Skein module valued open GW invariants and large N duality

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WPC symposium, Nov 7-9, 2018
DESY Hamburg

Plan

Knot invariants from Chern-Simons theory

Connections to topological string

Large N duality - from open to closed

The mechanism - stretching the complex structure

Further consequences, quantum curves, refinement . . .

M closed 3-manifold.

A $U(N)$-connection on M

$$CS(A) = \int_M \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$Z_{cs}(M) = \int \mathcal{D}A \; e^{\frac{ik}{4\pi} CS(A)}$$

Expand around flat connections in $1/k$.

$K \subset M$ knot or link.

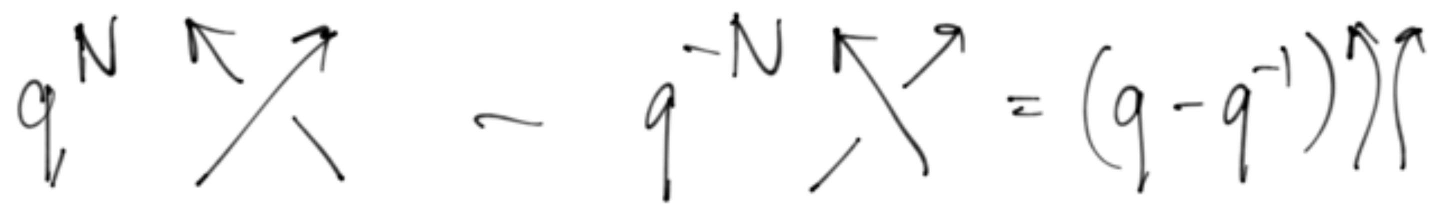
$$\langle K \rangle = \int \mathcal{D}A \, e^{\frac{ik}{4\pi} CS(A)} \operatorname{tr}_\rho \left(\mathrm{Hol}_A(K) \right)$$

Following Witten we compute by cut & paste:

$\dim \mathcal{H} = 2$ gives

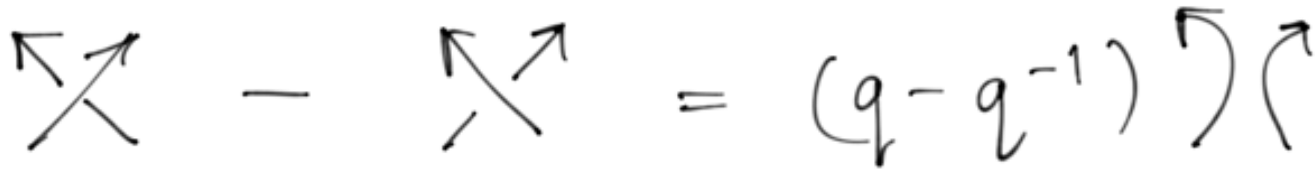
Skein relation

Constants for $U(N)$



$$q = \frac{2\pi i}{k+N}$$

Framed skein module



In S^3 this gives the framed HOMFLY-PT polynomial.

Adding an unknot

$(q-q^{-1})$  $=$  $-$  $= (a-a^{-1})$ 

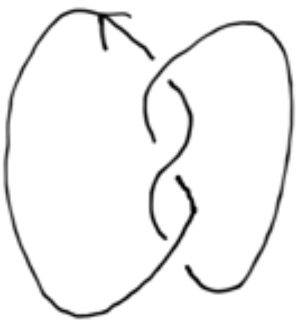
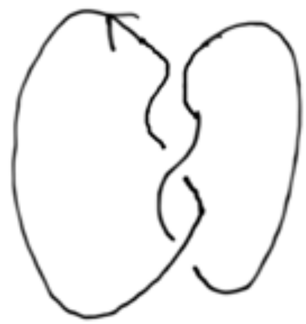
$$\Rightarrow \quad \langle K \sqcup \bigcirc \rangle = \frac{a-a^{-1}}{q-q^{-1}} \langle K \rangle$$

Unknot:

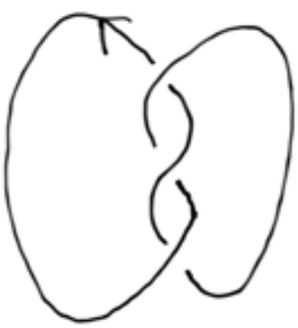
a  $- \; a^{-1}$  $=$  $-$ 

$= (q - q^{-1})$  

Trefoil:

 $-$  $= (q - q^{-1})$ 

$= (q - q^{-1}) \Big($  $+ (q - q^{-1})$  $\Big)$

$\Rightarrow$  $=$  $\left(q^2 a - q^{-2} a - a^{-1} \right)$

$$M \subset T^*M$$

Lagrangian submanifold

symplectic form

$$\omega = dp_i \wedge dq_i = d(p\,dq)$$

T^*M is CY.

Open string for N branes on M is approximated by $U(N)$ gauge theory.

In topological string this approximation is exact (Witten via string field theory)

$$Z(M, T^*M) = Z_{cs}(M)$$

N branes

U(N)

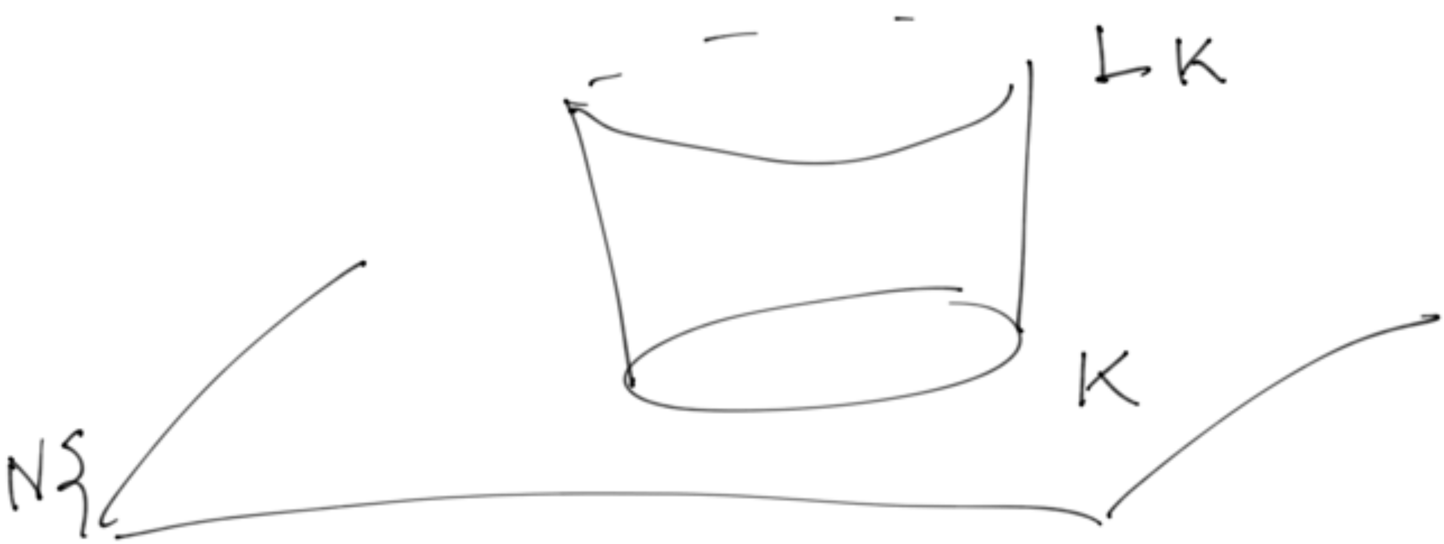
$$g_s = \frac{2\pi i}{k+N}$$

Add knot.

