Introduction to the calculation of Feynman Integrals

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Outline

Outline

Introduction

- Feynman integrals come in different shapes and colors
  - one loop $\leftrightarrow$ many loops
  - many legs $\leftrightarrow$ no legs
  - many scales $\leftrightarrow$ no scales
- for many types of diagrams many results are known especially one-loop is solved
- at three loops and more, massive tadpoles and massless propagators have been studied in great detail
- at two loops, much progress has been made for integrals relevant for $2 \to 2$ scattering processes, but every new process requires a new study of the integrals involved
- $2 \to 3$ at two loops starts to look promising

Introduction

- many different methods have been invented over the years to calculate the needed integrals
- most methods work well for certain classes but fail for others
- the ultimate method/tool is still missing
- will present here only an overview of *personal* selection of methods

Outline

Gamma function

Defining property

$$z\Gamma(z) = \Gamma(z+1) = z!$$

Integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$$

We see immediately from the properties that $\Gamma(-n)$ is singular for $n = 0, 1, 2, \ldots$.
The singularities are simple poles

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)$$

Series expansion of the Gamma function

In many applications one needs more than the pole of the Γ-function. This is best done by using the derivative of $\log\Gamma(z)$ and defines the digamma function $\Psi(z)$

$$\Psi(z) = \frac{\mathrm{d}\log(\Gamma(z))}{\mathrm{d}z} = \frac{\Gamma'(z)}{\Gamma(z)}$$

Therefore,

$$\begin{aligned}\Gamma(z-z_0) &= \Gamma(z_0) + \Gamma(z_0)\Psi(z_0)(z-z_0) \\ &+\frac{1}{2}\left(\Gamma(z_0)\Psi'(z_0) + \Gamma(z_0)\Psi^2(z_0)\right)(z-z_0)^2\end{aligned}$$

for regular points z_0.

The Digamma-function $\Psi(z)$

The digamma function satisfies the relation

$$\Psi(z+1) = \Psi(z) + \frac{1}{z}$$

For positive integer values the digamma function evaluates to

$$\begin{aligned} \Psi(1) &= -\gamma_E\,, \\ \Psi(2) &= 1 - \gamma_E \\ &\cdots \\ \Psi(n+1) &= \sum_{k=1}^{n} \frac{1}{k} - \gamma_E \end{aligned}$$

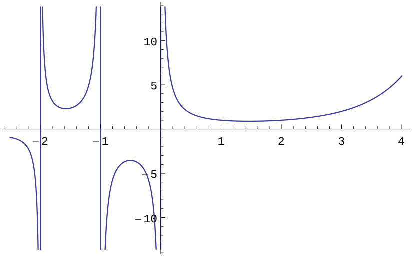
with the Euler-Mascheroni constant $\gamma_E = 0.577216\ldots$

The Digamma-function $\Psi(z)$ cont'd

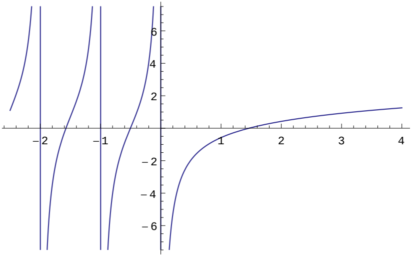
For non-positive integers the digamma functions evaluates again to simple poles

$$\Psi(-n+\epsilon) = -\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)$$

$\Gamma(z)$



$\Psi(z)$



Schwinger Parametrization

From the definition of the Γ function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$$

follows immediately the Schwinger Parametrization

$$\frac{1}{(-k^2 + M^2)^z} = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t \, t^{z-1} e^{-(M^2 - k^2)t}$$

by performing the substitution $t \to t' = (M^2 - k^2)t$

Simple example

Consider simplest diagram possible,
the one-loop tadpole (vacuum diagram)

$$I_1 = \int \mathrm{d}^4 k \frac{1}{-k^2 + M^2}$$

Either introduce an explicit parametrization of the measure or use the Schwinger parametrization for $\alpha = 1$

$$I_1 = \int \mathrm{d}^4 k \int_0^\infty \mathrm{d}t \, e^{-(M^2 - k^2)t}$$

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Either introduce an explicit parametrization of the measure or use the Schwinger parametrization for $\alpha = 1$

$$I_1 = \int_0^\infty \mathrm{d}t \, e^{-M^2 t} \int \mathrm{d}^4 k \, e^{k^2 t}$$

Simple example

perform Wick rotation

$$k_0 \to ik_0$$

with the result that

$$k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2 \to -k_0^2 - k_1^2 - k_2^2 - k_3^2$$

and we get

$$I_1 = i \int_0^\infty \mathrm{d}t\, e^{-M^2 t} \int \mathrm{d}^4 k\, e^{-k^2 t}$$

Simple example

$$I_1 = i \int_0^\infty \mathrm{d}t \, e^{-M^2 t} \int \mathrm{d}^4 k \, e^{-k^2 t}$$

doing the Gaussian integral we get

$$I_1 = i\pi^2 \int_0^\infty \mathrm{d}t \, \frac{e^{-M^2 t}}{t^2}$$

This integral does not converge for $t \to 0$.
$\Rightarrow$ first need a way to give meaning to these kind of integrals.

Dimensional regularization

- Most Feynman integrals are not convergent in four space time dimensions.
- Common way out is the use of dimensional regularization, where the four-dimensional space time is extended to d dimensions.
- Divergences of the integrals then become manifest as poles in $d-4$.
- d dimensional integrals behave identical to their four-dimensional counterparts

d-dimensional integration

d-dimensional integrals have to fulfil these axioms

Linearity

$$\int \mathrm{d}^d k \, (af(k) + bg(k)) = a \int \mathrm{d}^d k \, f(k) + b \int \mathrm{d}^d k \, g(k)$$

Scaling

$$\int \mathrm{d}^d k \, f(s\,k) = s^{-d} \int \mathrm{d}^d k \, f(k)$$

Translational invariance

$$\int \mathrm{d}^d k \, f(k+p) = \int \mathrm{d}^d k \, f(k)$$

d-dimensional integration – properties

Pro: dimensional regularization regularizes both UV and IR singularities

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Con: dimensional regularization regularizes both UV and IR singularities in the same way

- scaleless integrals vanish

$$\int \mathrm{d}^d k (k^2)^\alpha = 0$$

- integration by parts

$$\int \mathrm{d}^d k \frac{\partial}{\partial k^\mu} f(k) = 0$$

- Interchange of integrations

$$\int \mathrm{d}^d p \int \mathrm{d}^d k\, f(p,k) = \int \mathrm{d}^d k \int \mathrm{d}^d p\, f(p,k)$$

Gaussian integrals in d dimensions

For many purposes the problem of d-dimensional integrations can be reduced to one specific integral:
the Gaussian integral in d dimensions

$$\int \mathrm{d}^d k \, e^{-k^2} = \pi^{\frac{d}{2}}$$

which is the most natural generalization of the integer dimension one.

