

# ① Scattering Amplitudes I: Color ordering and spinor helicity formalism

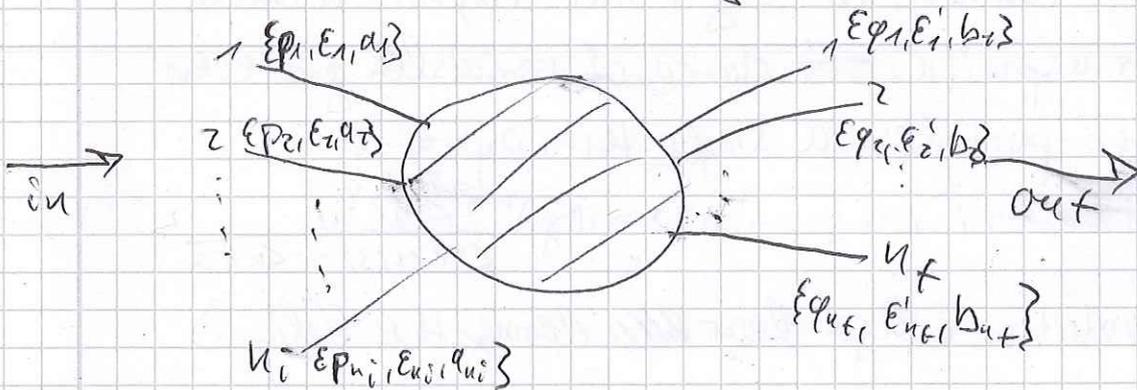
References:

- Dixon 9601359v2
- Iten, Plefka: "Scattering Amplitudes in Gauge Theories"

## 1. Motivation for studying scattering amplitudes

- Main observables in particle physics: cross section for scattering processes in colliders, directly proportional to probability of one state becoming the other ( $|\langle \text{out} | \text{in} \rangle|^2$ ).

- $\langle \text{out} | \text{in} \rangle$  is the scattering amplitude



Particles of the  $|\text{in}\rangle$  state are described by their state at  $t \rightarrow -\infty$  &  $|\text{out}\rangle$  by their state at  $t \rightarrow \infty$

$\{p_i, e_i, a_i\}$  are the quantum numbers of a particle.

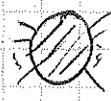
In this case  $p_i, q_i$ : momenta

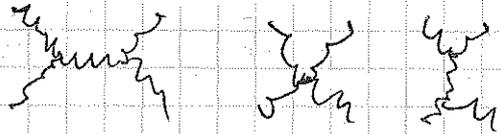
$e_i, e_i'$ : polarizations

$a_i, b_i$ : colors

- In the the S-matrix happens, that relates initial and final states. Therefore the amplitude  $f_{\text{in} \rightarrow \text{out}}$  is:

$$f_{\text{in} \rightarrow \text{out}} = \langle \{q_1, e_1', b_1\}, \dots, \{q_n, e_n', b_n\} | S | \{p_1, e_1, a_1\}, \dots, \{p_n, e_n, a_n\} \rangle$$

- The  is traditionally described with Feynman rules, that allow one to write down all possible Feynman diagrams of a ~~process~~ process. e.g. for  $2 \rightarrow 2$  gluon scattering at tree level



Each diagram is translated to formulae: Herules for  $2 \rightarrow 2$  but for

$2 \rightarrow 3$  : 25 diagrams

$2 \rightarrow 4$  : 220 diagrams

:

$2 \rightarrow 8$  > 1000 000 diagrams

- Stark contrast to e.g. Parke-Taylor formula: For a specific set choice of variables get even for  $n$ -particles a single term expression:

$$A(i^+, \dots, j^-, \dots, k^-, \dots, n^+) = i(g)^{n-2} \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

(notation will get dear ~~then~~ during the talk)

Goal: Find formalism in which the necessary cancellations become accessible.

Main objectives of SA series:

1. Efficient amplitude computation
2. Contact with corrections to inspiraling BH potential postnewtonian

Outline of this lecture

1. Separating out color information
2. Choosing "good" variables (spinor helicity formalism)
3. Example computation  $2 \rightarrow 2$  gluons.