$K \subset M \rightsquigarrow L_K \subset T^*M$

conormal Lagrangian

$$L_K = \{(q,p) \in T^*M : q \in K, p(T_K) = 0\}$$



$$Z_K = \sum_{n \geq 1} H_n \, e^{nx}$$

counts string
states in

$nx \in H_1(L_K)$.

$$H_n = \underbrace{\square\square\square\square\square\square\square}_{n} \text{ rep HOMFLY.}$$

in particular H_1 = HOMFLY.

Open topological string localizes on holomorphic curves and

$$Z_{\text{string}} = Z_{GW}$$

count of holomorphic curves

For $L_K \subset T^*M$

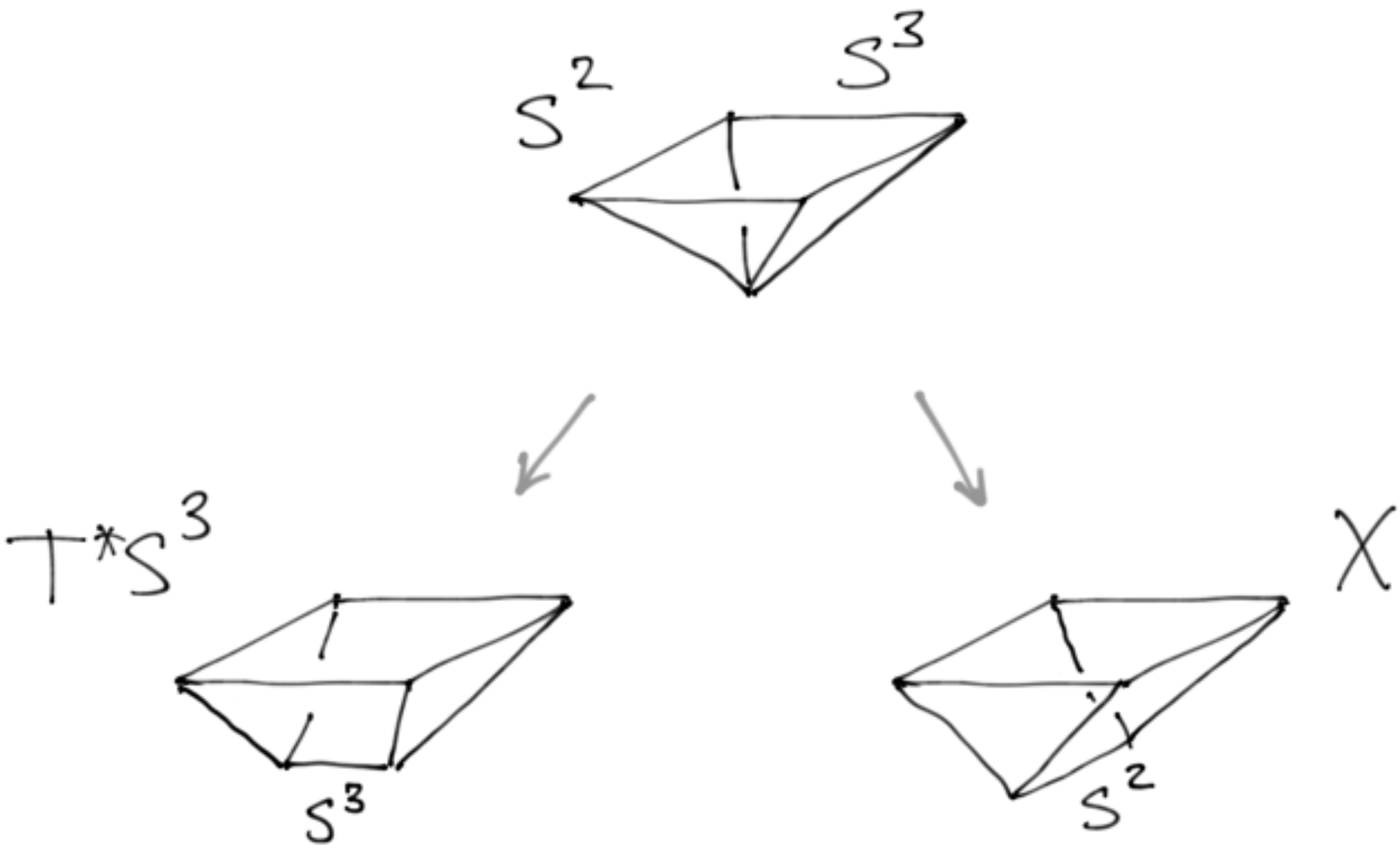
$$\text{area}(u) = \int_{\partial u} p\,dq = 0$$

$\Rightarrow$ all curves constant.

Ooguri - Vafa large N duality :

We move to resolved conifold X and turn open strings to closed, keeping partition functions.

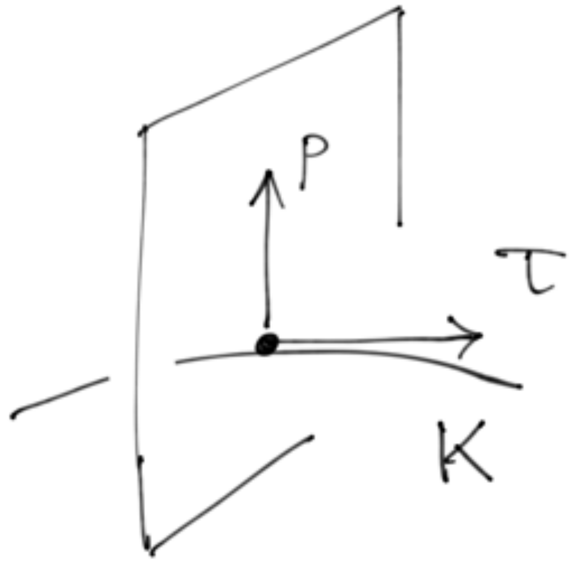
$$\{ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \}$$