Gaussian integrals in d dimensions

For many purposes the problem of d-dimensional integrations can be reduced to one specific integral:
the Gaussian integral in d dimensions

$$\int \mathrm{d}^d k \, e^{-Ak^2} = \left(\frac{\pi}{A}\right)^{\frac{d}{2}}$$

which is the most natural generalization of the integer dimension one.
The dependence on A follows by rescaling $k \to \frac{k}{\sqrt{A}}$

Simple example – Improved

$$I_1 = \int \mathrm{d}^d k \frac{1}{-k^2 + M^2}$$

$$I_1 = i \int_0^\infty \mathrm{d}t \, e^{-M^2 t} \int \mathrm{d}^d k e^{-k^2 t}$$

$$I_1 = i\pi^{d/2} \int_0^\infty \mathrm{d}t \, \frac{e^{-M^2 t}}{t^{d/2}}$$

Simple example – Improved

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$$I_1 = i \int \mathrm{d}t \, e^{-M^2 t} \int \mathrm{d}^d k \, e^{-k^2 t}$$

$$I_1 = i\pi^{d/2} \int_0^\infty \mathrm{d}t \, t^{-d/2} e^{-M^2 t}$$

$$\begin{aligned} I_1 &= i\pi^{d/2} (M^2)^{d/2-1} \int_0^\infty \mathrm{d}t \, t^{-d/2} e^{-t} \\ &= i\pi^{d/2} (M^2)^{d/2-1} \Gamma(-d/2+1) \end{aligned}$$

Simple example – Improved

$$
\begin{aligned}
I_1 &= \int \mathrm{d}^d k \frac{1}{-k^2 + M^2} \\
&= i\pi^{d/2} (M^2)^{d/2-1} \Gamma(-d/2+1)
\end{aligned}
$$

$(M^2)^{d/2-1}$ overall mass dimension of the integral, could be read off from the original integral

$\Gamma(-d/2+1)$ contains the real information
singular for $d \to 4$

$$
\begin{aligned}
\Gamma(-d/2+1) &= -\frac{1}{\epsilon} + (\gamma_E - 1) \\
&\quad + \frac{1}{12}\left(-6\gamma_E^2 + 12\gamma_E - \pi^2 - 12\right)\epsilon
\end{aligned}
$$

Simple example – Improved

$$\begin{aligned}\Gamma(-d/2+1) &= -\frac{1}{\epsilon}+(\gamma_E-1)\\ &\quad +\frac{1}{12}\left(-6\gamma_E^2+12\gamma_E-\pi^2-12\right)\epsilon\end{aligned}$$

not a very compact result, better to choose a suitable normalization

$$\Gamma(-d/2+1)/\Gamma(1+\epsilon) = -\frac{1}{\epsilon}-1-\epsilon$$

or

$$\Gamma(-d/2+1)/\exp(-\gamma_E\epsilon) = -\frac{1}{\epsilon}-1+\left(-1-\frac{\pi^2}{12}\right)\epsilon$$

Simple example – Extended

$$I_1 = \int \mathrm{d}^d k \frac{1}{(-k^2 + M^2)^\alpha}$$

$$I_1 = i\frac{1}{\Gamma(\alpha)} \int \mathrm{d}t\, t^{\alpha-1} e^{-M^2 t} \int \mathrm{d}^d k e^{-k^2 t}$$

$$I_1 = i\frac{\pi^{d/2}}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}t\, t^{\alpha-d/2-1} e^{-M^2 t}$$

$$\begin{aligned} I_1 &= \frac{i\pi^{d/2}}{\Gamma(\alpha)}(M^2)^{d/2-\alpha} \int_0^\infty \mathrm{d}t\, t^{\alpha-d/2-1} e^{-t} \\ &= \frac{i\pi^{d/2}}{\Gamma(\alpha)}(M^2)^{d/2-\alpha}\Gamma(\alpha - d/2) \end{aligned}$$

Also the integral with $\alpha = 2$ is divergent

not that simple example: One-loop propagator

Consider the one-loop propagator

$$P_1 = \int \mathrm{d}^4 k \frac{1}{-k^2(-(k+q)^2)}$$

introduce Feynman parameters

$$\frac{1}{D_1^{k_1} \cdots D_n^{k_n}} = \frac{\Gamma(\sum_i k_i)}{\prod_i \Gamma(k_i)} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_n \frac{\delta(\sum_i x_i - 1) \prod_i x_i^{k_i - 1}}{(\sum_i x_i D_i)^{\sum_i k_i}}$$

in their simplest form

$$\frac{1}{D_1 D_2} = \int_0^1 \mathrm{d}x \frac{1}{[xD_1 + (1-x)D_2]^2}$$

not that simple example: One-loop propagator

$$
\begin{aligned}
P_1 &= \int \mathrm{d}^4k \frac{1}{-k^2(-(k+q)^2)} \\
&= \int \mathrm{d}^4k \int_0^1 \mathrm{d}x \frac{1}{[-k^2 - 2xk\cdot q - xq^2]^2}
\end{aligned}
$$

complete the square

$$
P_1 = \int \mathrm{d}^4k \int_0^1 \mathrm{d}x \frac{1}{[-(k+xq)^2 + x^2q^2 - xq^2]^2}
$$

shift $k = k + (1-x)q$

$$
P_1 = \int \mathrm{d}^4k \int_0^1 \mathrm{d}x \frac{1}{[-k^2 + x^2q^2 - xq^2]^2}
$$

not that simple example: One-loop propagator

$$P_1 = \int \mathrm{d}^4k \int_0^1 \mathrm{d}x \frac{1}{[-k^2 + x(x-1)q^2]^2}$$

doing the momentum integration gives

$$P_1 = i\pi^{d/2}\Gamma(2-d/2) \int_0^1 \mathrm{d}x \left[x(x-1)q^2\right]^{d/2-2}$$

let's assume $q^2 < 0$

$$P_1 = i\pi^{d/2}\Gamma(2-d/2)(-q^2)^{d/2-2} \int_0^1 \mathrm{d}x \, [x(1-x)]^{d/2-2}$$

not that simple example: One-loop propagator

$$P_1 = i\pi^{d/2}\Gamma(2-d/2)(-q^2)^{d/2-2}\int_0^1 \mathrm{d}x\,[x(1-x)]^{d/2-2}$$

what is left is special case of the Beta-function

$$B(a,b) = \int \mathrm{d}t\, t^{a-1}(1-t)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$P_1 = i\pi^{d/2}\Gamma(2-d/2)(-q^2)^{d/2-2}\frac{\Gamma^2(d/2-1)}{\Gamma(d+2)}$$

Cuts

- branch cuts appear in an Feynman integral when the particles in the loop can be produced as real particles
- in the propagator example before the factor

$$(-q^2)^{d/2-2}$$

  could have again be predicted from mass dimension of the the analytic properties of the diagram
- the imaginary parts then corresponds to the total cross section belonging to the respective cut
- much information can be gained from studying cuts of Feynman integrals
- the full results can be obtained from the imaginary part by dispersion integrals