## ② 1. Gauge Theory Preliminaries/Reminder

- In this session we will discuss processes in a non-abelian gauge theory created by local  $SU(N_c)$  transformations.  $N_c$  will be the number of colors.

From requiring local gauge invariance i.e.  $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{N_c} \end{pmatrix} \rightarrow e^{i\alpha^a T^a} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{N_c} \end{pmatrix}$

(with  $T^a$  the  $N_c^2 - 1$  generators of the Lie group  $SU(N_c)$ )

for the Lagrangian of the theory we get

- A covariant derivative  $(D_\mu)_{ij} = \delta_{ij} \partial_\mu - ig A_\mu^a (T^a)_{ij}$

- A term for the new fields  $A_\mu^a$ :  $-\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu})$

with  $F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] \Rightarrow F_{\mu\nu} = F_{\mu\nu}^a T^a$  with  
 $F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + \sqrt{2} g f^{abc} A_\mu^a A_\nu^b$   
structure constants of Lie group

- Get Lagrangian  $\mathcal{L}_{QCD} = i \bar{\psi}_i \not{D}_i \psi_j - \frac{1}{4} F_{\mu\nu}^c F_c^{\mu\nu}$

• From the Lagrangian get the Feynman rules

-  $i \bar{\psi}_i \delta_{ij} \not{\partial} \psi_j$  becomes  $i \frac{\not{k}}{k} \delta_{ij}$

-  $i \bar{\psi}_i (-ig) A_\mu^a (T^a)_{ij} \psi_j$  becomes  $i g \not{\gamma}^\mu (T^a)_{ij}$

- From  $\frac{1}{4} F_{\mu\nu}^c F_c^{\mu\nu}$  we get:

terms with ~~AAAA~~  $\Delta A \Delta A$

$$\begin{matrix} \mu & & \nu \\ a & \rightsquigarrow & b \\ & & \end{matrix} \quad \frac{f^{abc}}{k^2 + i0}$$

terms with  $\Delta A A A$

$$\begin{matrix} \mu & & \nu \\ a & \rightsquigarrow & b \\ & & \end{matrix} \quad g f^{abc} [(q-r)_\mu r_\nu + (r-p)_\nu r_\mu + (p-q)_\nu r_\mu]$$

terms with  $A A A A$

$$-ig^2 [f^{abe} f^{cde} (\eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\nu\rho} \eta_{\mu\sigma}) + f^{ace} f^{dbe} (\eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\nu\rho} \eta_{\mu\sigma}) + f^{ade} f^{bce} (\eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\nu\rho} \eta_{\mu\sigma})]$$

(Note that we work with massless quarks and Feynman gauge)

- Choose Normalization:  $\text{Tr}(T^a T^b) = \delta^{ab}$

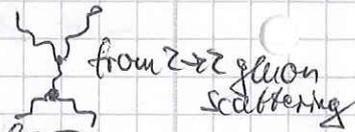
$$[T^a, T^b] = i\sqrt{2} f^{abc} T^c$$

- Everything is massless so helicity is the Lorentz invariant quantity we can use to describe polarization.

It is defined by 
$$h := \frac{\hat{p} \cdot \hat{S}}{|p|}$$

$p$ : momentum  
 $S$ : spin

## 2. Color management

If we translate for example the diagram  into formula, we get a part which depends on color and a part that depends on kinematics.

Our goal is to separate the two.

We use the two relations

$$f^{abc} = \frac{i}{\sqrt{2}} (\text{Tr}(T^a T^b T^c) - \text{Tr}(T^a T^c T^b)) \quad (1)$$

$$\sum_a (T^a)_{ij} (T^a)_{kl} = \delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \quad (2)$$

To make all color factors into just one trace of  $T^a$ -matrices (At tree level for  $n$ -gluon amplitudes) for each term.

(1) can be used to write structure constants as traces and (2) to deal with expressions of the form  $\text{Tr}(\dots T^e \dots) \text{Tr}(\dots T^e \dots)$  and

$\xrightarrow{\text{sum implied}}$

make

3

them into a single trace.  
 these considerations can also be done in a diagrammatic approach.