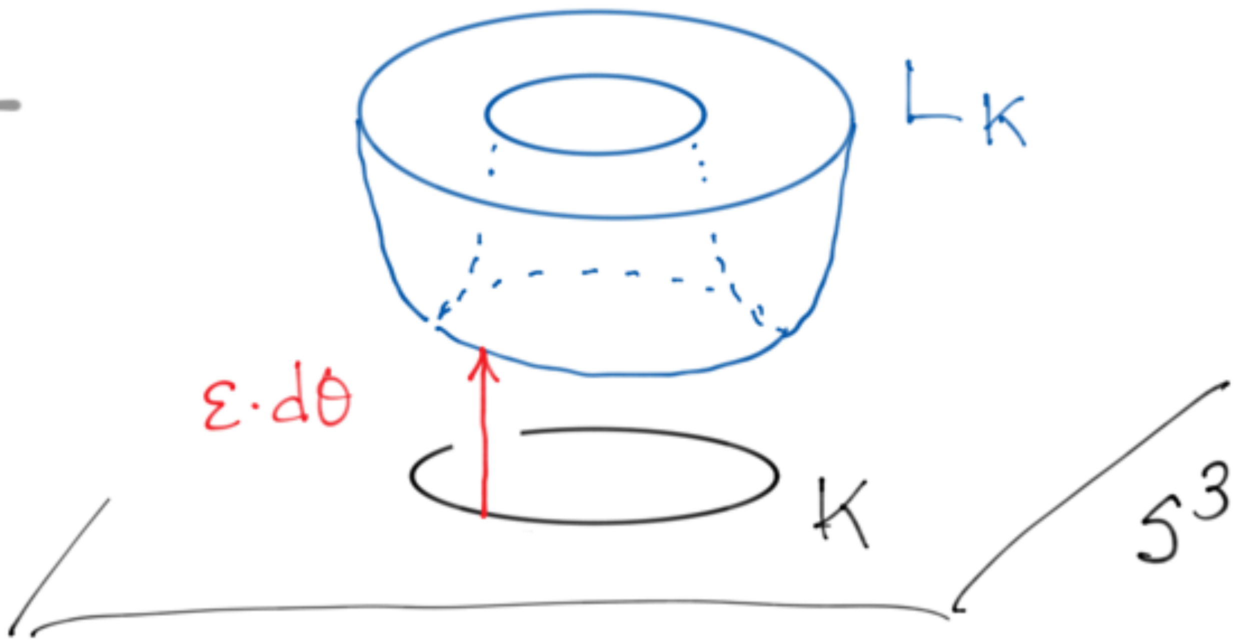
$$\begin{array}{c} \mathcal{O}(-1)^{\oplus 2} \\ \downarrow \\ \mathbb{CP}^1 \end{array}$$

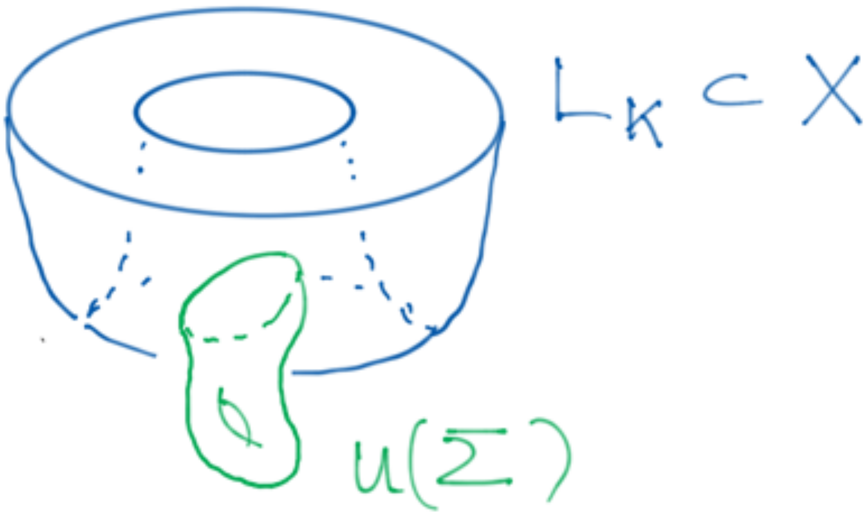
Knot conormals

$$L_K = \{(q,p) : q \in K,\ p(\tau) = 0\}$$



Shift along $\varepsilon . d\theta$





$$\begin{cases} u : (\Sigma, \partial\Sigma) \longrightarrow (X, L_K) \\ du + J \circ du \circ i = 0 \end{cases}$$

$$\Psi_K(Q, g_s) = \exp\left(\sum_{u \in \mathcal{M}} w(u)\, g_s^{-\chi(u)}\, Q^{k(u)}\, e^{\ell(u) x} \right)$$

Ooguri - Vafa large N

$$\Psi_K(Q, g_s) = \sum_n H_n(Q^{1/2}, e^{g_s/2}) e^{nx}$$

For example

$$\Psi_U^{(1)}(Q, g_s) = \frac{Q^{1/2} - Q^{-1/2}}{e^{g_s/2} - e^{-g_s/2}} =$$

$$= \frac{1}{g_s}\left(Q^{1/2} - Q^{-1/2}\right) + g_s \frac{1}{24}\left(Q^{1/2} - Q^{-1/2}\right)$$

$$+ \cdots$$

We approach large N duality with two tools:

1.) Open GW with values in framed skein module

2.) Stretching (from Symplectic Field Theory SFT)

X C-Y , $L \subset X$
Lagrangian.

$\mathcal{M}$ moduli space of
holomorphic curves,
boundary on L.

$u : (\Sigma, \partial\Sigma) \to (X, L)$
holomorphic

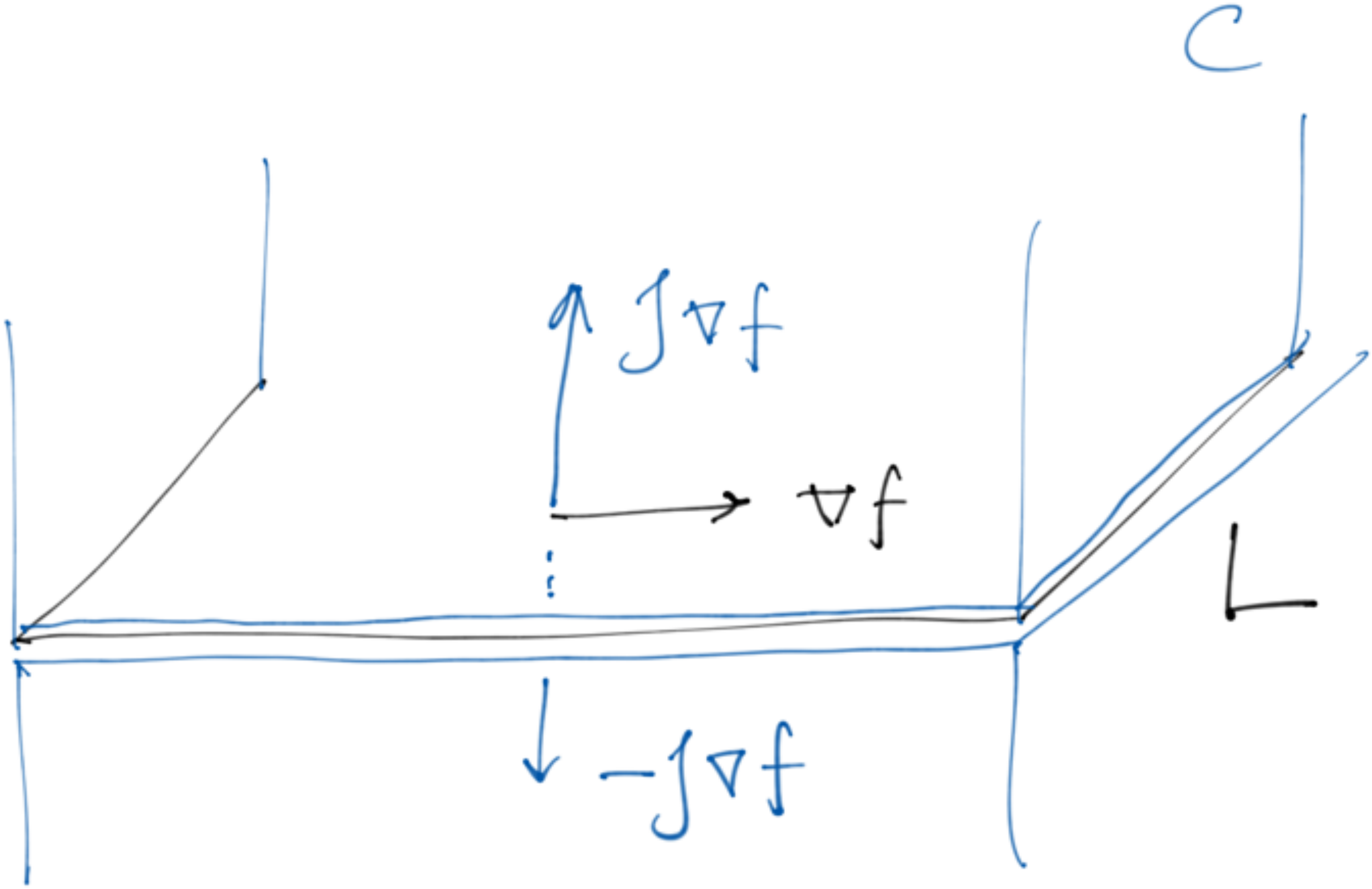
$u(\partial\Sigma)$ link in L

and u deforms C-S on
L.

