Quantum entanglement in many-body systems – Entanglement and topology detected by partial transpose –

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- Quantum entanglement. [Einstein-Podolsky-Rosen (1935)]

$$|\Psi\rangle = \frac{|\uparrow_A\rangle|\downarrow_B\rangle - |\downarrow_A\rangle|\uparrow_B\rangle}{\sqrt{2}}$$

- *"... spooky actions at a distance"* [– Albert Einstein –]
- *"I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."* [– Erwin Schrödinger –]

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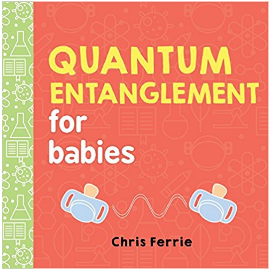
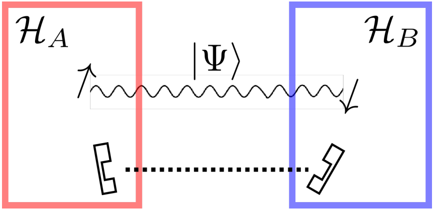
- 

Quantum entanglement; basic setup



- Quantum entanglement = What cannot be generated by Local quantum operations and classical communications (LOCC):

$$\rho \longrightarrow (U_A \otimes U_B)\,\rho\,(U_A \otimes U_B)^\dagger$$

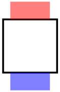
$$\text{but not} \quad \rho \longrightarrow U_{AB}\,\rho\,U_{AB}^\dagger.$$

Quantum entanglement; how to quantify it?

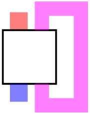
- The reduced density matrix by taking partial trace:

$$\rho_A := \mathrm{Tr}_B\left(\rho_{A\cup B}\right)$$

$\rho_{A\cup B}$



$\mathrm{Tr}_B\left(\rho_{A\cup B}\right)$



- von-Neumann entanglement entropy:

$$S_A := -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

- S_A for pure state $\rho_{A\cup B} = |\Psi\rangle\langle\Psi|$ decreases monotonically under LOCC.

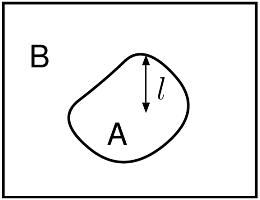
Quantum entanglement in many-body physics

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m} + \sum_{i,j} V(|\vec{r}_i - \vec{r}_j|)$$

- Characterization of quantum states beyond order parameter paradigm
- Computational difficulties
- Renormalization group flow and quantum entanglement
- Non-equilibrium physics, eigenstate thermalization, quantum information scrambling, etc.
- ....

Entanglement entropy in many-body systems

- Scaling of S_A as a function of subsystem size A can tell different phases, and computational difficulties.



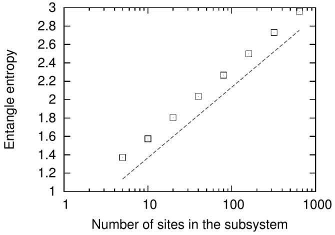
E.g., Area/volume laws

$$S_A \sim Area_A, \quad S_A \sim Vol_A$$

Examples

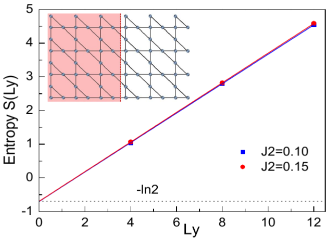
- (1+1)D CFT

$$S_A = \frac{c}{3} \log \ell$$



- (2+1)D quantum spin liquid (topologically ordered phases)

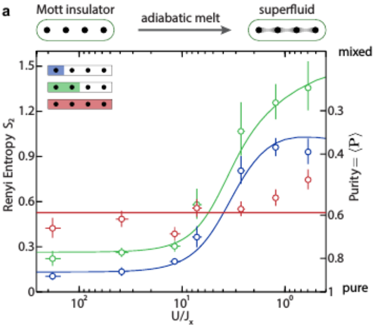
$$S_A = const. \frac{\ell}{\epsilon} - \gamma$$