Tensor integrals

So far we have only dealt with scalar integrals, i.e. integrals with no vectors with free indices in the numerator

$$I^{\mu_1 \cdots \mu_n} = \int d^d k \frac{k^{\mu_1} \cdots k^{\mu_n}}{D_1 \cdots D_N}$$

At one-loop it has long been worked out, how to reduce tensor integrals to scalar integrals $\Rightarrow$ Passarino Veltman reduction
Huge progress in automating one-loop calculations
keywords: unitary, integrand reduction, OPP

Projectors

- Most of the time it is more convenient not to have open Lorentz indices from the very start
- To avoid this the use of projectors is very convenient
- As an example let us consider corrections to the photon propagator $\Pi^{\mu\nu}(q)$
- from Lorentz covariance we know it can be written in the form

$$\Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi_T(q^2) + q^\mu q^\nu \Pi_L$$

- to calculate Π_T and Π_L we can use the projectors

$$P_T = (q^2 g^{\mu\nu} - q^\mu q^\nu)/(q.q)^2/(d-1),\; P_L = q^\mu q^\nu/(q.q)^2$$

Polylogarithms

Function classes appearing in Feynman integral calculations are the Polylogarithms

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

which for $z \to 1$ give the ζ values

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$$\mathrm{Li}_n(1) = \zeta_n$$

$$\zeta_2 = \frac{\pi^2}{6}, \zeta_3 = 1.2020\ldots, \zeta_4 = \frac{\pi^4}{90}, \cdots, \zeta_{2n} \propto \pi^{2n}$$

Polylogarithms

They appear in the calculations of integrals over logarithms

$$\begin{aligned}
\int \mathrm{d}x \log x/(1-x) &= \mathrm{Li}_2(1-x) \\
\int \mathrm{d}x\, \mathrm{Li}_2(x)/x &= \mathrm{Li}_3(1-x) \\
\int \mathrm{d}x\, \mathrm{Li}_2(1-x)/x &= -2\mathrm{Li}_3(x) + \mathrm{Li}_2(1-x)\log(x) \\
&\quad +2\mathrm{Li}_2(x)\log(x) + \log(1-x)\log^2(x)
\end{aligned}$$

The Polylogarithms are not very systematic and many relations between them exist.

Harmonic Polylogarithms

A more systematic approach are the harmonic polylogarithms (HPLs) defined as iterated integrals over the alphabet

$$f_{-1} = \frac{1}{1+x},\ f_0 = \frac{1}{x},\ f_1 = \frac{1}{1-x}$$

$$\int_0^x \mathrm{d}x' \mathrm{HPL}(\vec{n}; x') f_a(x') = \mathrm{HPL}(a, \vec{n}; x)$$

$$\begin{aligned}\mathrm{HPL}(1,x) &= -\log(1-x),\ \mathrm{HPL}(-1,x) = \log(1+x)\\ \mathrm{HPL}(0,x) &= \log(x),\ \mathrm{HPL}(0.\ldots,0,x) = \frac{1}{n!}\log^n(x)\end{aligned}$$

Harmonic Polylogarithms analytic properties

- all HPLs are well defined for

$$x \in [0,1] \cup \mathbb{C} \backslash \mathbb{R}$$

- for the remainder of the real axis one has to be careful to specify the correct imaginary part ($\delta = \pm 1, \epsilon \ll 1$)
  - $x = -t + \delta\epsilon i$, $0 < t < 1$

$$H(-1;x) = -H(1;t), \quad H(0;x) = H(0;t) + i\delta\pi, \quad H(1;x) = -H(-1;t)$$

  - $x = 1/t + \delta\epsilon i$, $x > 1$, e.g.

$$H(1;x) = H(1;t) + H(0;t) + i\delta\pi$$

Harmonic Polylogarithms properties

- HPLs obey the shuffle relation

$$H(\vec{a};x)H(\vec{b};x) = \sum_{\vec{c}\in\text{shuffles of }\vec{a},\vec{b}} H(\vec{c};x)$$

e.g.

$$\begin{aligned} H(a_1,a_2;x)H(b_1,b_2;x) = H(a_1,a_2,b_1,b_2;x) + H(a_1,b_1,a_2,b_2;x) \\ +H(a_1,b_1,b_2,a_2;x) + H(b_1,a_1,a_2,b_2;x) \\ +H(b_1,a_1,b_2,a_2;x) + H(b_1,b_2,a_1,a_2;x) \end{aligned}$$

which can be proved by taking the derivative and using that they result is correct for $x = 0$

Harmonic Polylogarithms properties cont'd

$$\frac{\partial}{\partial x}\left(H(a_1, a_2; x)H(b_1, b_2; x)\right) =$$
$$\frac{\partial}{\partial x}\left(H(a_1, a_2, b_1, b_2; x) + H(a_1, b_1, a_2, b_2; x) + H(a_1, b_1, b_2, a_2; x)\right) +$$
$$\frac{\partial}{\partial x}\left(H(b_1, a_1, a_2, b_2; x) + H(b_1, a_1, b_2, a_2; x) + H(b_1, b_2, a_1, a_2; x)\right)$$

Harmonic Polylogarithms properties cont'd

$$\frac{\partial}{\partial x}\left(H(a_1,a_2;x)H(b_1,b_2;x)\right) =$$
$$\frac{\partial}{\partial x}\left(H(a_1,a_2,b_1,b_2;x)+H(a_1,b_1,a_2,b_2;x)+H(a_1,b_1,b_2,a_2;x)\right)+$$
$$\frac{\partial}{\partial x}\left(H(b_1,a_1,a_2,b_2;x)+H(b_1,a_1,b_2,a_2;x)+H(b_1,b_2,a_1,a_2;x)\right)$$

$$f_{a_1}(x)H(a_2;x)H(b_1,b_2;x)+f_{b_1}(x)H(a_1,a_2;x)H(b_2;x) =$$
$$f_{a_1}(x)(H(a_2,b_1,b_2;x)+H(b_1,a_2,b_2;x)+H(b_1,b_2,a_2;x))+$$
$$f_{b_1}(x)(H(a_1,a_2,b_2;x)+H(a_1,b_2,a_2;x)+H(b_2,a_1,a_2;x))$$

Harmonic Polylogarithms properties

We can use the shuffle relations to construct a *basis* for the HPLs. Conventionally one replaces all HPLs with leading ”1”s (or ”-1”s) or trailing ”0”s by products of simpler HPLs. E.g.

$$
\begin{aligned}
H(-1,0;x) &= H(-1;x)H(0;x) - H(0,-1;x) \\
H(1,0,1;x) &= H(1;x)H(0,1;x) - 2H(0,1,1;x)
\end{aligned}
$$

Besides these HPLs several other HPLs can be chosen to be eliminated.

Hypergeometric functions

- Since HPLs, and their extensions, are iterated integrals they naturally appear when dealing first-order factorizing differential equations.
- In more complicated cases, or when a closed solution in d is needed, higher functions come into play.

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum \frac{\prod_i (a_i)_n}{\prod_j (b_j)_n} \frac{z^n}{n!} $$

with the Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$

- The hypergeometric function ${}_pF_q$ fulfils a second order differential equation.