To define the invariant we fix extra data.

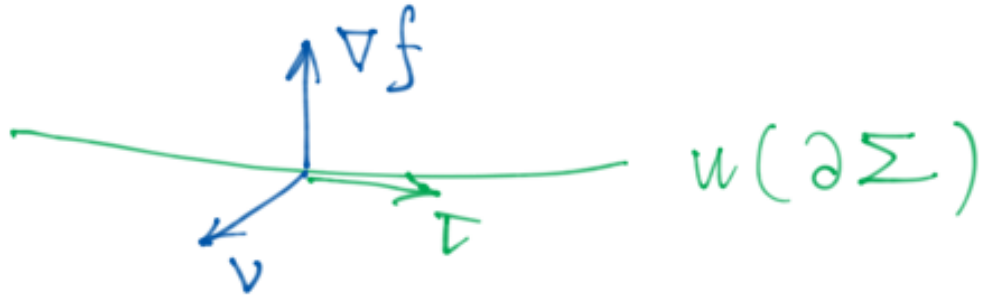
- A Morse function $f : M \longrightarrow \mathbb{R}$, on S^3

- A 4-chain C with $\partial C = 2 \cdot L$ and $C = \int \cdot \nabla f$ near ∂C.

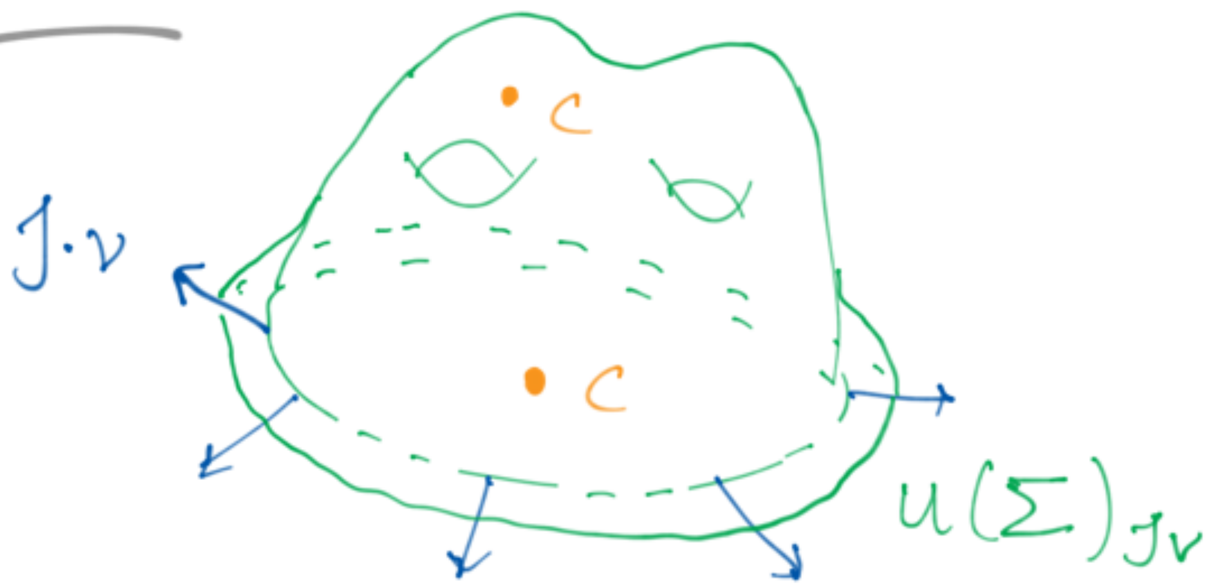


C
$J\nabla f$
∇f
L
$-J\nabla f$

The boundary in the Lagrangian



In the ambient space



$$\mathrm{slk}_L(u) = u(\Sigma)_{J\nu} \cdot C$$

$$:= u \cdot C$$

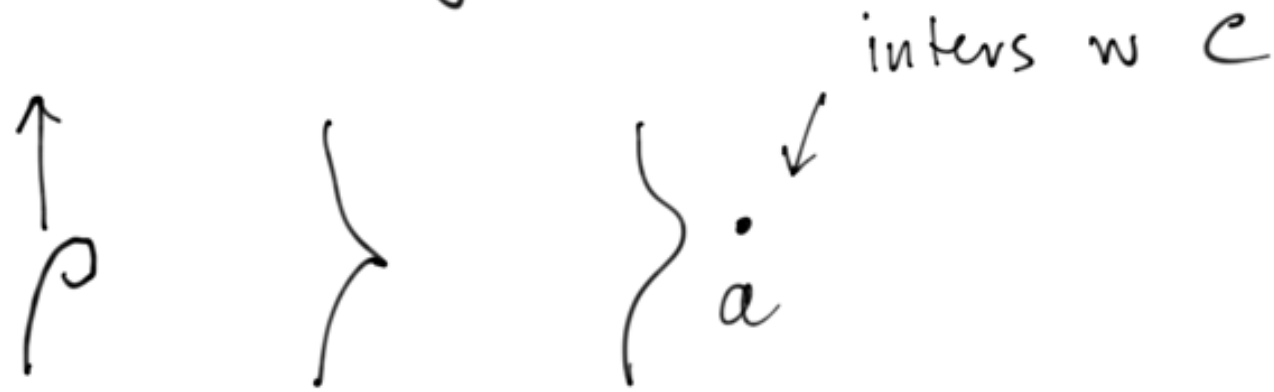
$GW^{sk}(L) =$

$$\sum_{u \in \mathcal{M}} w(u)\, g_s^{-\chi(u)}\, a^{u \cdot c} \langle \partial u \rangle (a, q)$$

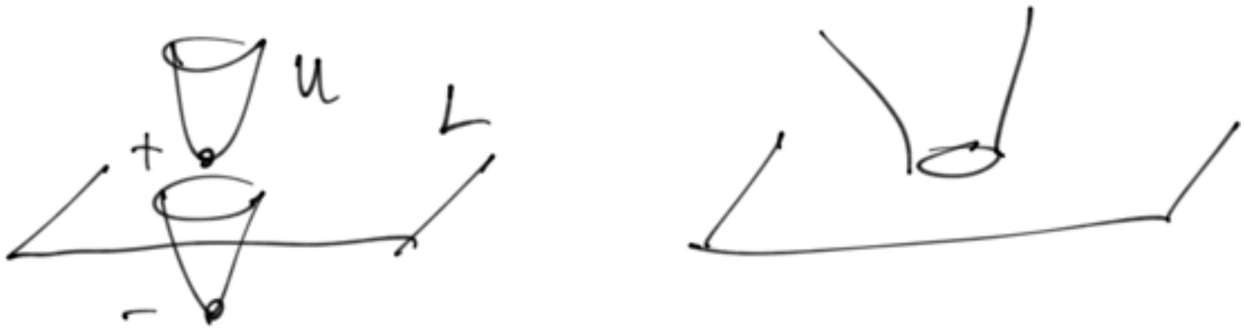
<u>Thm</u> (E - Shende)

If $q = e^{g_s/2}$ then $GW^{sk}(L)$ is invariant under deformation.