We use diagrams to express their associated color factors. i.e.

$$\begin{array}{c} a \\ \diagdown \\ \text{---} \\ \diagup \\ b \end{array} \text{---} c \sim f^{abc} \quad \text{---} \text{---} \text{---} (T^a)_{ij}$$

$$\text{---} \sim \delta_{ij} \quad \text{---} \text{---} \text{---} \text{---} \sim \delta_{ab}$$

then (1) becomes

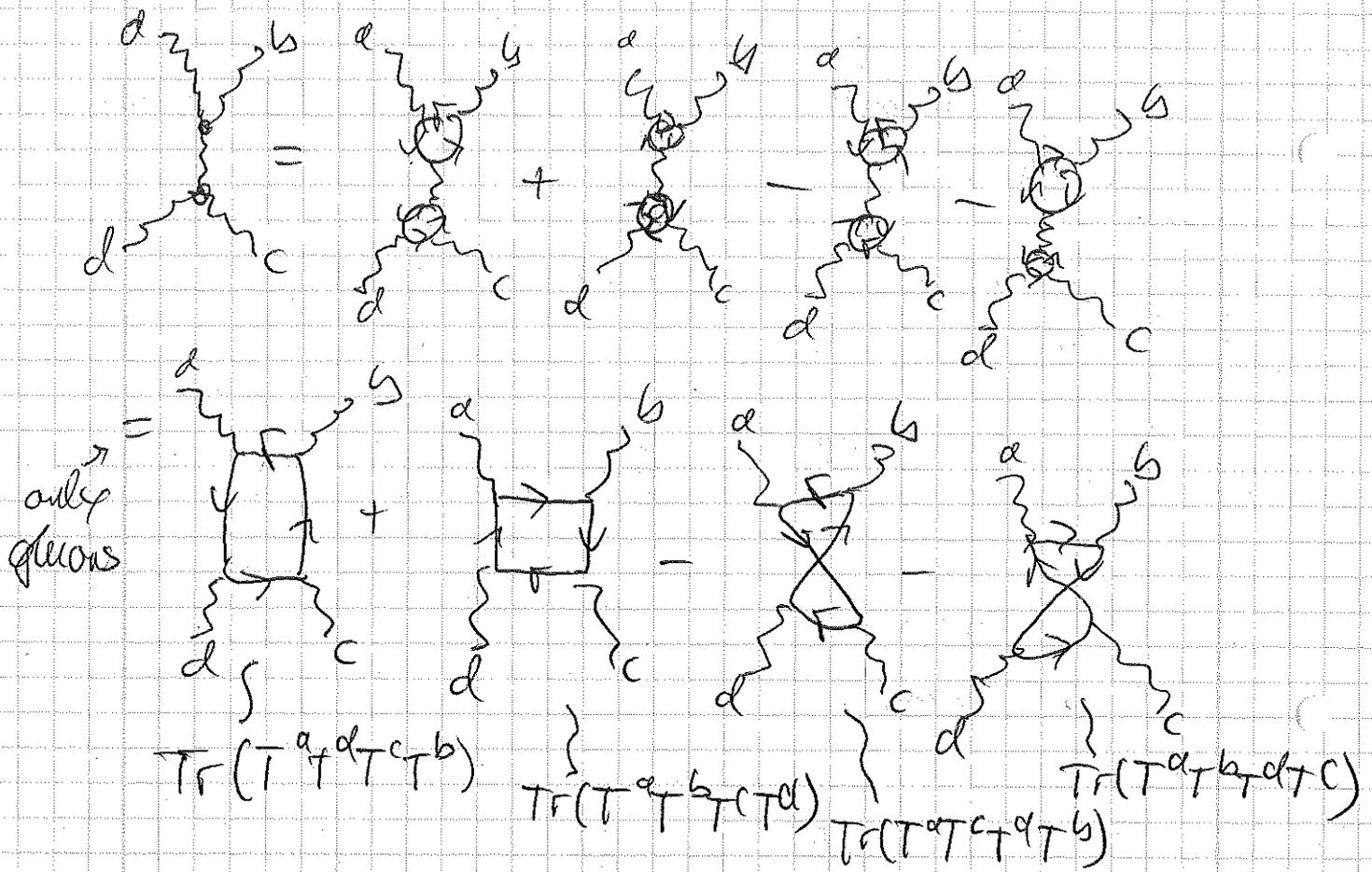
$$\begin{array}{c} a \\ \diagdown \\ \text{---} \\ \diagup \\ b \end{array} \text{---} c = \begin{array}{c} a \\ \diagdown \\ \text{---} \\ \diagup \\ b \end{array} \begin{array}{c} c \\ \diagdown \\ \text{---} \\ \diagup \\ \end{array} - \begin{array}{c} a \\ \diagdown \\ \text{---} \\ \diagup \\ b \end{array} \begin{array}{c} c \\ \diagdown \\ \text{---} \\ \diagup \\ \end{array}$$