Hypergeometric functions

- The ${}_2F_1$ is given by the integral representation

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int \mathrm{d}x \, x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} $$

- expansions in z around z_0 are rather simple, cp. series representation, esp. $z_0 = 0$
- expansions in a, b, c can be obtained by expanding the Pochhammer symbols in the series representation and resumming the resulting expression.
  `HypExp, HypExp 2` [Huber,Maitre]

Hypergeometric functions

- Certain arguments result in simpler (?) functions, e.g.
  - ${}_2F_1(1,1;2;x) = -\frac{\log(1-x)}{x}$
  - ${}_2F_1(1/2,1/2;1;x) = \frac{2}{\pi}K(x)$
- Here we have the elliptic integral of the first kind

$$K(x) = \int_0^1 \mathrm{d}t \, \frac{1}{\sqrt{(1-t^2)(1-xt^2)}}$$

- If there is a first kind, there must be a second kind, too

$$E(x) = \int_0^1 \mathrm{d}t \, \sqrt{\frac{(1-xt^2)}{(1-t^2)}}$$

Outline

Generalized formula and why we need more tools

Consider a general L-loop integral with N internal lines with momenta q_i and masses m_i and E external lines with momenta p_e

$$G_L = \frac{1}{(i\pi^{d/2})^L} \int \frac{\mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L}{(q_1^2 - m_1^2)^{\nu_i} \cdots (q_N^2 - m_N^2)^{\nu_N}}$$

The denominators are of the form

$$d_i = \left(\sum_\ell \alpha_{i\ell} k_\ell - P_i \right)^2 - m_i^2 = \left(\sum_\ell \alpha_{i\ell} k_\ell - \sum_e \beta_{ie} p_e \right)^2 - m_i^2$$

the numerator in the Feynman parameter representation before the momentum integration can thus be written in the form

$$\mathcal{N} = \sum_i x_i d_i = kMk - 2kQ + J$$

Generalized formula and why we need more tools

$$\mathcal{N} = \sum_i x_i d_i = kMk - 2kQ + J$$

$$\begin{aligned} M_{\ell\ell'} &= \sum_i x_i \alpha_{i\ell} \alpha_{i\ell'} \\ Q_\ell &= \sum_i x_i \alpha_{i\ell} P_i \\ J &= \sum_i x_i (P_i^2 - m_i^2) \end{aligned}$$

before we can do the momentum integration we need to diagonalize and rescale in k space. This gives rise to the determinant of M.

$$U(x) = \det M$$

Generalized formula and why we need more tools

Shifting the loop momenta to complete the square gives rise to the polynomial

$$F(x) = -(\det M)J + Q\tilde{M}Q$$

with

$$\tilde{M} = (\det M)M^{-1}$$

and we thus obtain the final Feynman parameter representation

$$G_L = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{d}{2}L)}{\prod_i \Gamma(\nu_i)} \int \prod_i \mathrm{d}x_i x_i^{\nu_i - 1} \delta(1 - \sum x_i) \times \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

$U(x)$ is homogeneous function of degree L
$F(x)$ is homogeneous function of degree $L + 1$

Generalized formula and why we need more tools

to proceed with the calculation in the next step the Feynman parameter integral

$$\int \prod_i \mathrm{d}x_i x_i^{\nu_i - 1} \delta(1 - \sum x_i) \times \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

has to be performed.
Problem: The polynomials in the Feynman parameters x_i (one variable per line of the integral) become too complicated to be integrated very fast.

Where do we go from here?

Problem: we have a completely general representation for Feynman integral properly regularized in *d* dimensions. It can not (easily) evaluated any further.

Possible solutions?

- expand in $\epsilon = (4 - d)/2$

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  most of the time not possible, since Feynman integrals still do not converge for $\epsilon \to 0$

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Possible solutions?

- expand in $\epsilon = (4 - d)/2$
  most of the time not possible, since Feynman integrals still do not converge for $\epsilon \to 0$
- integrate numerically

Where do we go from here?

Problem: we have a completely general representation for Feynman integral properly regularized in d dimensions. It can not (easily) evaluated any further.

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- expand in $\epsilon = (4 - d)/2$
  most of the time not possible, since Feynman integrals still do not converge for $\epsilon \to 0$
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  cannot do that in *d* dimensions
- make the objects simpler again
  can be done, but for a prize ...

Mellin-Barnes representation

One way relation to achieve this goal is the Mellin-Barnes representation [Smirnov; Tausk]

$$\frac{1}{(A+B)^{\lambda}} = \frac{B^{-\lambda}}{2\pi i\Gamma(\lambda)} \int_{-i\infty}^{i\infty} \mathrm{d}z A^{z} B^{-z} \Gamma(-z)\Gamma(\lambda+z)$$

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Note: If this relation is applied finely enough the Feynman integration is guaranteed to become doable

Note: This might not be the best idea ...

Mellin-Barnes representation

Very important is how the integration contour has to be chosen.

$$\frac{1}{(A+B)^{\lambda}} = \frac{B^{-\lambda}}{2\pi i \Gamma(\lambda)} \int_{-i\infty}^{i\infty} \mathrm{d}z A^{z} B^{-z} \Gamma(-z) \Gamma(\lambda + z)$$

There are left poles coming from

$$\Gamma(\lambda + z)$$

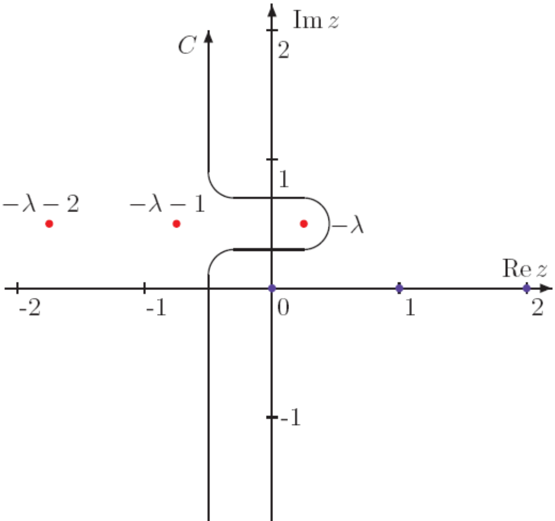
and right poles coming from

$$\Gamma(-z)$$

The integration contour has to separate these poles

Mellin-Barnes representation

$$\Gamma(-z)\Gamma(\lambda + z)$$



Mellin-Barnes representations: Singularities

In this approach singularities (in ϵ) arise, when for $\epsilon \to 0$ left and right poles coincide such that no valid integration contour can be chosen.

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In this approach singularities (in ϵ) arise, when for $\epsilon \to 0$ left and right poles coincide such that no valid integration contour can be chosen.

This problem can be solved by explicitly taking residues when necessary.

in the example, instead of the complicated contour shown take a straight line but explicitly include the residue of the first left pole

Mellin-Barnes representations: How to continue?