Pf Framing

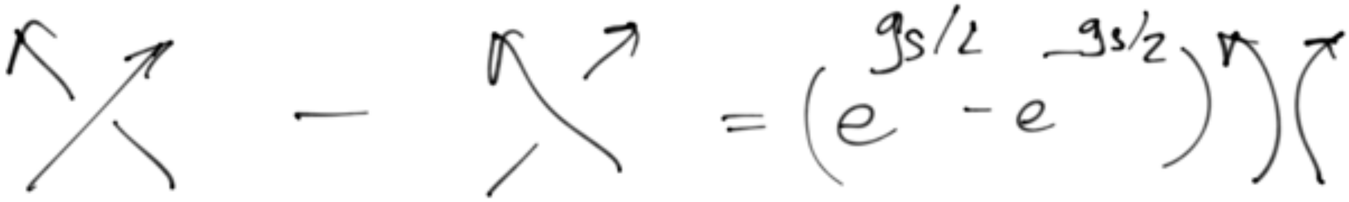


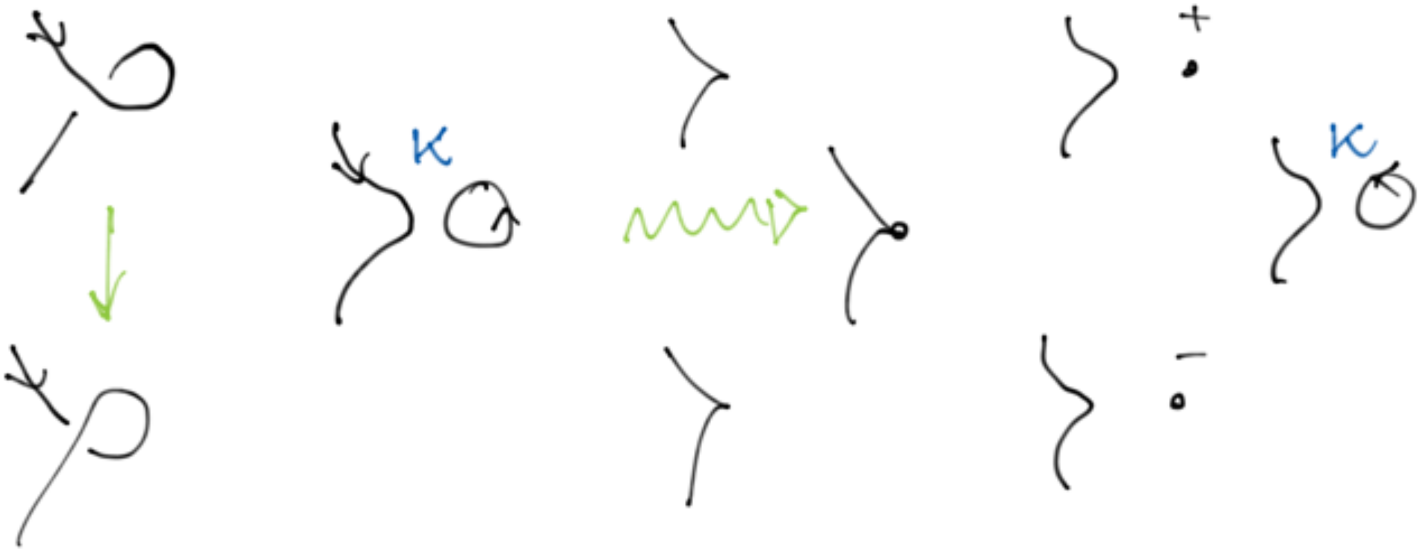
Two splittings



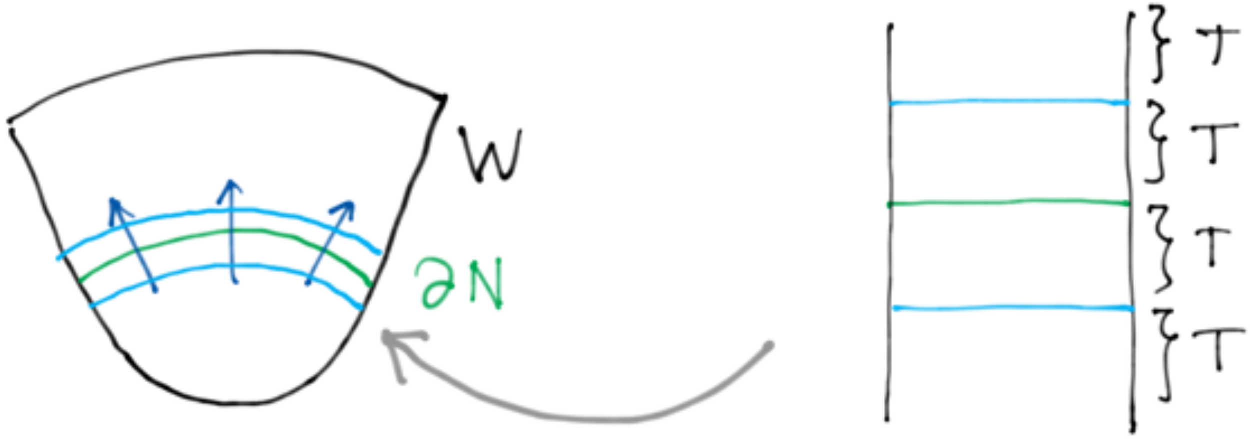
$$a\cdot - a^{-1}\cdot = \left(e^{g_s/2} - e^{-g_s/2}\right) \bigcirc$$

join $\quad -(-1)^k \dfrac{g_s^k}{2^k k!}$



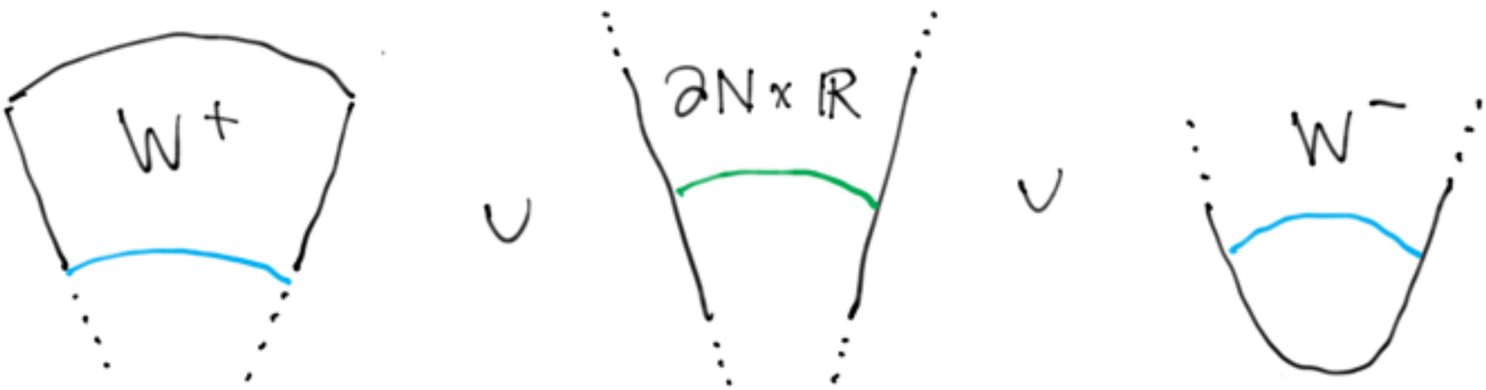


Using this invariance together with SFT-stretch gives large N.