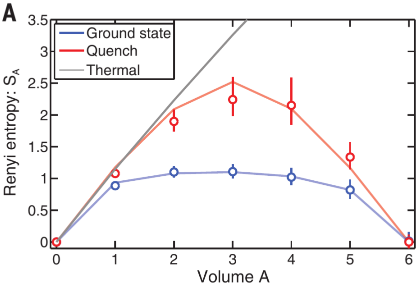


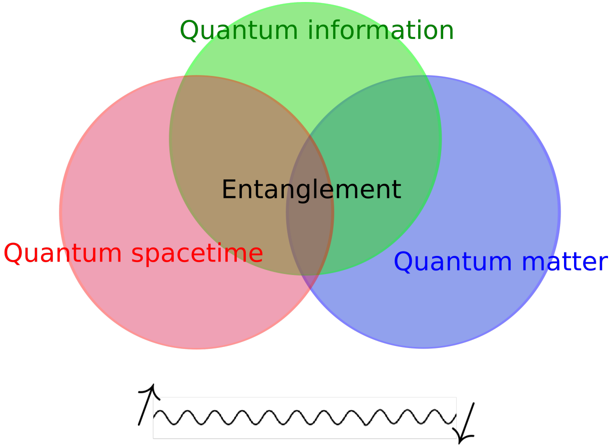
[Jiang-Wang-Balents (12)]

Can we measure it experimentally?

- [R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. N. Rispoli, M. Greiner, Nature (2015)] [AM Kaufman, ME Tai, A Lukin, M Rispoli, R Schittko, PM Preiss, M Greiner, Science (2016)]






Quantum information
Entanglement
Quantum spacetime
Quantum matter

Outline

- Introduction
- Quantum entanglement and partial transpose
- Fermionic quantum entanglement
- Topological phases detected by partial transpose
- Future problems

Entanglement in mixed states?

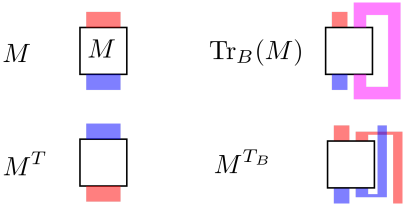
- How to quantify quantum entanglement between A and B when $\rho_{A\cup B}$ is *mixed* ? E.g., finite temperature, A, B is a part of bigger system.
- The entanglement entropy is an entanglement measure only for pure states. For mixed states, it is not monotone under LOCC.

Partial transpose (bosonic case)

- Definition: for an operator M, its partial transpose M^{T_B} is

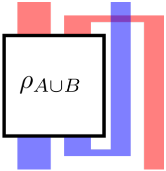
$$\langle e_i^{(A)} e_j^{(B)} | M^{T_B} | e_k^{(A)} e_l^{(B)} \rangle := \langle e_i^{(A)} e_l^{(B)} | M | e_k^{(A)} e_j^{(B)} \rangle$$

where $|e_i^{(A,B)}\rangle$ is the basis of $\mathcal{H}_{A,B}$.



Partial transpose and entanglement

$$\rho_{A\cup B}^{T_B}$$



Partial transpose and quantum entanglement

- Bell pair:$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01|]$$

- Partial transpose:

$$\rho^{T_2} = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - \underline{|00\rangle\langle 11|} - \underline{|11\rangle\langle 00|}]$$

- Entangled states are badly affected by partial transpose: Negative eigenvalues:

$$\{\lambda_i\} = Spec(\rho^{T_B}) = \{1/2, 1/2, 1/2, -1/2\}$$

- C.f. For a classical state: $\rho = \frac{1}{2}[|00\rangle\langle 00| + |11\rangle\langle 11|] = \rho^{T_B}$