Once the optimal MB-representation has been found one can continue with one or more of the following

- further analytical manipulations: e.g. Barnes Lemmas

Mellin-Barnes representations: How to continue?

Once the optimal MB-representation has been found one can continue with one or more of the following

- further analytical manipulations: e.g. Barnes Lemmas
- preparation of a series representation and application of summation techniques
- preparation for numerical integration, e.g. MB.m

Barnes Lemmas

First Barnes Lemma

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z)\mathrm{d}z$$
$$=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

Barnes Lemmas

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Second Barnes Lemma

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(1-d-z)\Gamma(-z)}{\Gamma(e+z)}\mathrm{d}z$$
$$=\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1-d+a)\Gamma(1-d+b)\Gamma(1-d+c)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}$$

with $e=a+b+c-d+1$

Towards a series representation

If the integrand $f(z)$ of the MB integral

$$\int_{-i\infty}^{i\infty} \mathrm{d}z f(z)$$

is vanishing fast enough for $|z| \to \infty$ we can close the integration contour either to the right or to the left and use the residue theorem and write

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \mathrm{d}z\, f(z) = \sum Res(f(z))$$

AMBRE - Automatic Mellin-Barnes Representation

AMBRE written by I. Dubovyk, J. Gluza, K. Kajda, T. Riemann, which can be obtained from the webpage
http://prac.us.edu.pl/˜gluza/ambre/
is a tool for the automatic construction of a good (best) MB representation.

MB.m

MB.m Mathematica package by M. Czakon

- can be used to extract poles from Mellin Barnes representations
- prepare code for numerical integration
- and run it

Caveats:

- as provided on webpage written to use f77
- uses the old Cuba API
- needs parts of the CERNLIB

Outline

Integration by parts

- state of the art calculations require the calculation of $\mathcal{O}(10^3)$ - $\mathcal{O}(10^7)$ Feynman integrals
- especially multi-loop or calculations involving expansions require the calculation of many integrals
- individual calculation of all these integrals is not feasible
- the number of integrals can be greatly reduced by applying the so-called integration-by-parts identities

Integration by parts

Integration-by-parts identities are based on the property

$$0 = \int \mathrm{d}^d k \, \frac{\partial}{\partial k_i^\mu} \frac{1}{D_1^{k_1} \ldots D_n^{k_n}}$$

which being the integral of a total derivative evaluates to a surface term and can be shown to vanish.

Integration by parts

To make them more manageable contract with either an external or a loop momentum

$$0 = \int \mathrm{d}^d k \, \frac{\partial}{\partial k_i^\mu} \frac{\{k^\mu, q_j^\mu\}}{D_1^{k_1} \cdots D_n^{k_n}}$$

which then yields

$$\#\text{loops} \times (\#\text{loops} + \#(\text{indep ext momenta}))$$

relations

Integration by parts

Integration-parts-relations can either be used by

- constructing a set of symbolic relations reducing the number of propagators

  `LiteRed` [Lee]

- explicitly applying the relations to a set of integrals and solving the resulting system of linear equations

  `Air` [Anastasiou, Lazopoulos]

  `FIRE` [Smirnov]

  `Reduze` [v. Manteuffel, (Studerus)]

  `KIRA` [Maierhöfer,Usowitch,Uwer]

Simple example

Consider the class of integrals

$$J(n) = \int \mathrm{d}^d k \frac{1}{(k^2 - M^2)^n}$$

Applying the only IBP relation leads to

$$\begin{aligned} 0 &= \int \mathrm{d}^d k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2 - M^2)^n} \\ &= dJ(n) - 2n \int \mathrm{d}^d k \frac{k^2}{(k^2 - M^2)^{n+1}} \end{aligned}$$

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Notation

For the presentation of IBP-identities a notation using lowering and raising operators is commonly used.

$$J(n_1, \ldots, n_N) = \int \mathrm{d}^d k \frac{1}{D_1^{n_1} \cdots D_N^{n_N}}$$

The operators i^+ and i^- act on $J(n_1, \ldots, n_N)$ as

$$\begin{aligned} i^+ J(n_1, \ldots, n_N) &= J(n_1, \ldots, n_i + 1, \ldots, n_N) \\ i^- J(n_1, \ldots, n_N) &= J(n_1, \ldots, n_i - 1, \ldots, n_N) \end{aligned}$$

Note: i^+ always comes together with an n_i and therefore the n_i is sometimes included in the definition of the operator.

Simple example cont'd

Using the new notation the IBP identity can be written as

$$0 = ((d - 2n_1) - 2n_1 M^2 1^+)J(n_1)$$

or simply

$$(d - 2n_1) - 2n_1 M^2 1^+$$

implying the application to an integral and omitting the left-hand side

Simple example cont'd

In this simple case the IBP identities can easily be solved leading to

$$J(n+1) = \frac{d-2n}{2nM^2} J(n)$$

and explicitly to

$$\begin{aligned} J(2) &= \frac{d-2}{2M^2} J(1) \\ J(3) &= \frac{d-4}{4M^2} J(2) = \frac{(d-2)(d-4)}{8M^4} J(1) \end{aligned}$$

IBPs: Challenges

- Either
  huge system of equations $\mathcal{O}(100 \cdot 10^6 - 10^9)$
  or
  many invariants leading to very complicated rational functions
- algorithm used for solving the system of equations:
  Gauss elimination, scales like $\mathcal{O}(N^3)$
  BUT only if the cost of every operation is constant
- possible way out: map everything to finite fields,
  use Chinese remainder theorem to reconstruct the full solution.

Outline

Recap of IBP identities

IBP identities are very useful because they allow us to express any integral I_j as linear combination of master integrals M_i

$$I_j = \sum_i C_{ji}(d, m_i, s_{ij}) M_i$$

The set of master integrals is obtain by exploiting all available IBP identities (and possibly also symmetry relations). The number of master integral is fixed but there is a freedom to choose the integrals. The idea is of course to reduce more complicated integrals, i.e. more lines, more dots, to simpler ones, i.e. fewer lines, fewer dots.

Differential equations

Feynman integrals are functions of masses and kinematical variables

$$I(m_i, s_{ij})$$

As such one can try to find a differential equation for the integral, e.g.

$$\frac{\partial}{\partial z} I(m_i, s_{ij}) = f(m_i, s_{ij}) I(m_i, s_{ij}) + R(m_i, s_{ij}), \quad z \in \{m_i, s_{ij}\}$$

and find a solution for it.

[Kotikov; Remiddi]

Construction of the differential equation

One way to construct the differential equation is by means of a master formula

$$\mathcal{D} I(m_i, s_{ij}) = 2\left(m_i^2 \frac{\partial}{\partial m_i^2} + s_{ij} \frac{\partial}{\partial s_{ij}} \right) I(m_i, s_{ij})$$

where $\mathcal{D}$ denotes the mass dimension of the integral. m_i and $s_{ij} = (p_i + p_j)^2$ denote internal masses and kinematical variables, respectively.