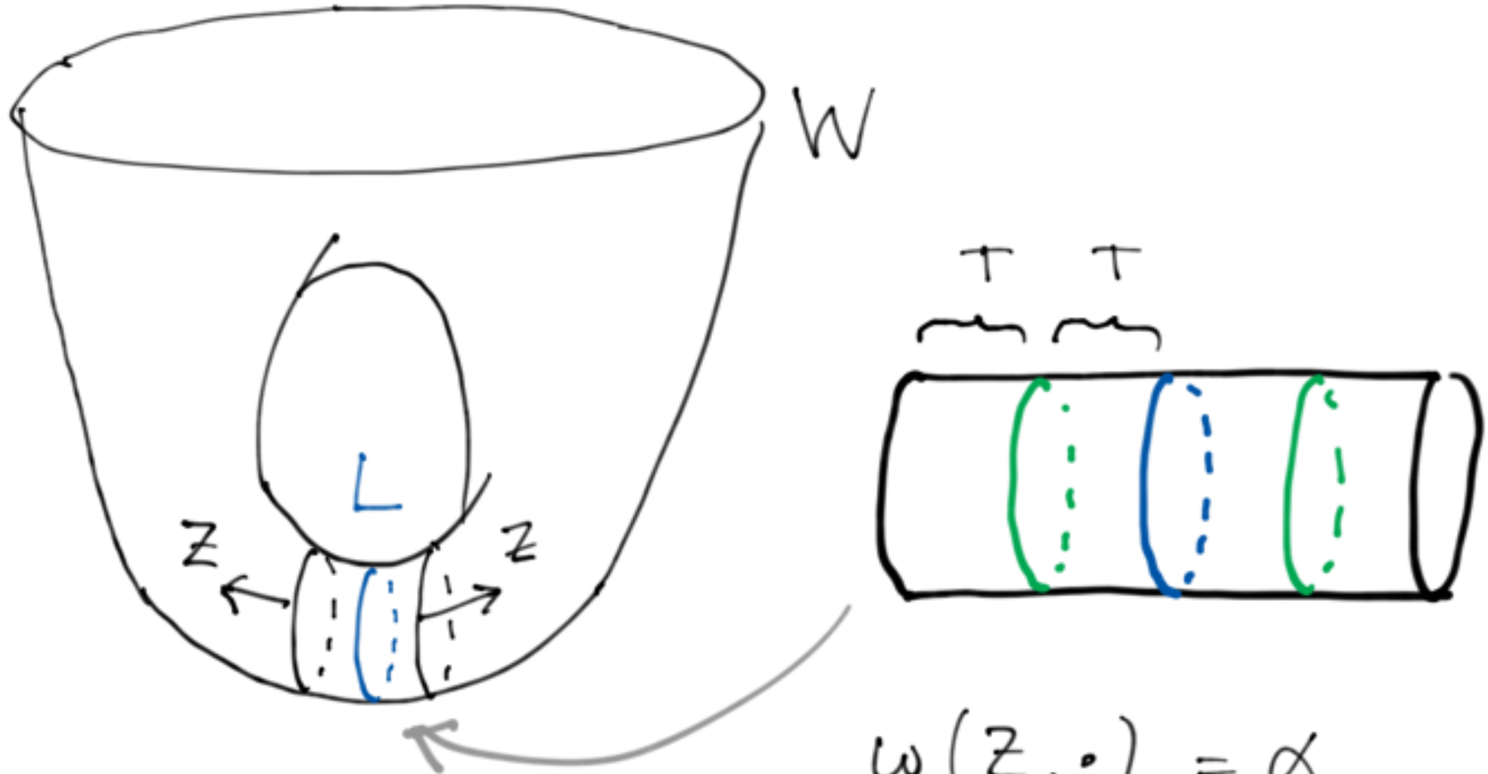
(In our case $W = T^*S^3$ and $\partial N = S_\varepsilon T^*S^3$)