Partial transpose and Entanglement negativity

- *Entanglement negativity* and *logarithmic negativity*, using *partial transpose*,

$$\mathcal{N}(\rho) := \sum_{\lambda_i<0} |\lambda_i| = \left(||\rho^{T_B}||_1 - 1 \right)/2,$$

$$\mathcal{E}(\rho) := \log(2\mathcal{N}(\rho) + 1) = \log ||\rho^{T_B}||_1.$$

[Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]

- For mixed states, Negativity extracts quantum correlations only.
- The logarithmic negativity is an entanglement monotone (but not convex). [Plenio (2005)]

Partial transpose and negativity in fermionic systems

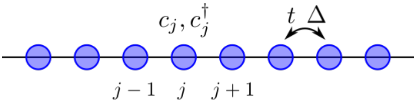
- Partial transpose is useful to detect entanglement in many-body states.
- How about fermion systems? E.g., the Kitaev chain
- Collaborators: Hassan Shapourian (U Chicago), Ken Shiozaki (RIKEN)

- Based on: arXiv:1611.07536; 1804.08637; 1807.09808

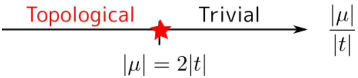
The Kitaev chain

- The Kitaev chain

$$H = \sum_j \Big[-t c_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$



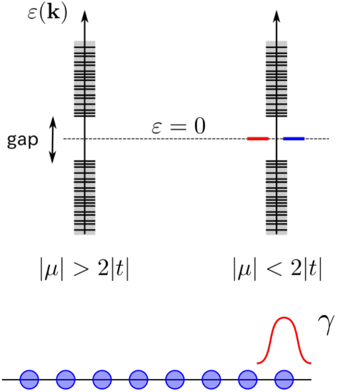
- Phase diagram: there are only two phases:



- Topologically non-trivial phase is realized when $2|t| \geq |\mu|$.

Majorana end states

- Characteristic to the topological SC phase are zero-energy states at each end.

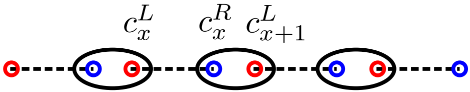


- The end state is a *Majorana* fermion. $\gamma^\dagger = \gamma$

Ground state; Majorana dimers

- Fractionalizing an electron into two Majoranas:

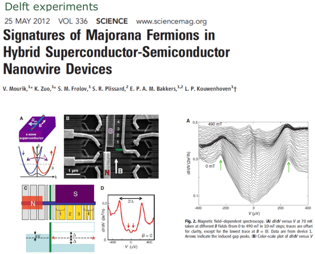
$$c_x = c_x^L + i c_x^R, \quad c_x^\dagger = c_x^L - i c_x^R.$$

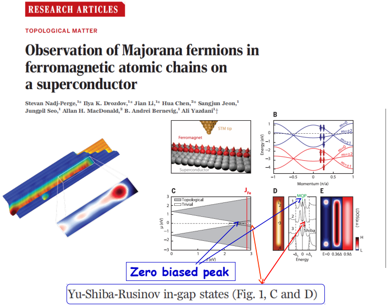


Ground state consists of "Majorana dimers"

Experiments

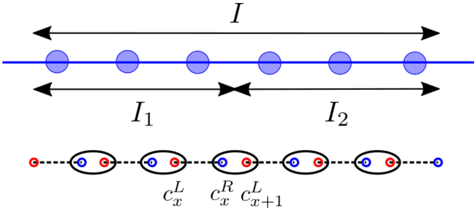
- Majorana fermions are useful for topological quantum computation.
- Proximitized spin-orbit quantum wire [Lutchyn et al (10), Oreg et al (10), Mourik et al (12), ...], magnetic adatomes on the surface of an s-wave superconductor [Nadj-Perge et al (14)]





