Gaudin models and integrable field theories

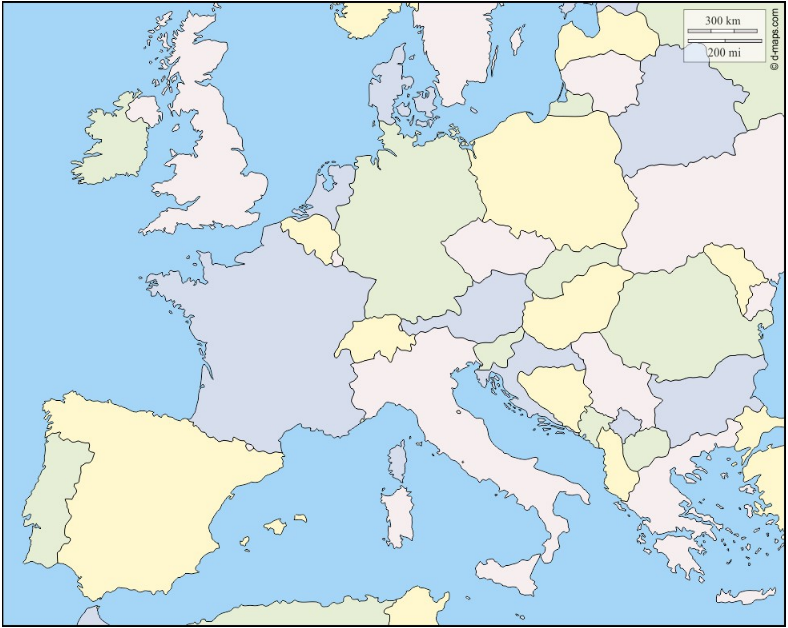
Sylvain Lacroix

DESY Theory Fellows Meeting

November 20th, 2018

A little bit about myself

- Nationality: French and Swiss
- Born in 1991