$T \to \infty$, $W \to W^+ \cup \partial N \times \mathbb{R} \cup W^-$



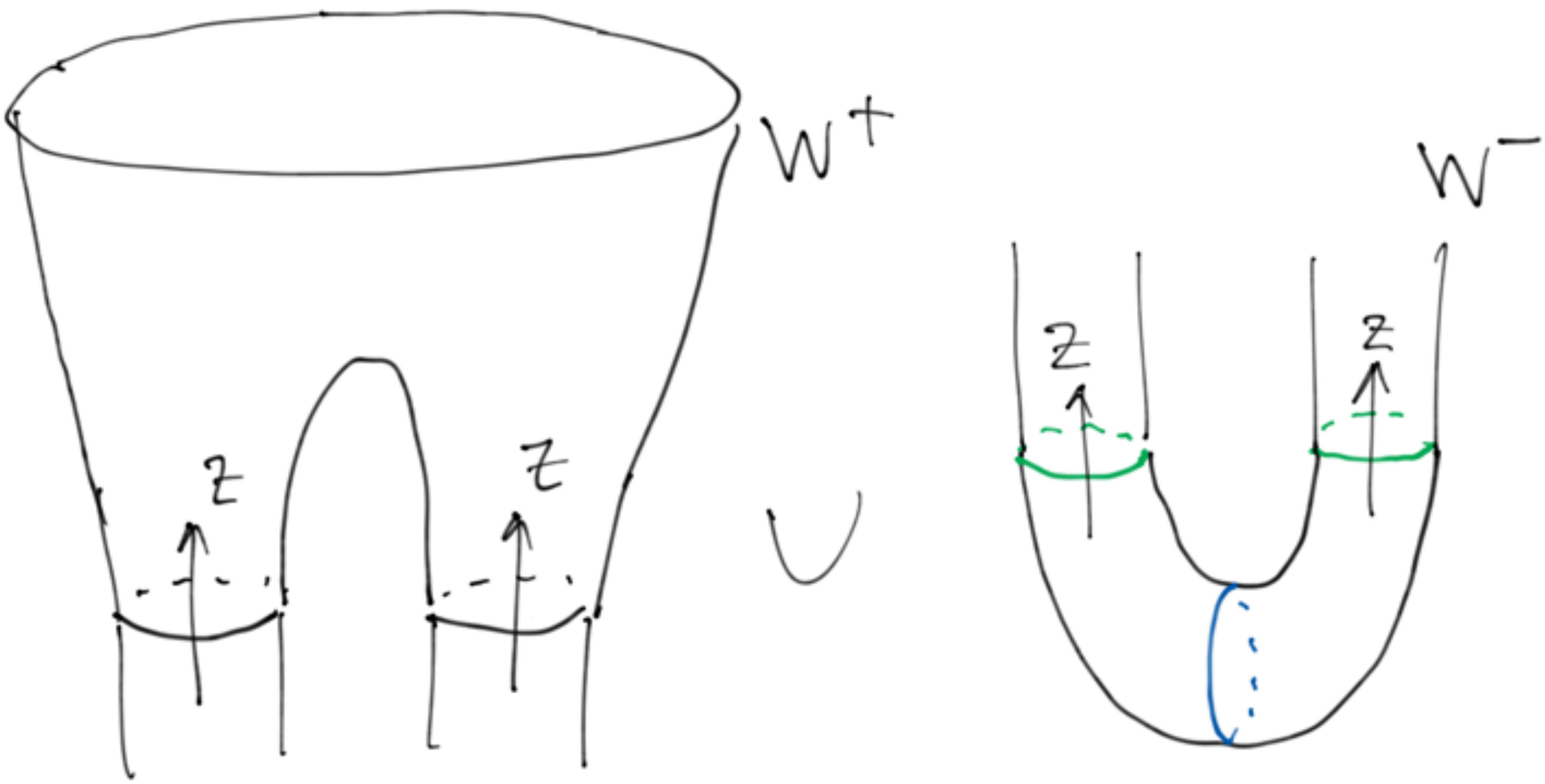
$(W^+ = T^*S^3 \setminus S^3)$

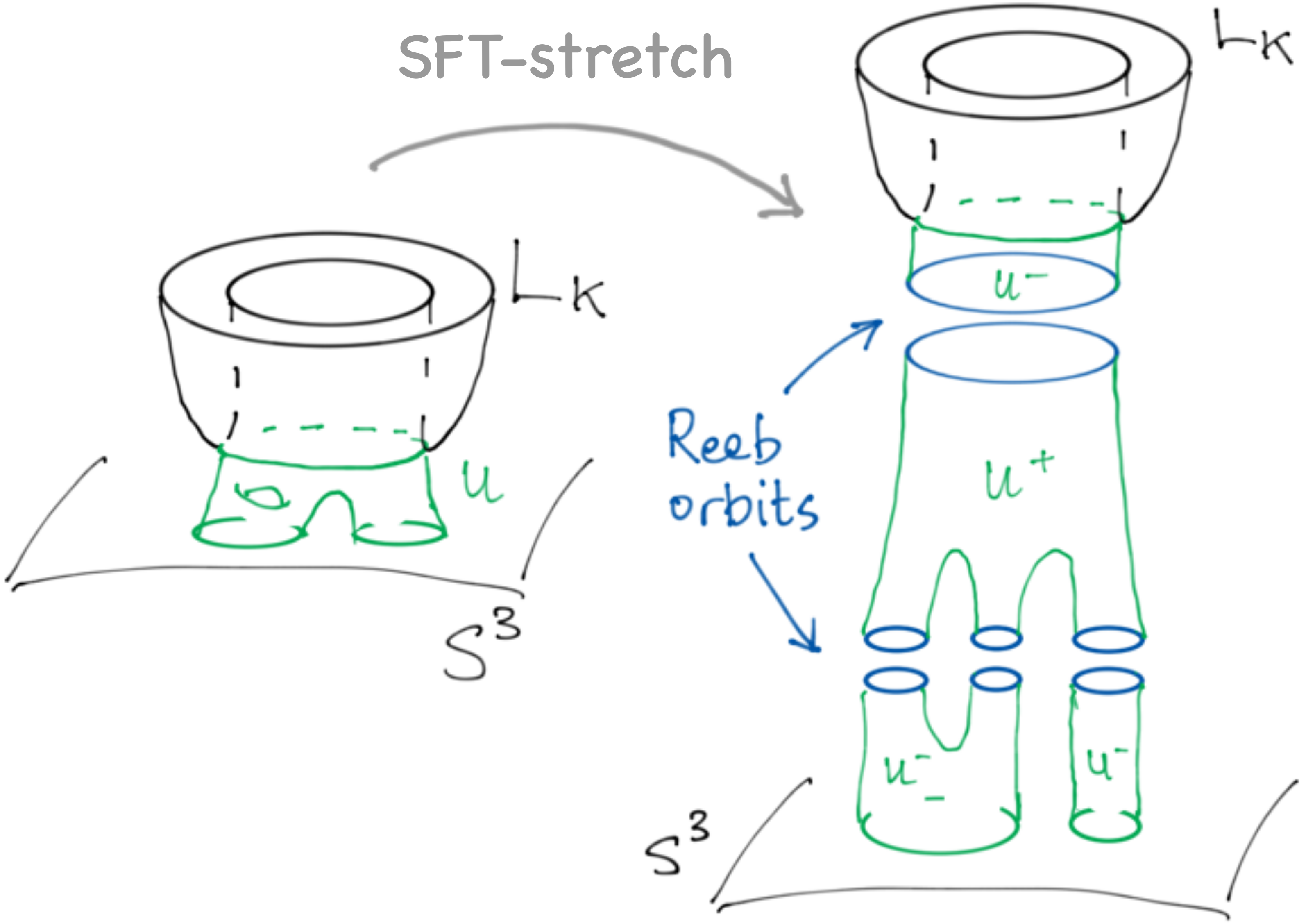
$(W^- = T^*S^3)$



$$\omega(Z,\cdot) = \alpha$$

$$T \to \infty, \quad W = W^+ \cup W^-$$




SFT-stretch
L_K
u
S^3
L_K
u^-
Reeb orbits
u^+
u^-_-
u^-
S^3

Using a Riemannian metric on L the Reeb flow on ST^*L is just the geodesic flow and

$\dim\Big($ 

$\Big) = \sum \operatorname{ind}(\gamma)$

$\dim\Big($ 

$\Big) = \sum - \operatorname{ind}(\beta)$.

Where ind = Morse ind.

- Round metric on S^3
  ind ≥ 2.

- Metric on $S^1 \times \mathbb{R}^2$, one
  closed geod ind $= 0$.

Proof of large N.

Stretch around $S^1 \times \mathbb{R}^2$ standardize
curves there

Reeb
geod

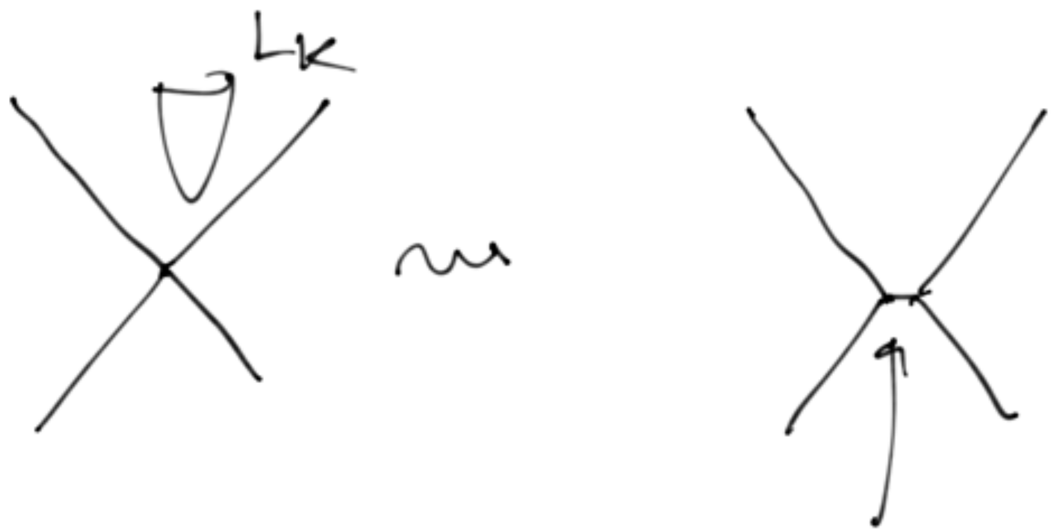
For small shift

$$\Psi_K^1 = \langle K \rangle (a,q)$$

Under stretching all curves must leave W^-

So

$$\Psi^1_K = \left(\sum_{u \in \mathcal{M}} w(u)\, g_s^{\chi(u)}\, Q^{u \cdot c} \Big|_{Q=a^2} \right) e^{X}$$