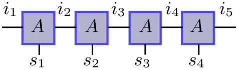
Negativity in the Kitaev chain

- log negativity $\mathcal{E}$ for two adjacent intervals I_1 and I_2

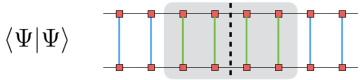


- Wave function;

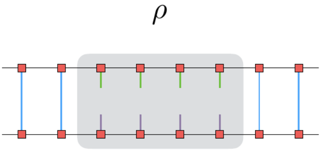
$$\Psi(s_1, s_2, \cdots) = \sum_{\{i_n=1,\cdots\}} A^{s_1}_{i_1 i_2} A^{s_2}_{i_2 i_3} A^{s_3}_{i_3 i_4} \cdots \quad s_a = \uparrow, \downarrow$$



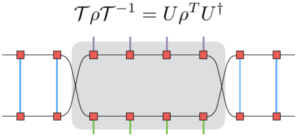
- Overlap ("Partition function"):



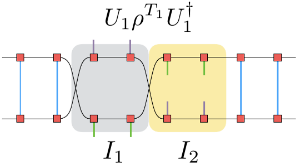
- Reduced density matrix:



Full transpose:

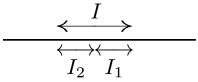


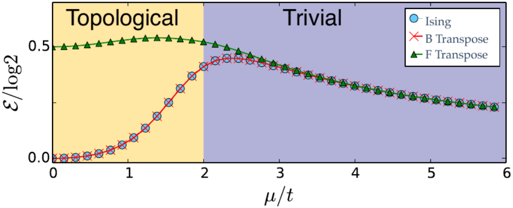
Partial transpose:



Issues in fermionic systems

- Log negativity $\mathcal{E}$ for two adjacent intervals I_1 and I_2 of equal length. $(L = 4\ell = 8)$





- (Blue circles and Red corsses) is computed by Jordan-Wigner + bosonic partial transpose

- Log negativity fails to capture Majorana dimers.

Issues in fermionic systems (2)

- Partial transpose of bosonic Gaussian states is still Gaussian; easy to compute by using the correlation matrix
- Partial transpose of fermionic Gaussian states are not Gaussian
  - ρ^{T_1} can be written in terms of two Gaussian operators $O_\pm$:

$$\rho^{T_1} = \frac{1-i}{2} O_+ + \frac{1+i}{2} O_-$$

  - Negativity estimators/bounds using $\mathrm{Tr}\left[\sqrt{O_+ O_-}\right]$ [Herzog-Y. Wang (16), Eisert-Eisler-Zimborás (16)]
  - Spin structures: [Coser-Tonni-Calabrese, Herzog-Wang]

Partial transpose for fermions – our definition

[Shiozaki-Shapourian-SR (16)]

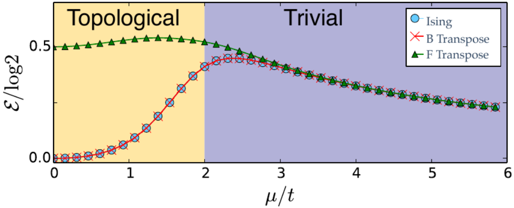
- Fermion operator algebra does not trivially factorize for $\mathcal{H}_A \otimes \mathcal{H}_B$.
- In Fock space basis $|\{n_j\}_A, \{n_j\}_B\rangle$,

$$\begin{aligned}&\langle \{n\}_A, \{n\}_B | \rho^{T_B} | \{m\}_A, \{m\}_B \rangle \\ &= (-1)^{\phi(\{n\},\{m\})} \langle \{n\}_A, \{m\}_B | \rho | \{m\}_A, \{n\}_B \rangle\end{aligned}$$

- C.f. fermionic matrix product states [Bultinck et al]
- Gaussian states stay Gaussian under our partial transpose; very computable

Comparison with previous definitions

[Shiozaki-Shapourian-SR (16)]



- (Blue circles and Red crosses): Old (bosonic) definition
- (Green triangles and Orange triangles) Our definition;
- At critical point: agrees with CFT prediction [Calabrese-Cardy-Tonni].