- Hobbies: table tennis, movies, series, music, concerts ...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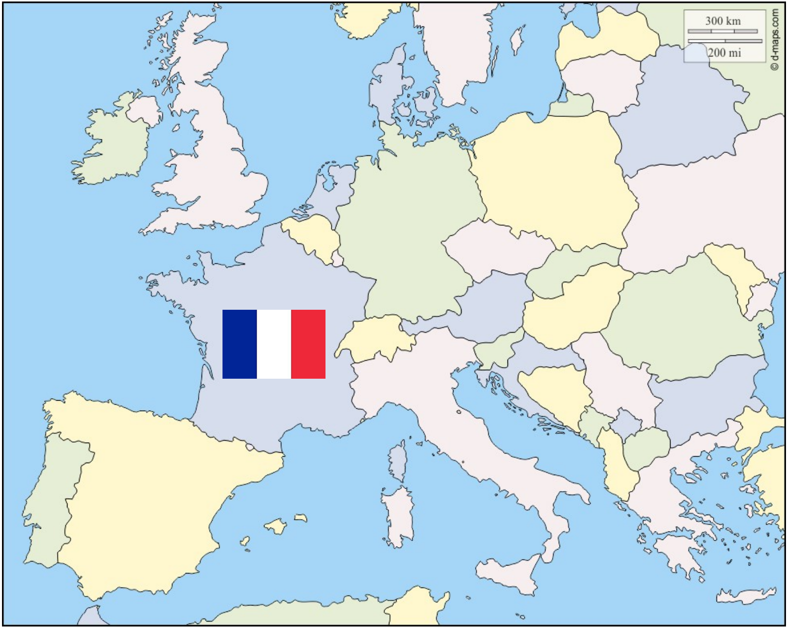
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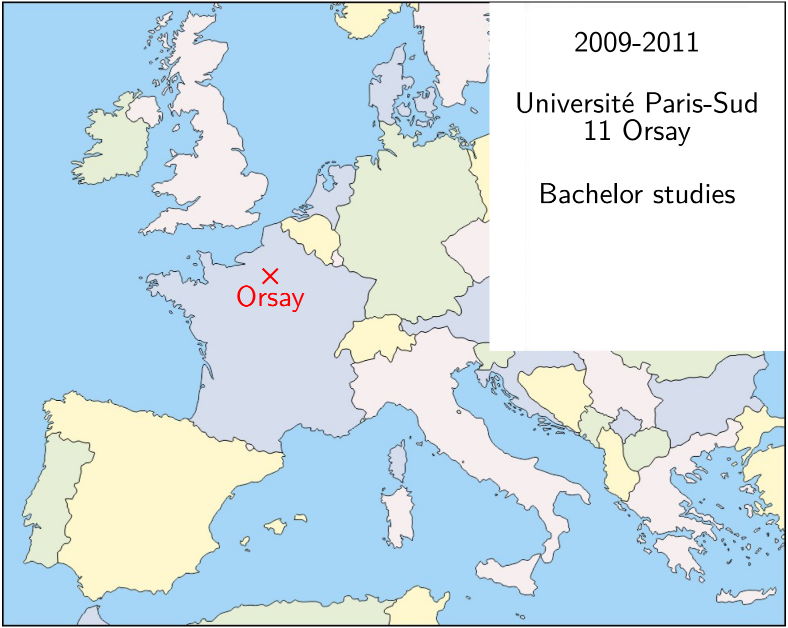
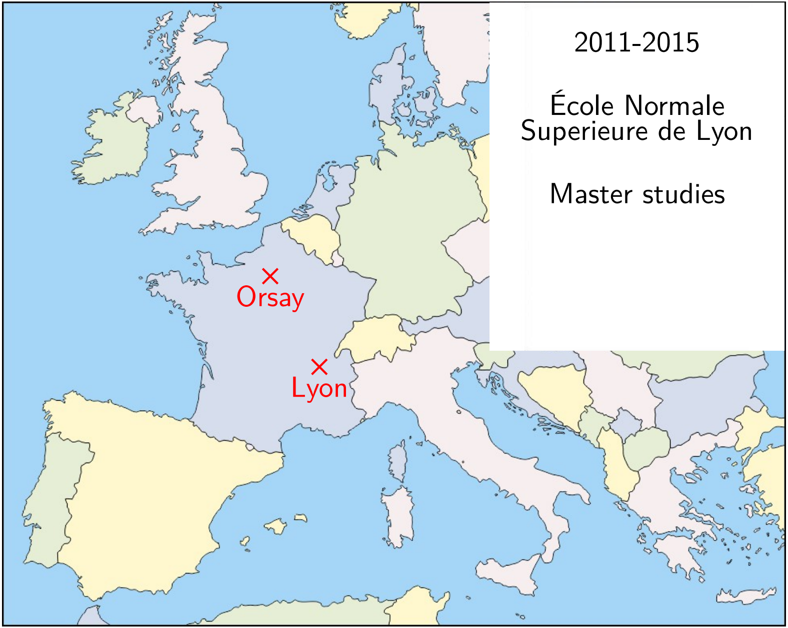