and (2) becomes

$$\begin{array}{c} j \\ \swarrow \\ \text{---} \\ \searrow \\ l \end{array} \begin{array}{c} k \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array} \begin{array}{c} l \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array} = \begin{array}{c} j \\ \swarrow \\ \text{---} \\ \searrow \\ l \end{array} \begin{array}{c} k \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array} - \frac{1}{N_C} \begin{array}{c} k \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array} \begin{array}{c} l \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array}$$

For diagrams only involving gauge bosons (gluons)  
 we can assume (2) to be  $\begin{array}{c} j \\ \swarrow \\ \text{---} \\ \searrow \\ l \end{array} \begin{array}{c} k \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array} \begin{array}{c} l \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array} = \begin{array}{c} j \\ \swarrow \\ \text{---} \\ \searrow \\ l \end{array} \begin{array}{c} k \\ \swarrow \\ \text{---} \\ \searrow \\ \end{array}$   
 since this is the relation for  $U(N_C)$   
 The generator of  $U(1)$ , which is  $\frac{1}{N_C} \mathbb{1}_{N_C \times N_C} = T_0$ ,  
 commutes with the  $T^a$   $a \in \{1, \dots, N_C^2 - 1\}$ , therefore  
 therefore the  $U(1)$  generator doesn't couple to  
 the gluons

Knowing all this we can discuss the color structure of the example diagrams  $\begin{array}{c} a \\ \diagdown \\ \text{---} \\ \diagup \\ b \end{array} \text{---} c$   
 $\begin{array}{c} d \\ \diagdown \\ \text{---} \\ \diagup \\ e \end{array} \text{---} f$



This motivates the color decomposition of the  $n$ -gluon tree amplitude

$$A_n^{\text{tree}}(\{k_i, \lambda_i, \alpha_i\}) = g^{n-2} \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{\alpha_{\sigma(1)}} \dots T^{\alpha_{\sigma(n)}})$$

$Z_n$ : Cyclic permutations

$2 \rightarrow n \rightarrow 2$  scattering with all momenta going outwards

$$\times A_n^{\text{tree}}(\{p_{\sigma(1)}, \epsilon_{\sigma(1)}\}, \dots, \{p_{\sigma(n)}, \epsilon_{\sigma(n)}\})$$

Now the color factors are separated and we get color-ordered partial amplitudes that only contain kinematical information.

Notation:  $A_n^{\text{tree}}(\{p_{\sigma(1)}, \epsilon_{\sigma(1)}\}, \dots, \{p_{\sigma(n)}, \epsilon_{\sigma(n)}\}) =: A_n^{\text{tree}}(1^+, \dots, n^-)$   
 where  $+/-$  are chosen depending on helicity of the  $i$  particle

At first this seems like we now have to compute even more terms!

② Fortunately there are plenty of relations between the  $A_n^{\text{tree}}$ :

- Parity allows one to switch all helicities at once
- Using the fact that amplitudes containing the extra  $U(1)$  particle vanish we get from inserting  $T^0$  into the color decomposition that

$$0 = A_n^{\text{tree}}(1, 2, 3, \dots, n) + A_n^{\text{tree}}(2, 1, 3, \dots, n) + \dots + A_n^{\text{tree}}(2, 3, \dots, 1, n)$$

after collecting terms with same color structure.

Also a lot of the  $A_n^{\text{tree}}$  will vanish.

The  $A_n^{\text{tree}}$  are computed using color-ordered Feynman-rules.