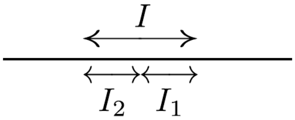
Motivation behind our construction

- Partial transpose can change the topology of spacetime: path integral on an unoriented spacetime [Pollmann-Turner, Calabrese-Cardy-Tonni, Shiozaki-SR]

$$\mathrm{Tr}\,(e^{-\beta H}) = \int \mathcal{D}[\phi] e^{-S(M\times S^1_\beta)},$$
$$\mathrm{Tr}\,(\rho\rho^{T_B}) = \int \mathcal{D}[\phi] e^{-S(?)}$$

- For topological phases, path integral = topological invariant.
- Once we "calibrate" the definition of T_B, one can use it for any system.

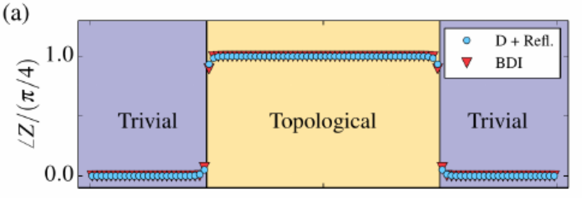
Partial transpose and topological invariant



- <u>Step 1:</u> The reduced density matrix for an interval I,
  $\rho_I := \mathrm{Tr}_{\bar{I}} |\Psi\rangle\langle\Psi|$.
- <u>Step 2:</u> Bipartition I into two *adjacent* intervals, $I = I_1 \cup I_2$.
- <u>Step 3:</u> Take *fermionic partial transpose* acting only on I_1;
  $\rho_I \longrightarrow \rho_I^{T_1}$.
- <u>Step 4:</u> The invariant is given by the phase of:

$$Z = \mathrm{Tr}(\rho_I \rho_I^{T_1}) = e^{2\pi i\nu/8}.$$

Numerics

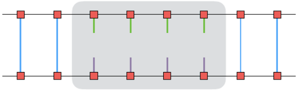


- The phase of Z is quantized to the 8th root of unity. Consistent with $\mathbb{Z}_8$ classification: [Fidkowski-Kitaev(10)]
- Similar constructions of *many-body* topological invariants for other fermionic SPT phases; e.g. $\mathbb{Z}_2$ time-reversal symmetric topological insulators.

What does the topological invariant computes?

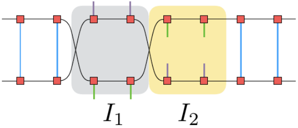
Reduced density matrix

ρ



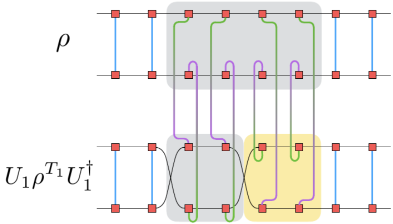
Partial transpose:

$U_1 \rho^{T_1} U_1^\dagger$



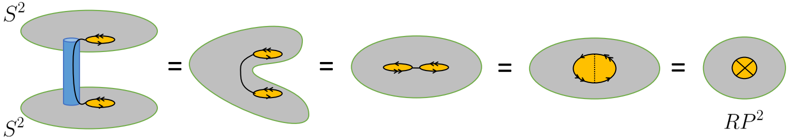
Topological invariant:

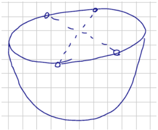
$\mathrm{Tr}(\rho U_1 \rho^{T_1} U_1^\dagger)$



It computes the partition function on unoriented surface

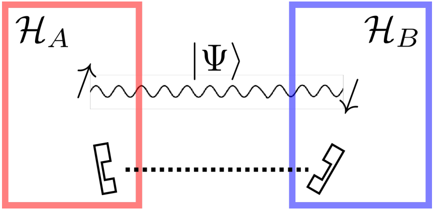
- The invariant simulates the path integral on real projective plane $\mathbb{R}P^2$: [Shiozaki-Ryu (16)]