tiny $\mathbb{CP}^1$

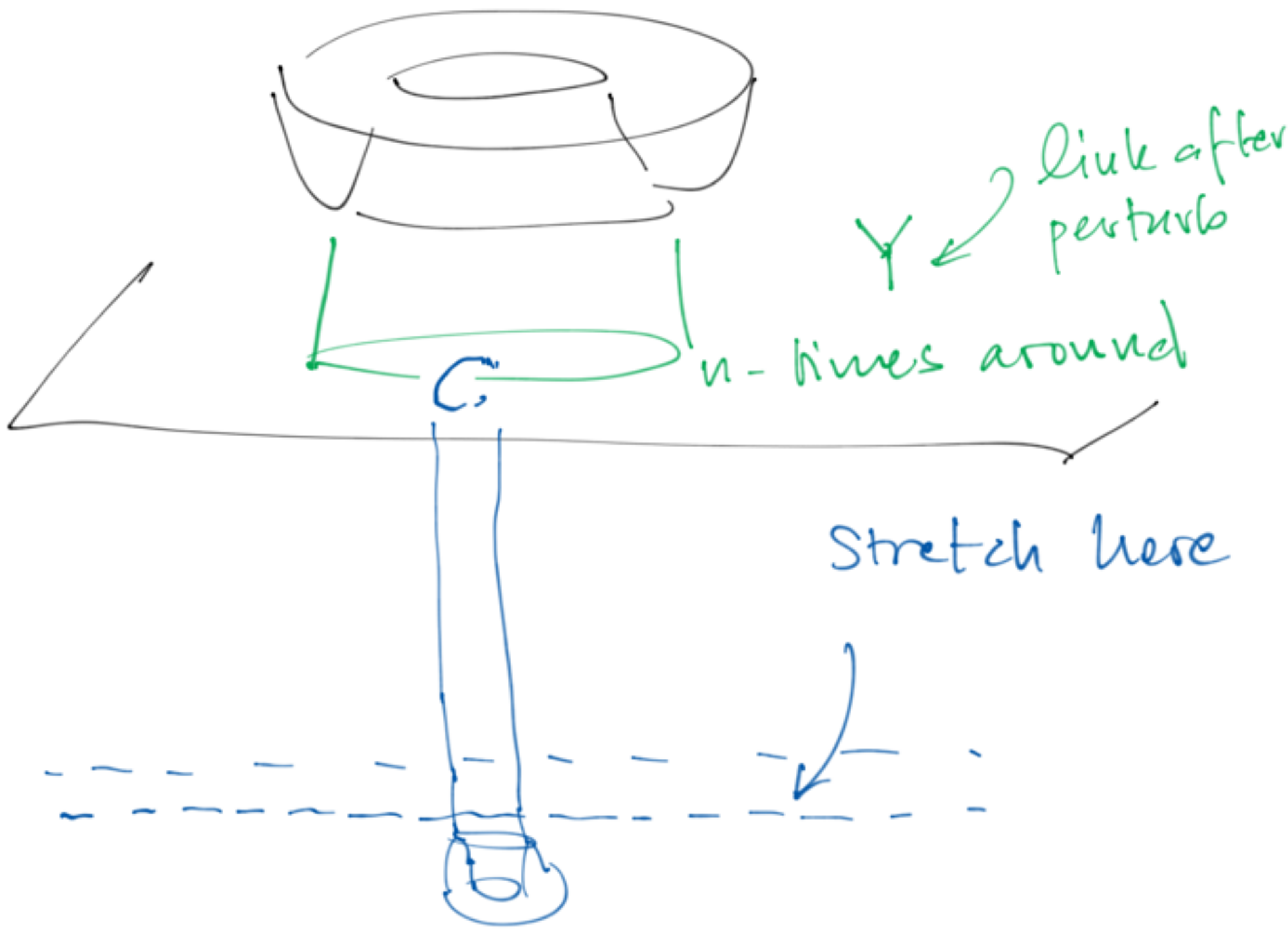
$\Downarrow$

curves same as in $T^*S^3 \setminus S^3$.

Symmetric colors:

$$H_n(K) = H_1(K * Y_n)$$

eigen link of φ operator in $S^1 \times D^2$



The green curves are unchanged

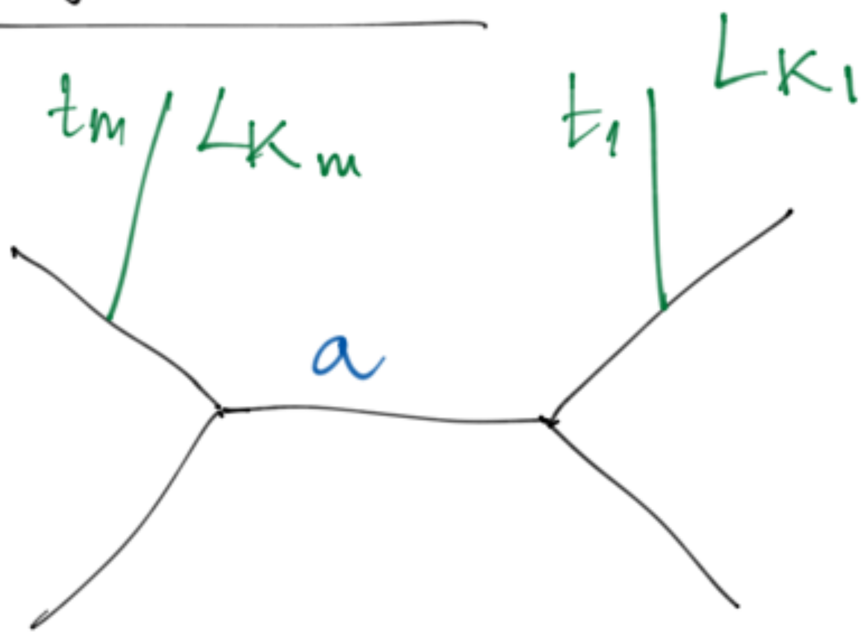
and $\quad (Y) = P(a,q)\, Y .$

Note Y universal

check on unknot

(e.g. with recursion).

Refinement



Stretch around L_{K_j}

$$\Rightarrow \quad GW = \sum_u w(u)\, g_s^{\chi(u)}\, a^{k(u)}\, t_1^{k_1(u)} \cdots t_m^{k_m(u)}$$

Adding boundaries back

t_j is the framing

HOMFLY variable on L_{K_j}.

Can in principle be computed by elimination from infinity

$$\hat{A}_\ell(q, e^{\hat{p}_i}, e^{\hat{x}_i}, t_j, a)\, \Psi_K = 0$$