Color decomposition:  $V_{\mu\nu\rho}^{abc} = \frac{-ig}{\sqrt{2}} [\text{Tr}(T^a T^b T^c) - \text{Tr}(T^a T^c T^b)] \cdot [ (q-r)_\mu \eta_{\nu\rho} + (r-p)_\nu \eta_{\rho\mu} + (p-q)_\rho \eta_{\mu\nu} ]$

Let's us read off the color-ordered rule

$$V_{\mu\nu\rho}^{abc} = \frac{-ig}{\sqrt{2}} [ (q-r)_\mu \eta_{\nu\rho} + (r-p)_\nu \eta_{\rho\mu} + (p-q)_\rho \eta_{\mu\nu} ]$$

The G-vertex looks considerably nicer in this version:

$$V_{\mu\nu\rho}^{abc} = i\eta_{\mu\nu} \eta_{\rho\sigma} - \frac{i}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\sigma} \eta_{\mu\rho})$$

these ~~kinematic~~ partial amplitudes can be very efficiently computed using the spinor helicity formalism.

# Spinor helicity formalism

Before doing anything else let us consider some objects:

- We write down spinors that transform as

$$\begin{aligned} & \left(\frac{1}{\sqrt{2}}, 0\right) \text{ or } \lambda^\alpha \quad \alpha \in \{1, 2\} \\ \text{or } & \left(0, \frac{1}{\sqrt{2}}\right) \quad \tilde{\lambda}^{\dot{\alpha}} \quad \dot{\alpha} \in \{1, 2\} \end{aligned}$$

- We also write down  $U(1)$ -vectors like  $p^\mu$  as  $2 \times 2$ -

$$\text{metrics } p^{\dot{\alpha}\alpha} \text{ via } p^\mu \rightarrow p^{\dot{\alpha}\alpha} = \overline{\sigma}^{\dot{\alpha}\alpha\mu} p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

with  $\overline{\sigma}^{\dot{\alpha}\alpha\mu} = (\mathbb{1}^{\dot{\alpha}\alpha}, (\vec{\sigma})^{\dot{\alpha}\alpha})$  consider only massless particles

$$\Rightarrow \det(p^{\dot{\alpha}\alpha}) = m^2 \stackrel{!}{=} 0$$

$$\Rightarrow \text{rank}(p^{\dot{\alpha}\alpha}) \leq 1 \Rightarrow p^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}} \lambda^\alpha$$

Some properties:  $(\lambda^\alpha)^\dagger = \tilde{\lambda}^{\dot{\alpha}}$  for real momenta  
 $\lambda_\alpha := \epsilon_{\alpha\beta} \lambda^\beta; \tilde{\lambda}_{\dot{\alpha}} := \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}}$$

A possible numerical choice for  $\lambda^\alpha$  and  $\tilde{\lambda}^{\dot{\alpha}}$  is

$$\lambda^\alpha = \frac{1}{\sqrt{p^0 + p^3}} \begin{pmatrix} p^0 + p^3 \\ p^1 + ip^2 \end{pmatrix}, \quad \tilde{\lambda}^{\dot{\alpha}} = \frac{1}{\sqrt{p^0 + p^3}} \begin{pmatrix} p^0 + p^3 \\ p^1 - ip^2 \end{pmatrix}$$

- We define scalar products notation:

$$\langle \lambda_i | \lambda_j \rangle := \lambda_i^\alpha \lambda_{j\alpha} = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta = -\langle \lambda_j | \lambda_i \rangle =: \langle ij \rangle$$

$$[\tilde{\lambda}_i | \tilde{\lambda}_j] := \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_{j\dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}} = -[\tilde{\lambda}_j | \tilde{\lambda}_i] =: [ij]$$

$$\Rightarrow [ii] = \langle ii \rangle = 0$$

$$\text{These evaluate to } \langle ij \rangle = \sqrt{|s_{ij}|} e^{i\phi_{ij}} \quad \text{with } s_{ij} = (p_i + p_j)^2$$