2009-2011

Université Paris-Sud
11 Orsay

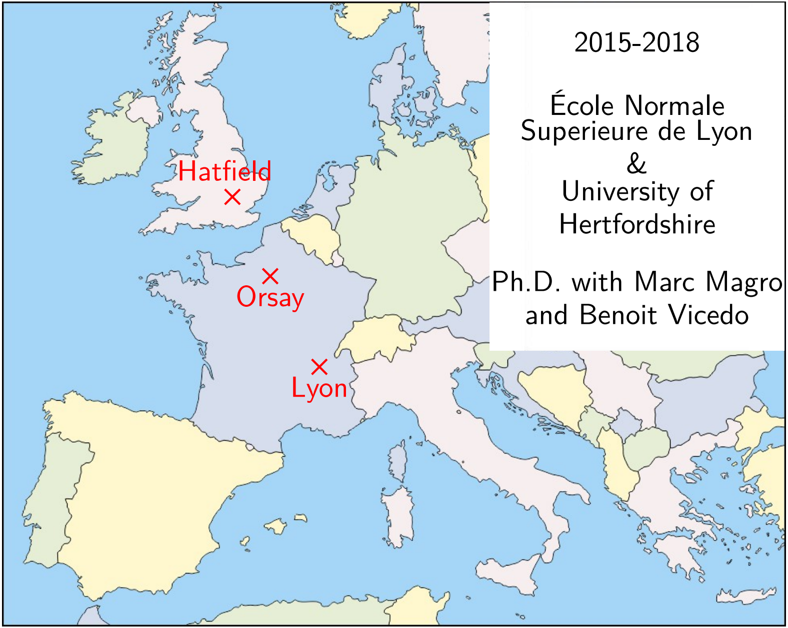
Bachelor studies



2011-2015

École Normale Superieure de Lyon

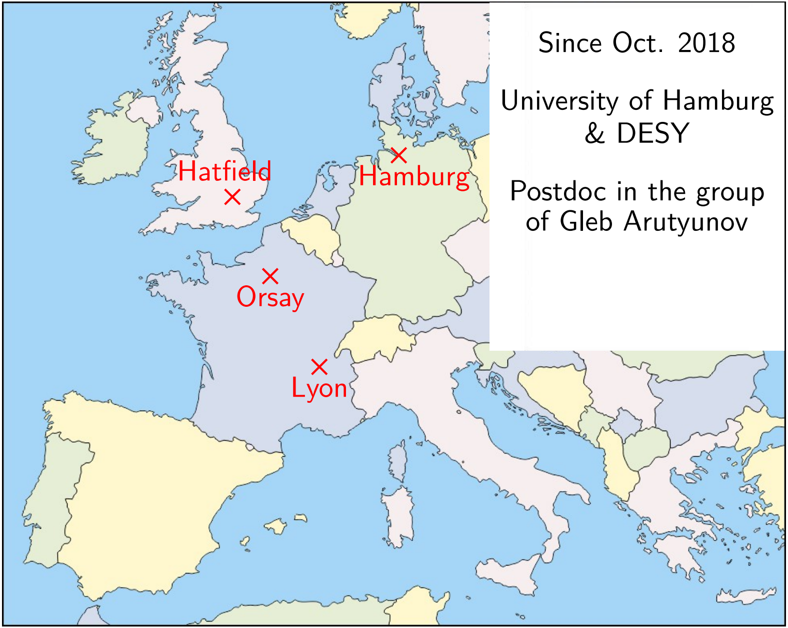
Master studies



2015-2018

École Normale Superieure de Lyon
&
University of Hertfordshire

Ph.D. with Marc Magro and Benoit Vicedo



Since Oct. 2018

University of Hamburg & DESY

Postdoc in the group of Gleb Arutyunov

And now physics

And now physics

... and maths

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... but hopefully not too much !

Integrable quantum systems (vague definition)

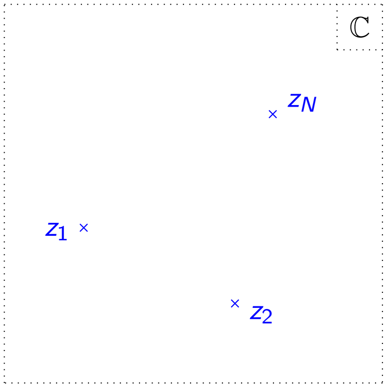
- Quantum system: Hamiltonian $\mathcal{H}$ (operator on a Hilbert space)
- Integrability: *a lot* of conserved commuting quantities

$$\frac{\mathrm{d}\mathcal{Q}_i}{\mathrm{d}t} = [\mathcal{H}, \mathcal{Q}_i] = 0 \quad \text{and} \quad [\mathcal{Q}_i, \mathcal{Q}_j] = 0$$

$\rightarrow$ systems with a lot of symmetries

- Exactly solvable: computation of spectrum, eigenvectors, correlation functions, ...

- Main subject today: Gaudin models
  → historically introduced as a spin chain [Gaudin 76']



- Positions $z_1, z_2, \cdots, z_N$ in $\mathbb{C}$

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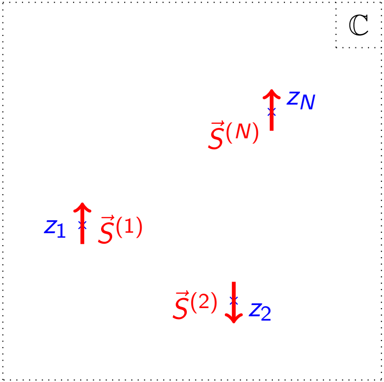

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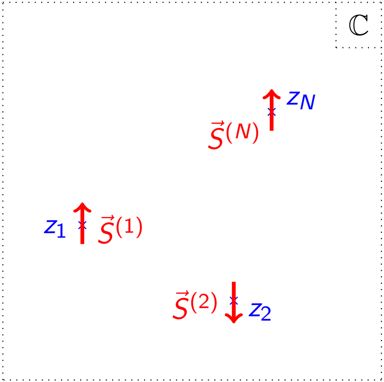

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$$\mathcal{H}_i = \sum_{j \neq i} \frac{\vec{S}^{(i)} \, . \, \vec{S}^{(j)}}{z_i - z_j}$$

