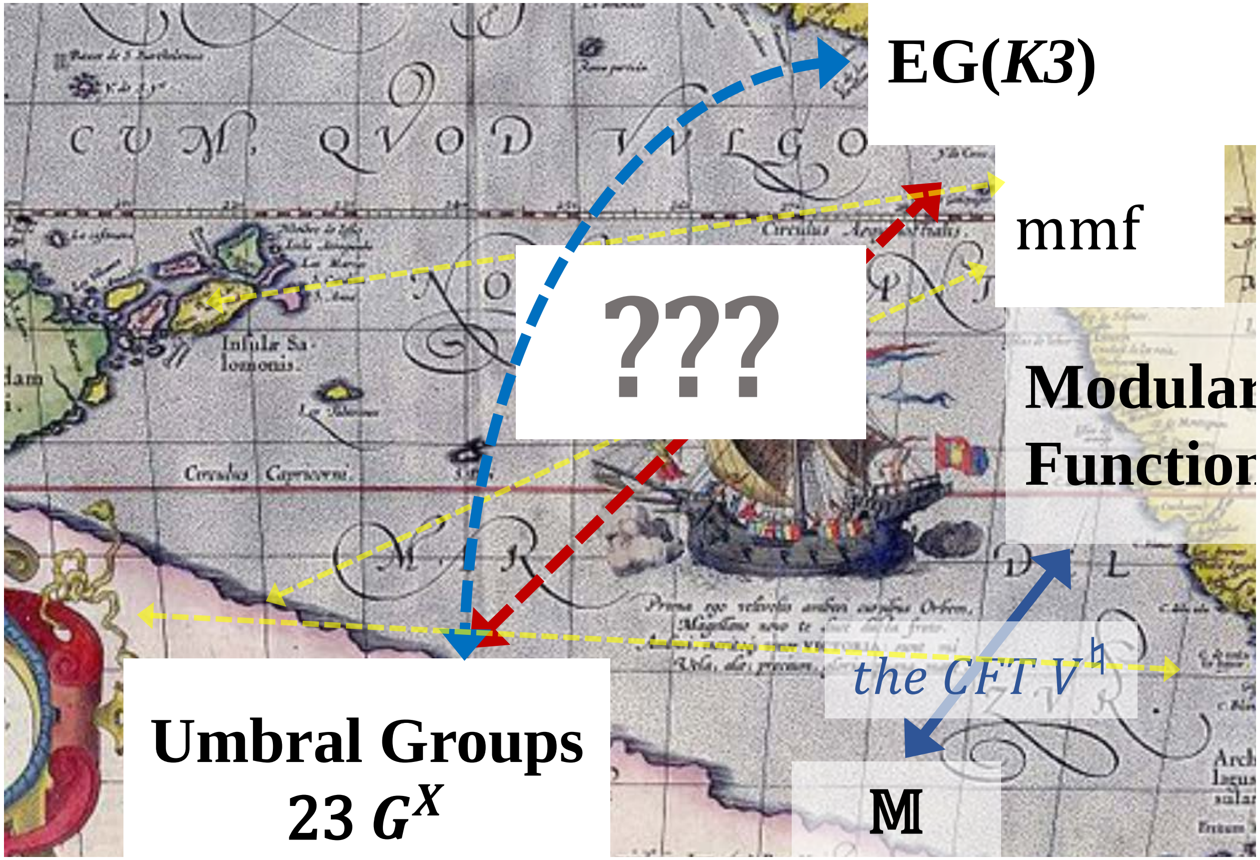
$$2h_{11}(\tau) - \frac{11}{5}g_{11} = -2 + O(q).$$

where $g_{11} = q^3 + O(q^4)$ is the unique cusp form at level 11.

The connection to arithmetic geometry goes way further. Since the coefficients of the cusp forms give interesting information on the arithmetic geometry of the associated elliptic curve (over rationals) via the celebrated Birch–Swinnerton–Dyer conjecture.


EG($K3$)
Mock Modular Forms
Modular Functions
???
the CFT $V^\natural$
Umbral Groups
23 G^X
$\mathbb{M}$

What's the landscape?



Outlook:

⋆ Connections between modular objects and finite groups seem ubiquitous.

⋆ It could be that these two are intrisically connected. We need to find out what the structure is.

⋆ Can finite group connections shed new light on arithmetic geometry?

⋆ Are these “post-moonshine” also related to physics?

Thank You!