In the simple case of one mass parameter m and one kinematic one q^2 this turns into

$$\mathcal{D} I(m, q^2) = 2\left(m^2 \frac{\partial}{\partial m^2} + q^2 \frac{\partial}{\partial q^2} \right) I(m, q^2)$$

Construction of the differential equation

$$\mathcal{D}I(m,q^2) = 2\left(m^2\frac{\partial}{\partial m^2} + q^2\frac{\partial}{\partial q^2}\right) I(m,q^2)$$

the derivative with respect to m^2 can be taken directly leading to integrals with raised powers of propagators and we remain only with the differential with respect to q^2.

Now we can use IBP-relations to rewrite the new integrals in terms of the original one.
Thus, we obtain a differential equation for the integral

System of differential equation

In general, there is more than one master integral that we want to calculate. Thus we get a system of coupled differential equations.

$$\frac{\partial}{\partial z} I_j(z,d) = C_j(z,d) I_j + \sum_{k \neq j} D_{jk} I_k(z,d)$$

The integrals in the inhomogeneity are by construction at most as complicated as the integral we are looking at.
In the best case they are all simpler than the original one.
In the worst case we obtain a system of coupled differential equation within a sector, i.e. involving integrals where the same lines have positive powers.

Solving the differential equation

A first-order differential equation can be solved by using the method "variation of constants".
Consider the first-order differential equation

$$\frac{\partial}{\partial z} f(z) = a(z) f(z) + b(z)$$

then

$$\tilde{f}(z) = C e^{A(z)}, \text{ with } A(z) = \int \mathrm{d}z\, a(z)$$

is a solution of the homogeneous equation and

$$f(z) = e^{A(z)} \left(\int_{z_0}^{z} b(z') e^{-A(z')} \mathrm{d}z' + C \right)$$

a solution of the inhomogeneous equation.

Solving the system of differential equation

In practice, it is best to first perform the expansion in $\epsilon = (4 - d)/2$ and then to solve the differential equations order by order in ϵ. Make an ansatz for the master integrals in the form

$$I_j(z) = \sum_{k=-n}^{m} I_{jk}(z)\epsilon^k$$

This leads to a system of coupled differential equations for the coefficients of the Laurent expansion of the master integrals.
The system of differential equations can then be solved in bottom up approach.
This approach leads naturally to iterated integrals like HPLs.
In case there are several integrals in a sector one can try to decouple them in ϵ be a suitable choice of the master integrals.

Canonical basis

It was recently proposed [Henn] and demonstrated that in many cases a canonical basis of master integrals can be found in which the differential equations take the form

$$\frac{\partial}{\partial x}\vec{I}(x,\epsilon) = \epsilon A(x)\vec{I}(x,\epsilon)$$

where A is a $n \times n$ matrix. This makes the solution of differential equations trivial. In addition the alphabet of functions can be read off from the entries of the matrix.
Furthermore, [Lee] recently proposed an algorithm how to obtain such a representation. There are a few impletation of the algorithm and several extensions available.

Example: Massive one-loop propagator

use

$$\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

and derive the differential equations for the two masters J1 and J2.

$$\begin{aligned}
-\frac{x^2 {J_1}'(x)}{(x-1)(x+1)} &= 0 \\
-\frac{x^2 {J_2}'(x)}{(x-1)(x+1)} &= -\frac{(d-2)x^2 J_1(x)}{(x-1)^2(x+1)^2} \\
&\quad -\frac{x\left(dx^2-2dx+d-4x^2+4x-4\right)J_2(x)}{2(x-1)^2(x+1)^2}
\end{aligned}$$

- The first equation does not give much information since J1 does not depend on x. $J_1 = C(m,d) = c_{1,-1}/\epsilon + c_{1,0} + c_{1,1}\epsilon$

Example: Massive one-loop propagator

Inserting an ansatz in form of a Laurant series in ϵ we get

$$
\begin{aligned}
0 &= 2c_{1,-1} - \left(x^2 - 1\right) J'_{2,-1}(x) - 2J_{2,-1}(x) \\
0 &= x\left(-2c_{1,0} + 2c_{1,-1} + \left(x^2 - 1\right) J'_{2,0}(x)\right) \\
&\quad +2xJ_{2,0}(x) + (x-1)^2 J_{2,-1}(x)
\end{aligned}
$$

For the solutions one gets

$$
J_{2,-1}(x) = \frac{kx + k + 2c_{1,-1}}{1-x}
$$

There is no singularity for $x \to 1$ thus (with $c_{1,-1} = 1$)

$$
k = -1 \quad \Rightarrow \quad J_{2,-1}(x) = 1
$$

Example: Massive one-loop propagator

$$0 = -2xJ_{2,0}(x) - x\left(x^2 - 1\right)J'_{2,0}(x) - (x-1)^2$$

with the solution

$$J_{2,0}(x) = \frac{kx + k + x\log(x) + \log(x) + 4}{1 - x}$$

To get a regular solution at $x = 1$ we need $k = -2$ and get

$$J_{2,0}(x) = \frac{-2x + x\log(x) + \log(x) + 2}{1 - x}$$

Outline

Motivation

Quite often a problem is to complex at the start to be tackled directly.

In this case it is often possible to expand in some small or large parameter to simplify the problem.

These expansions sometimes lead to more than a simple power series and more work than taking a simple Taylor series has to be done.

In general, the procedure than goes by the name of asymptotic expansion.

There are two procedures to perform an asymptotic expansion:

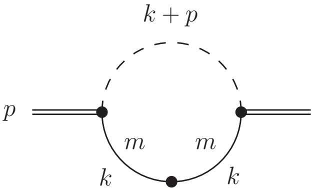
- expansion by regions
- expansion by subgraphs

Expansion by regions: The idea

1. Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a Taylor series with respect to the parameters that are considered small there.
2. Integrate the integrand, expanded in the appropriate way in every region, over the *whole integration domain* of the loop momenta.
3. Set to zero any scaleless integral.

The peculiar thing here is the second step since naively this could lead to double counting problems.
The problematic thing is how do we find *all* these regions

Example I: large momentum expansion



$$F = \int \mathrm{d}^d k \, I$$

with the integrand $I = I_1 I_2$ and the propagators

$$I_1 = \frac{1}{\left((k+p)^2\right)^{n_1}} = \frac{1}{(k^2 + 2k \cdot p + p^2)^{n_1}} \quad \text{and} \quad I_2 = \frac{1}{(k^2 - m^2)^{n_2}}$$

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We want to consider the case where

$$|p| \gg m$$

What are the regions?