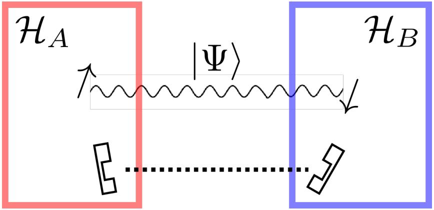
- The effective action prediction: $Z(\mathbb{R}P^2, \eta) = e^{2\pi i \mathrm{Brown}(\eta)/8}$ where η is one of two Pin_- structures on $\mathbb{R}P^2$. [Kapustin et al (14-15), Freed-Hopkins (14-15), Witten (15), and others]

Monotoncity under LOCC



- For bosonic systems, negativity is LOCC monotone
- I.e., what cannot be generated by LOCC = "quantum entanglement".
- von-Neumann entanglement entropy decreases monotonically at $T = 0$, but *not* at $T > 0$.

Monotoncity under LOCC



- Our fermionic version of partial transpose, and negativity, is it a good entanglement measure? Is it monotone under LOCC?
- In [Shapourian-SR (18)], we proved that fermionic entanglement negativity is monotone, if LOCC are taken to be fermion number parity preserving.
- Our negativity does not increase under: local unitaries, adding ancilla, local projective measurements, tracing ancilla

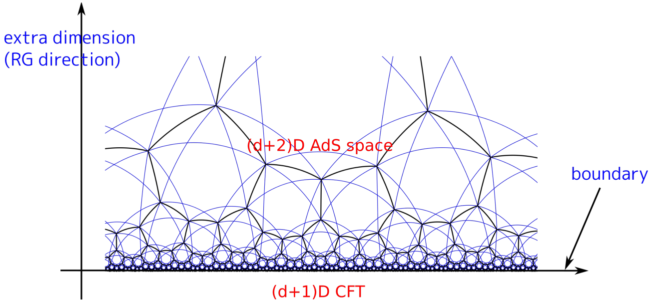
Summary

- Introduced partial transpose for fermionic systems. “Fermionic partial transpose”
- The (log) negativity using the fermionic partial transpose can capture the formation of Majorana dimers in the Kitaev chain.
- Proved that the fermionic negativity is a proper entanglement measure for fermionic systems.
- Partial transpose can be used to construct topological invariants of topological phases of matter.

Future problems

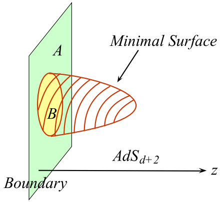
- Can we measure entanglement negativity and topological invariants?
- How useful is entanglement negativity (over other quantities)? Topological order, non-equilibrium phenomena (thermalization, quantum chaos, etc.)
- Geometric/holographic interpretation of negativity?
  - C.f. Holographic entanglement entropy formula
  - Connection to entanglement wedge cross section? "Entanglement negativity and minimal entanglement wedge cross sections in holographic theories", Jonah Kudler-Flam and SR, arXiv:1808.00446

Holographic quantum states



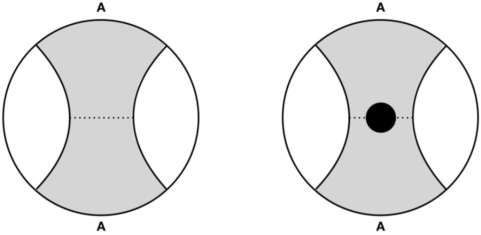
- Holographic entanglement entropy formula

$$S_A = \frac{\mathit{Area}\ \gamma_A}{4G_N}$$



Related to Entanglement wedge?