$$[ij] = -\sqrt{|s_{ij}|} e^{i\phi_{ij}} = (p_i + p_j)^2$$

$i, j$  count through the momenta involved in the scattering

⑤

$$\text{with } \cos \phi_{ij} = \frac{P_i^1 P_j^1 - P_i^0 P_j^0}{\sqrt{|S_{ij}|} \sqrt{P_i^1 P_j^1}}$$

$$\sin \phi_{ij} = \frac{P_i^2 P_j^2 - P_i^3 P_j^3}{\sqrt{|S_{ij}|} \sqrt{P_i^1 P_j^1}}$$

$$P_i^{\pm} = P_i^0 \pm P_i^3$$

• Now back to physics:

$$|p\rangle := \begin{pmatrix} 1 \\ \lambda \\ 0 \end{pmatrix} \quad |p]\ := \begin{pmatrix} 0 \\ \tilde{\lambda} \\ 1 \end{pmatrix} \quad \text{solve } \overset{\text{the}}{\gamma} \text{ massless}$$

Dirac equation  $\not{p}|p\rangle = 0 = \not{p}|p]$  for  $\gamma_\mu$  in the chiral representation

$\Rightarrow$  We can use these as massless Dirac ~~equation~~ <sup>spinors</sup>

$$u_+(p) = v_-(p) = |p\rangle \quad u_-(p) = v_+(p) = |p]$$

$$\bar{u}_+(p) = \bar{v}_-(p) = (0 \ \tilde{\lambda}) =: \langle p|$$

$$\bar{u}_-(p) = \bar{v}_+(p) = (\lambda \ 0) =: \langle p|$$

Here  $|p\rangle$  and  $\langle p|$  correspond to  $-\frac{1}{2}$  helicity states and  $|p]$  and  $\langle p|$  correspond to  $+\frac{1}{2}$  helicity states

⌈ We get useful identities

$$\langle k | \gamma^\mu | p \rangle = 0 = [k | p^\mu | p]$$

$$\langle p | \gamma^\mu | p \rangle = 2p^\mu = [p | \gamma^\mu | p \rangle$$

$|i\rangle, |j]$  is def. as  $|p_i\rangle, |p_j]$

$$[1 | \not{q} | 2 \rangle = \tilde{\lambda}_{1\dot{\alpha}} q^{\dot{\alpha}\alpha} \lambda_{2\alpha}$$

$$\langle 2 | \not{q} | 1 ] = \lambda_{2\dot{\alpha}} q^{\dot{\alpha}\alpha} \tilde{\lambda}_{1\alpha}$$

$$[i | \gamma^\mu | j \rangle = \langle j | \gamma^\mu | i ]$$

$$\text{and } [i | \gamma^\mu | j \rangle \langle l | \gamma_\mu | k ] = 2 [ik] \langle lj \rangle$$

these allow

this allows us to express all Lorentz-invariant quantities made of spinors ~~and~~, momenta and Dirac matrices ~~all~~ as combinations of  $\langle \cdot \rangle$  and  $[\cdot]$ .

- Polarization vectors for gluons ~~with~~ have helicity  $\pm 1$  and can be expressed as

$$E_{+,i}^{\alpha\dot{\alpha}}(p) = \sqrt{2} \frac{\tilde{\lambda}_i^{\dot{\alpha}} \mu_i^{\alpha}}{\langle \lambda_i \mu_i \rangle} \quad E_{-,i}^{\alpha\dot{\alpha}}(p) = \sqrt{2} \frac{\lambda_i^{\alpha} \tilde{\mu}_i^{\dot{\alpha}}}{[\tilde{\lambda}_i \tilde{\mu}_i]}$$

where  $\mu_i, \tilde{\mu}_i$  are arbitrary reference spinors. These refer to reference momenta  $q_i^{\alpha\dot{\alpha}} = \tilde{\mu}_i^{\dot{\alpha}} \mu_i^{\alpha}$  (these, which can be chosen as one wishes (but have to be massless) for each external gluon. This choice reflects gauge-invariance of freedom of the theory

We see that we have expressed the  $E_{\pm}^{\alpha\dot{\alpha}}$  in helicity spinors as well therefore also contractions involving these can be reduced to products of  $\langle \cdot \rangle [\cdot]$

~~we get~~

$E_{\pm}(p,q)$  refers to the polarization vector with momentum  $p$ , and reference momentum  $q$ .