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We want to consider the case where

$$|p| \gg m$$

What are the regions?
We find the following

hard region (h) $k \sim p$

soft region (s) $k \sim m$

Example I: hard region

In the hard region we have $k \sim p$ so

$$m^2 \ll k^2 \sim k \cdot p \sim p^2$$

so I_1 remains untouched and in I_2 we perform an expansion in m^2/k^2

$$I_2 \to T^{(h)} I_2 \equiv \sum_j T_j^{(h)} I_2 = \sum_{j=0}^{\infty} \frac{(n_2)_j}{j!} \frac{(m^2)^j}{(k^2)^{n_2+j}}$$

and we obtain a massless propagator

$$\int \mathrm{d}^d k \, \frac{1}{(k+q)^2 (k^2)^n}$$

Example I: soft region

In the soft region we have $k \sim m$ so

$$|k^2| \ll |p^2|, |2k \cdot p| \ll |p^2|$$

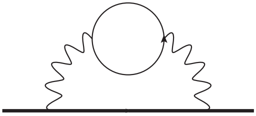
now I_2 is untouched and we have to expand I_1

$$I_1 \to T^{(s)} I_1 \equiv \sum_j T_j^{(s)} I_1 \equiv \sum_{j_1, j_2} T_{j_1, j_2}^{(s)} I_1 = \sum_{j_1, j_2 = 0}^{\infty} \frac{(n_1)_{j_{12}}}{j_1! \, j_2!} \frac{(-k^2)^{j_1} (-2k \cdot p)^{j_2}}{(p^2)^{n_1 + j_{12}}}$$

In this region we end with massive tadpoles

$$\int \mathrm{d}^d k \frac{k \cdot p^n}{k^2 - m^2}$$

Example II: On-shell integrals



Consider the typical on-shell integral ($q^2 = M^2$)

$$\int \mathrm{d}k^d \mathrm{d}l^d \frac{1}{(k^2 + 2k \cdot q)k^4((k-l)^2 - m^2)(l^2 - m^2)}$$

we get the regions

- $k^2, l^2 \approx M^2$: expand in $m^2 \to$ massless on-shell propagator
- $k^2 \approx M^2$, $l^2 \approx m^2$: one-loop on-shell $\times$ massive tadpole
- $k^2, l^2 \approx m^2$: expand in $k^2 \to$ new class of diagrams

$$\int \mathrm{d}^2k\, \mathrm{d}^d l\, \frac{1}{(k \cdot q)k^4((k-l)^2 - m^2)(l^2 - m^2)}$$

Tools for asymptotic expansions

To help with finding the regions of a Feynman integrals the Mathematica programs

- `asy.m` [Pak, Smirnov]
- `asy2.m` [Jantzen, Smirnov]

are useful.
Both programs are based on the program `Qhull`.

Outline

Recap

We derived the Feynman parameter representation

$$G_L = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{d}{2}L)}{\prod_i \Gamma(\nu_i)} \int \prod_i \mathrm{d}x_i x_i^{\nu_i - 1} \delta(1 - \sum x_i) \times \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

where $U(x)$ and $F(x)$ are homogeneous functions of x_i of degree L and $L+1$, respectively.

For euclidean kinematics $F(x)$ is positive semi definite and we only have to deal with end-point singularities.
The sector decomposition approach can be used to disentangle overlapping singularities.

Primary sectors

Let us set $\nu_i = 1$

$$\int_0^1 \prod_i \mathrm{d}x_i \delta(1 - \sum x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

First step is to integrate over the delta function

To do this split the integration into n parts to get the primary sectors G_ℓ

$$\int \mathrm{d}^n x = \int \mathrm{d}^n x \prod \theta(x_i \geq 0) = \sum_\ell \int \mathrm{d}^n x \prod_{i \neq \ell} \theta(x_\ell \geq x_i \geq 0)$$

explicit for 2 variables

$$\int \mathrm{d}x_1 \mathrm{d}x_2 = \int \mathrm{d}x_1 \mathrm{d}x_2 \theta(x_1 \geq x_2 \geq 0) + \int \mathrm{d}x_1 \mathrm{d}x_2 \theta(x_2 \geq x_1 \geq 0)$$

Primary sectors

In each primary sector G_ℓ do the variable transformation

$$x_j = \begin{cases} x_\ell t_j & j < \ell \\ x_\ell & j = \ell \\ x_\ell t_{j-1} & j > \ell \end{cases}$$

Then due to the homogeneity x_ℓ factors out and

$$F_\ell(x) \to F_\ell(t) x_\ell^{L+1}, \quad U_\ell(x) \to U_\ell(t) x_\ell^{L}$$

doing the integration over x_ℓ to eliminate the δ-function gives for each primary sector

$$G_\ell = \int_0^1 \prod_{i=1}^{N-1} \mathrm{d}t_i \frac{U_\ell(t)^{N_\nu - d(L+1)/2}}{F_\ell(t)^{N_\nu - dL/2}}$$

Iterated sector decomposition

There are many strategies to do the actual sector decomposition. I follow here the original version by Binoth and Heinrich

1. Determine a minimal set of parameters, say $S = \{t_{\alpha_1,\dots,t_{\alpha_r}}\}$, such that U_ℓ, respectively F_ℓ, vanish if the parameters of S are set to zero.
2. Decompose the corresponding r-cube into r subsectors.

$$\prod_{j=1}^{r} \theta(1 \geq t_{\alpha_j} \geq 0) = \sum_{k=1}^{r} \prod_{j \neq k} \theta(1 \geq t_{\alpha_k} \geq t_{\alpha_j} \geq 0)$$

3. remap to the unit cube in each subsector

$$t_{\alpha_j} \to \begin{cases} t_{\alpha_k} t_{\alpha_j}, & j \neq k \\ t_{\alpha_j}, & j = k \end{cases}$$

t_{α_k} factors from $U(t)$ and/or $F(t)$ and we get the form

$$G_{\ell k} = \int_0^1 \prod_{j=1}^{N-1} \mathrm{d}t_i \left(\prod t_j^{A_j - B_j \epsilon} \right) \frac{U_{\ell k}(t)^{N_\nu - d(L+1)/2}}{F_{\ell k}(t)^{N_\nu - dL/2}}$$

Extraction of poles

After the sector decomposition is complete we are left with expressions of the form

$$I_j = \int \mathrm{d}t_j \, t_j^{A_j - B_j\epsilon} \mathcal{I}(t_j, \epsilon)$$

if $A_j \geq 0$ we do not get a pole in ϵ from the t integration
otherwise we expand $\mathcal{I}(t_j, \epsilon)$ around $t_j = 0$

$$\mathcal{I}(t_j, \epsilon) = \sum_{p=0}^{|A_j]-1} \mathcal{I}_j^{(p)}(0, \epsilon) \frac{t_j^p}{p!} + R(t_j, \epsilon)$$

and obtain for the integral

$$I_j = \sum_{p=0}^{|A_j]-1} \frac{1}{|A_j| + p + 1 - B_j\epsilon} \frac{\mathcal{I}_j^{(p)}(0, \epsilon)}{p!} + \int_0^1 \mathrm{d}t_j t_j^{A_j - B_j\epsilon} R(t_j, \epsilon)$$

Example: one-loop triangle

Let us look at the one-loop triangle with propagators

$$\{-k^2, -(k+p_1)^2 - M^2, -(k+p_2)^2 - M^2\}, \; p_1^2 = p_2^2 = 0$$

we have the U and F polynomials

$$U(x) = x_1 + x_2 + x_3, \; F(x) = s x_3 x_2 + {x_2}^2 + x_1 x_2 + 2 x_3 x_2 + {x_3}^2 + x_1 x_3$$