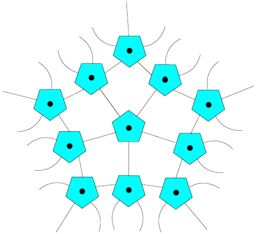
- Entanglement wedge = the bulk region corresponding to the reduced density matrix on the boundary [Headrick et al (14), Jafferis-Suh (14), Jafferis-Lewkowycz-Maldacena-Suh (15), ...]

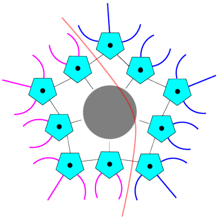


- Minimal entanglement wedge cross section and negativity [Kudler-Flam, SR, arXiv:1808.00446]
- C.f. Entanglement of purification: [Takayanagi-Umemoto(17), Nguyen-Devakul-Halbasch-Zaletel-Swingle (17)]

Perfect tensor holographic error correcting code

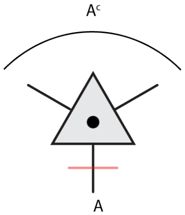
- Computed/explored entanglement negativity in a tensor network model of holographic duality. [Almheiri-Dong-Harlow (15), Harlow (17), Pastawski-Yoshida-Harlow-Preskill(15)]

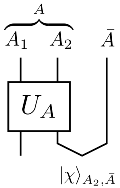




- This tensor network acts as an error correcting code encoding multiple "bulk" logical qubits into multiple "boundary" physical qubits
- Captures many aspects of holography; black holes, bulk reconstruction, subregion duality, holographic entanglement entropy, etc.

- It is crucial/enough to look at “smallest” possible network:





- In this case, the negativity is given by

$$\mathcal{E}(\rho_A) = \log(|A|).$$

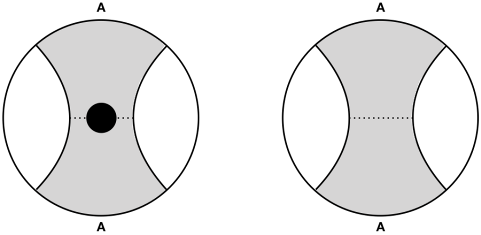
- Compared to the entanglement entropy:

$$S(\rho_A) = \log(|A|) + S(\tilde{\rho}),$$

the negativity does not pick up the bulk contribution.

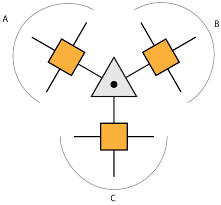
Negativity in perfect tensor holographic code

- Negativity avoids horizon.

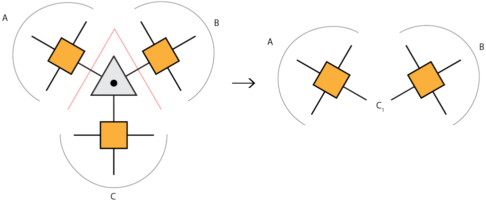


9-qutrit tensor network

- Simplest model: 9-qutrit model

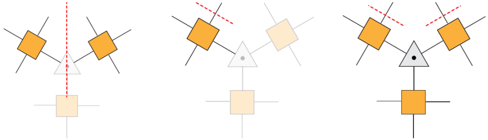


- Tracing out uninteresting degrees of freedom to get effective error correcting code:



Tensor network

- The traced degrees of freedom behave as bulk indices; negativity will not detect them.
- The negativity is given by the minimal entanglement wedge cross section



Negativity	0	$\log(3)$	$2\log(3)$
Entanglement Wedge	0	$\log(3)$	$2\log(3)$
Mutual Information	$\log(3) - S(\rho)$	$2\log(3)$	$4\log(3)$

AdS_3/CFT_2

- How about negativity in the full fledged AdS/CFT? No time to discuss ... but rather interesting.
- In holographic code models; many quantities are “degenerate”; mutual information, negativity, entanglement of purification.
- CAUTION Back reaction is expected; since, e.g., in certain case, negativity is Renyi entropy at $n = 1/2$ [Dong(16)]
- See our paper for more detailed comparisons.