We get useful relations  $(E_{\pm})^{\dot{\alpha}} = E_{\mp}$

$$p \cdot E_{\pm}(p,q) = 0 \quad E_{+}(p,q) \cdot E_{-}(p,q) = -1$$

and the concrete results

$$p_j \cdot E_{+,i} = \frac{1}{2} \lambda_{j\alpha} \tilde{\lambda}_{j\dot{\alpha}} E_{+,i}^{\alpha\dot{\alpha}} = \frac{-1}{\sqrt{2}} \frac{[\tilde{\lambda}_j \tilde{\lambda}_i] \langle \lambda_j \mu_i \rangle}{\langle \lambda_i \mu_i \rangle}$$

$$p_j \cdot E_{-,i} = \frac{1}{2} \lambda_{j\alpha} \tilde{\lambda}_{j\dot{\alpha}} E_{-,i}^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \frac{\langle \lambda_j \lambda_i \rangle [\tilde{\lambda}_j \tilde{\mu}_i]}{[\tilde{\lambda}_i \tilde{\mu}_i]}$$

where we used  $\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta}$

$$\epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma_{\alpha\dot{\alpha}}^{\mu} \sigma_{\beta\dot{\beta}}^{\nu}$$

6

$$\text{also } \epsilon_{+i} \cdot \epsilon_{+j} = \frac{\langle \mu_i \mu_j \rangle [\lambda_j \lambda_i]}{[\lambda_i \mu_j] \langle \lambda_j \mu_i \rangle}$$

$$\epsilon_{+i} \cdot \epsilon_{-j} = - \frac{\langle \mu_i \lambda_j \rangle [\mu_j \lambda_i]}{\langle \lambda_i \mu_j \rangle [\lambda_j \mu_i]}$$

$$\epsilon_{-i} \cdot \epsilon_{-j} = \frac{\langle \lambda_i \lambda_j \rangle [\mu_j \mu_i]}{[\lambda_i \mu_j] [\lambda_j \mu_i]}$$

from here we can already see, that a clever choice of reference momentum can lead to vanishing terms.

$$\epsilon_{\pm i}(q) \cdot q = 0$$

reference momentum  
↓

gluon momentum  
↖

$$\epsilon_{i,+}^*(q) \cdot \epsilon_{j,+}^*(q) = \epsilon_{i,-}^*(q) \cdot \epsilon_{j,-}^*(q) = 0$$

$$\epsilon_{i,+}^*(p_i) \cdot \epsilon_{j,-}^*(q) = \epsilon_{i,+}^*(q) \cdot \epsilon_{j,-}^*(p_j) = 0$$

Now we have everything we need to compute some amplitudes (at tree level).

### 4. Computing partial amplitudes

- We notice that  $A_n^{\text{tree}}(1^+, \dots, n^+)$  vanishes. To see this one chooses reference momenta  ~~$q \in n$~~   $q = q_1 = \dots = q_n$  and use that  $\epsilon_{i,+}(q) \cdot \epsilon_{j,+}(q) = 0$ .

Each term of the partial amplitude contains a least one factor of contracting gluon helicities. In the extreme case where we have a diagram made of only 3-gluon vertices we get  $n-2$  momenta and  $n$  gluon polarization vectors to work with. So 2 of these must contract!

Roughly the same argument also holds for  $A_n^{\text{tree}}(1^+, 2^-, \dots, n^-) = 0$ . Here we choose

reference momenta  $q_2 = \dots = q_n = p_1$ ,  $q_1 = q \neq p_1$ , then again all  $\epsilon_i \cdot \epsilon_j$  vanish.