Example: one-loop triangle

To obtain the first primary sector we use the rules

$$\{x_1 \to x_1, x_2 \to t_1 x_1, x_3 \to t_2 x_1\}$$

integrate over the delta function and obtain the new polynomials

$$U_1(t) = t_1 + t_2 + 1,\ F_1(t) = s t_2 t_1 + {t_1}^2 + 2 t_2 t_1 + t_1 + {t_2}^2 + t_2$$

$F(t)$ vanishes for $t_1, t_2 \to 0 \Rightarrow$ we get 2 subsectors

Example: one-loop triangle

We get the 2 subsectors by the transformations $t_2 \to t_1 t_2$ and $t_1 \to t_1 t_2$, respectively.

$$\begin{aligned} F_{1,1} &= s t_1 t_2 + t_1 {t_2}^2 + 2 t_1 t_2 + t_2 + t_1 + 1 \\ F_{1,2} &= s t_2 t_1 + t_2 {t_1}^2 + 2 t_2 t_1 + t_1 + t_2 + 1 \end{aligned}$$

both are now positive and we can stop here

primary sectors 2 and 3 have no subsectors

all singularities in the Feynman parameter integration are now made explicit

$$\int_0^1 \left(\prod_j t_j^{A_j - B_j \epsilon} \right) f(t)$$

and $f(t)$ is free of singularities

Non-Euclidean kinematics

- If we are not in the Euclidean region ($s_{ij} < 0$) the F-polynomial is no longer positive semi definite, but changes sign inside the domain of integration. Leading to (integrable) singularities.
- A necessary condition are the Landau equations

$$\begin{aligned} x_j(q_j^2 - m_j^2) &= 0 \quad \forall j \\ \frac{\partial}{\partial k^\mu} \sum_j x_j(q_j^2(k,p) - m_j^2) &= 0 \end{aligned}$$

- if there is a solution $x_i > 0$ for the Landau equations, we have the leading Laudau singularity, which is not integrable
- To perform these integrations contour deformation [Borowka, Heinrich] can be used.

Contour deformation

- Reparametrize the integration path

$$\int_0^1 \prod_{j=1}^{N} \mathrm{d}x_j \mathcal{I} x = \int_0^1 \prod_{j=1}^{N} \mathrm{d}x_j \left| \left(\frac{\partial z_k(x)}{\partial x_l} \right) \right| \mathcal{I}(z(x))$$

- a convenient choice is

$$\begin{aligned} \vec{z}(\vec{x}) &= \vec{x} - i\vec{\tau}(\vec{x}) \\ \tau_k &= \lambda x_k (1 - x_k) \frac{\partial F(\vec{x})}{\partial x_k} \end{aligned}$$

- F expressed in the new variables

$$F(\vec{z}(\vec{x})) = F(\vec{x}) - i\lambda \sum_j x_l (1 - x_j) \left(\frac{\partial F}{\partial x_j} \right)^2 + \mathcal{O}(\lambda^2)$$

Tools

The method of sector decomposition has been implemented in several tools

- FIESTA (http://git.sander.su/fiesta) [Smirnov]
- SecDec (https://secdec.hepforge.org/)
  [Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke]
- pySecDec (https://secdec.hepforge.org/)
  [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke]
- SectorDecomposition (http://wwwthep.physik.uni-mainz.de/˜stefanw/sector_decomposition/)
  [Bogner, Weinzierl]

Outline

Factorial Series

- The idea of the method goes back to Laporta who suggested the to calculate Feynman integrals in form of of a factorial series.
- Take an integral and raise the power of one propagator to the power x e.g. $I(1,1,1) \to I(x) = I(x,1,1)$
- Using IBP relations on can obtain a difference equation for the integral

$$\sum_{k+0}^{R} p_k(x) I(x+k) = \sum_i \sum_{k=0}^{R_i} p_{ik}(x) J_i(x+k)$$

  where J_i are integrals of simpler sectors
- Make an ansatz for $I(x)$ in terms of a factorial series (N.B. not the most general one)

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$

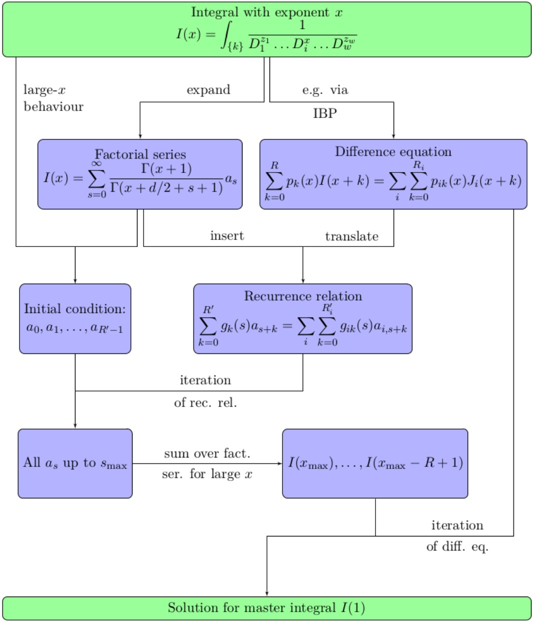
Factorial Series cont'd

- Inserting the ansatz into the difference equation results in a recurrence relation for a_s

$$\sum_{k=0}^{R'} g_k(s) a_{s+k} = \sum_i \sum_{k=0}^{R'_i} g_{ik}(s) a_{i,s+k}$$

- given the initial values $a_0, a_1, \ldots$ are known, an arbitrary number of values for a_n can be calculated.
- using the obtained values for a_n $I(x)$ can be calculated

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$
$$= \frac{\Gamma(x+1)}{\Gamma(x+d/2+1)} \Big(a_0 + \frac{a_1}{(x+d/2+1)} + \frac{a_2}{(x+d/2+1)(x+d/2+2)}$$
$$+ \cdots \Big)$$



Integral with exponent x
$I(x) = \int_{\{k\}} \frac{1}{D_1^{z_1} \dots D_i^{x} \dots D_w^{z_w}}$
large-x behaviour
expand
e.g. via IBP
Factorial series
$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$
Difference equation
$\sum_{k=0}^{R} p_k(x) I(x+k) = \sum_i \sum_{k=0}^{R_i} p_{ik}(x) J_i(x+k)$
insert
translate
Initial condition:
$a_0, a_1, \dots, a_{R'-1}$
Recurrence relation
$\sum_{k=0}^{R'} g_k(s) a_{s+k} = \sum_i \sum_{k=0}^{R'_i} g_{ik}(s) a_{i,s+k}$
iteration of rec. rel.
All a_s up to $s_{\max}$
sum over fact. ser. for large x
$I(x_{\max}), \dots, I(x_{\max} - R + 1)$
iteration of diff. eq.
Solution for master integral $I(1)$

Conclusion

- Many approaches exist for various classes of integrals.
- Two very stong options to obtain numerical results
- Amount of multi purpose tools for analytical integrals very limited.
- Different problems may require different methods even mithin the same project.

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Thank you very much for your attention!