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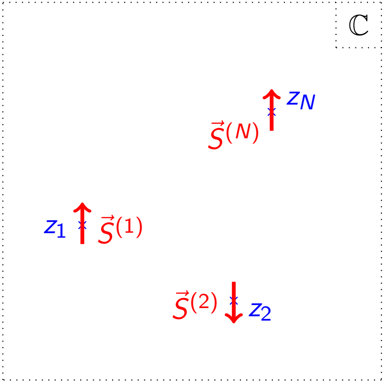

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$$\mathcal{H}_i = \sum_{j \neq i} \frac{\vec{S}^{(i)} \cdot \vec{S}^{(j)}}{z_i - z_j}$$

- Total energy:

$$\mathcal{H} = \sum_{i=1}^{N} a_i \mathcal{H}_i$$

Integrability

- Spin operators: $\vec{S} = (S_x , S_y , S_z)$

$$\left[S_\mu , S_\nu \right] = \epsilon_{\mu\nu\rho} S_\rho$$

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- Exact spectrum and common eigenvectors: Bethe ansatz

Gaudin model with arbitrary Lie algebra [Gaudin 83']

- Spin operators: $\mathfrak{su}(2)$ algebra

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with $\cdot$ scalar product on $\mathfrak{su}(2)$

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- Hamiltonians:

$$\mathcal{H}_i = \sum_{j \neq i} \frac{\kappa\left(I^{(i)}, I^{(j)}\right)}{z_i - z_j}$$

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- Scalar product invariant:

$$[S_\mu, S_\nu] \cdot S_\rho = S_\mu \cdot [S_\nu, S_\rho]$$

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- Integrability: $[\mathcal{H}_i, \mathcal{H}_j] = 0$?
- Invariant scalar product

$$\kappa([I_a, I_b], I_c) = \kappa(I_a, [I_b, I_c])$$

Finite Gaudin models

- Finite Gaudin models: finite dimensional semi-simple algebras
- Invariant scalar-product: Killing form
- Includes $\mathfrak{su}(2)$

- Higher degree Hamiltonians: $\mathcal{H}_i^d$ (degree d)

$$\left[\mathcal{H}_i^p, \mathcal{H}_j^q\right] = 0, \qquad\qquad \mathcal{H}_i^2 = \mathcal{H}_i$$

- Example: for $\mathfrak{su}(N)$, degrees $2, 3, \cdots, N$

- Diagonalisation of $\mathcal{H}_i^d$: Bethe ansatz

Quantum fields and affine algebras

- Quantum field theory on the circle: field operators $\phi(x)$'s ($x \in [0, 2\pi[$)
- Kac-Moody currents: $J(x) = J_a(x) I^a$ ($\mathfrak{g}$-valued, $\mathfrak{g}$ finite algebra)

$$\left[J_a(x), J_b(y)\right] = f_{ab}{}^{c}\, J_c(x)\, \delta(x-y)$$

- Fourier decomposition:

$$J_a(x) = \sum_{n \in \mathbb{N}} I_{a,n} e^{inx}$$

- Commutation relations of Fourier modes:

$$\left[I_{a,n}, I_{b,m}\right] = f_{ab}{}^{c}\, I_{c,n+m}$$

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$\rightarrow$ affine Kac-Moody algebra $\widetilde{\mathfrak{g}}$ (infinite-dimensional)

- Operators $I_{a,n}$ of algebra $\widetilde{\mathfrak{g}} \Leftrightarrow$ Kac-Moddy current (quantum field)

Affine Gaudin models as integrable field theories

[Feigin Frenkel '11, Vicedo '18]

- Affine gaudin model: associated with affine Kac-Moody algebra $\widetilde{\mathfrak{g}}$
- "Spin" $\widetilde{\mathfrak{g}}$-operators $I^{(i)}_{a,n}$ at each sites $\Leftrightarrow$ Kac-Moody currents $J^{(i)}(x)$
  $\rightarrow$ quantum field theory on the circle

- Quadratic commuting Hamiltonians: $[\mathcal{H}_i, \mathcal{H}_j] = 0$
- Diagonalisation through the Bethe ansatz (from [Schechtman Varchenko '91])

- Integrable QFT: infinite number of commuting conserved charges
- Commuting higher-degree charges ?

My research interests (past, present and future)

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  - $\rightarrow$ relation with the ODE/IM correspondence

Thank you for your attention !