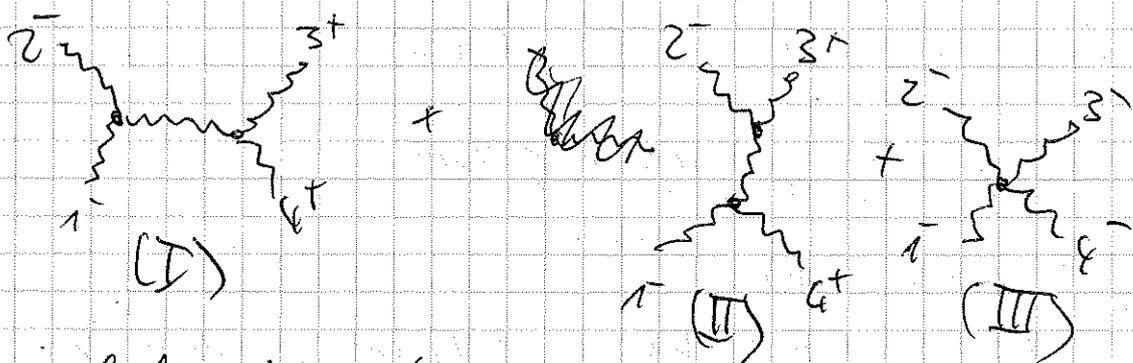
The first non-vanishing partial amplitude is the one with at least two flipped helicities e.g.  $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$ . It is also

called the maximally helicity violating <sup>(MHV)</sup> amplitude.

- let us now compute  $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$

$A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$  ~~is~~

Since it is a color ordered partial amplitude we write down the Feynman diagrams ~~and~~ and use their color ordered values.



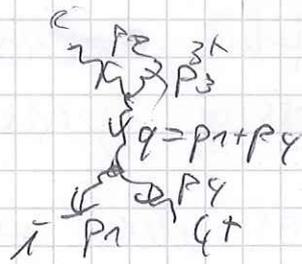
and fix the reference momenta  $q_1 = q_2 = p_4$   
 $q_3 = q_4 = p_1$

such that a lot of terms vanish.

With this choice we see that  $\epsilon_2 \cdot \epsilon_3$  is the only non-vanishing contraction.

From this we immediately get that diagram III doesn't contribute.

⑦ diagram (II) gives



$$\frac{ig^2}{2q^2} (\epsilon_2 \cdot \epsilon_3) \left[ \cancel{(\epsilon_2 \cdot p_1) (\epsilon_3 \cdot p_4)} + (\epsilon_1) \cdot (p_2 - p_3) (-q - p_1) \cdot (\epsilon_4) + (p_2 - p_3) \cdot (\epsilon_4) (p_4 + q) \cdot (\epsilon_1) \right] \stackrel{q}{=} 0$$

from choice  $E_q + p_1 = 0$   
of ref. momenta  $E_1 - p_4 = 0$

The only non-vanishing contribution comes from (I)

$$= \frac{ig^2}{2q^2} \epsilon_2 \cdot \epsilon_3 (p_2 + q) \cdot \epsilon_1 \times (-p_3 - q) \cdot \epsilon_4$$

$$= \frac{ig^2}{2s_{12}} \epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_1 p_3 \cdot \epsilon_4$$

$$= \left( -\frac{\langle 12 \rangle [34]}{\langle 21 \rangle [32]} \right) \left( \frac{1}{\sqrt{2}} \frac{\langle 12 \rangle [23]}{[14]} \right) \left( \frac{1}{\sqrt{2}} \frac{\langle 13 \rangle [34]}{[21]} \right)$$

$$= -ig^2 \frac{\langle 12 \rangle [21] [34]^2}{[12] [14] [21]} = \frac{\langle 12 \rangle^3}{\langle 22 \rangle \langle 34 \rangle \langle 21 \rangle} \cdot (-ig^2)$$

$$s_{12} = \langle 12 \rangle [21]$$

$$\Rightarrow s_{34} = \langle 34 \rangle [43]$$

momentum conservation

Which is our final result for today.

- Using this, ~~and~~ parity, cyclic permutation ~~and~~, crossing symmetry (pay attention ~~with~~ with helicities when replacing  $p \rightarrow -p$ ) and the vanishing amplitudes  $A_{\epsilon}^{\text{tree}}(1^+, 2^+, 3^+, 4^+) = A_{\epsilon}^{\text{tree}}(1^-, 2^+, 3^+, 4^+) = 0$  we have already computed all possible independent partial amplitudes.

## 5. Outlook

In the next sessions we will expand on this knowledge and then show, how to use the developed methods to describe the potential of inspiraling black holes. A problem very important to GW-astronomy.