BCFW

Refs:
- Elvang, Huang book 2015, 1308.1697
- B. Truijen master's thesis 2012
- Schwartz "QFT & SM" book 2013
- Henn, Plefka book 2014

* **Plan:** (page)
- reminder about color separation & helicity formalism ①
- on-shell Britto-Cachazo-Feng-Witten (BCFW) recursion ②
- Parke-Taylor formula ③
- remarks ④

* **Recap of prev. lecture:** scattering of n gluons

$$\mathrm{Amp}(\underbrace{\{p_1 \ldots p_n\}}_{\text{momenta}}, \underbrace{\{h_1 \ldots h_n\}}_{\text{helicity}}, \underbrace{\{a_1, \ldots a_n\}}_{\text{color}}) \in \mathbb{C}$$

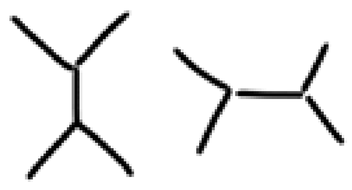
momenta: $p_i \in M^{\mathbb{C}}$, $p_i^2 = 0$; helicity: $h_i = \pm$

- **Color ordering:**

$$\sum_{\sigma \in S_n / \mathbb{Z}_n} \mathrm{tr}\left[t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}\right] \underbrace{\mathrm{Amp}(\{p_{\sigma(1)} \cdots p_{\sigma(n)}\}, \{h_{\sigma(1)} \ldots h_{\sigma(n)}\})}_{\text{color ordered partial amplitude}}$$

S_n: symm. group; $\mathbb{Z}_n$: cyclic

only planar graphs



- **Spinor helicity variables:**

$\{p, q, k\}$ are null

- $p^{\dot\alpha\alpha} = p_\mu (\bar\sigma^\mu)^{\dot\alpha\alpha} = -|p\rangle^{\dot\alpha} [p|^{\alpha}$ ($\dot\alpha$: 1, 2; α: 1, 2)
- $\langle pq\rangle [pq] = 2p \cdot q = (p+q)^2$
- $\langle p|k|q] = -\langle pk\rangle [kq]$
- if $p \in M^{\mathbb{C}}$, then $|p\rangle$ and $[p|$ are indep.
  if $p \in M^{\mathbb{R}}$, then $|p\rangle = ([p|)^*$
- $\langle pq\rangle = \langle p|_{\dot\alpha} |p\rangle^{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta} |p\rangle^{\dot\beta} |p\rangle^{\dot\alpha}$
- $\langle p|P|k] = \langle p|_{\dot\alpha} P^{\dot\alpha\beta} |k]_\beta$
- polarization vectors:

$$\epsilon_-^\mu (p; q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}\,[qp]}$$

$$\epsilon_+^\mu (p; q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\,\langle qp\rangle}$$

notation:

$$\underbrace{(0, \langle q|_{\dot\alpha})}_{\langle q|} \underbrace{\begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix}}_{\gamma^\mu} \underbrace{\begin{pmatrix} |p]_\beta \\ 0 \end{pmatrix}}_{|p]}$$

(bonus material)

* **3 gluon amps:** induction base for BCFW

- for real $p_i^2 = 0$ $i = 1, 2, 3$ corresp. $\mathrm{Amp}(p_1, p_2, p_3) = 0$ since

$$\underbrace{p_1 + p_2 + p_3 = 0}_{\text{mom. conservation}} \Rightarrow \begin{cases} (p_1 + p_2 + p_3)^2 = 0 \\ p_i (p_1 + p_2 + p_3) = 0 \end{cases} \Rightarrow p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1 = 0 \quad \left(\text{or } \langle ij\rangle [ij] = 0\right)$$

so there are no non-vanishing Mandelstam invs & the 3 gluon real Amp $= 0$

- for complex $p_i^2 = 0$ $|i\rangle$ & $[i|$ are indep. so $p_i \cdot p_j = 0$ is solved by either $\langle ij\rangle = 0$ or $[ij] = 0$

$|1\rangle \propto |2\rangle \propto |3\rangle$ or $[1| \propto [2| \propto [3|$

- from this one can show that (either using Feynman diagrams or more general dimensional arguments)

$$\mathrm{Amp}(1^- 2^- 3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

$$\mathrm{Amp}(1^+ 2^+ 3^-) = \frac{[12]^3}{[23][31]}$$

* **Parke-Taylor formula:** scattering of n-gluons @ tree level for the case where only 2 gluons have $h_i = h_j = -1$

$$\boxed{\mathrm{Amp}(1^+ \ldots i^- \ldots j^- \ldots n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}}$$

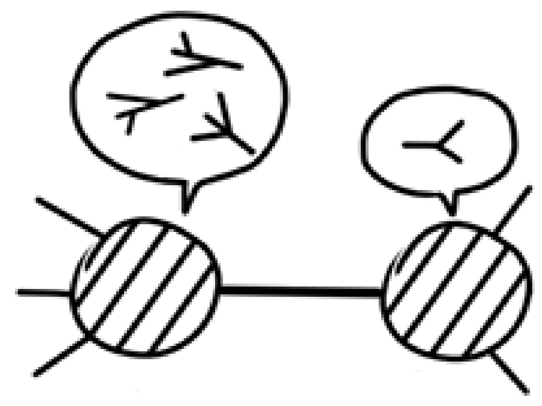
our main goal will be to prove this for $i = 1, j = 2$ (page ③)

BCFW

* BCFW recursion relation: pick 2 momenta p_i and p_j and define shifts for $z \in \mathbb{C}$

$$\left.\begin{array}{ll} |\hat{1}] = |1] + z|2] & |\hat{1}\rangle = |1\rangle \\ |\hat{2}] = |2] & |\hat{2}\rangle = |2\rangle - z|1\rangle \end{array}\right\} \Rightarrow \begin{array}{l} \hat{p}_1 = p_1 - z|1\rangle[2| \\ \hat{p}_2 = p_2 + z|1\rangle[2| \end{array}$$

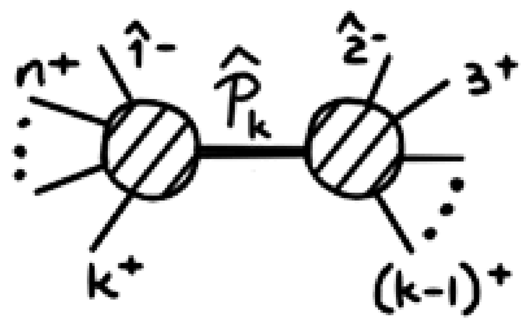
- Properties: $\hat{p}_1 + \hat{p}_2 = p_1 + p_2$ momentum conservation
  $\hat{p}_1^2 = \hat{p}_2^2 = 0$ shifted mom. is on-shell



- Define: $r = |1\rangle[2|$, $r^2 = 0$

- This changes $\text{Amp} \longmapsto \underbrace{\text{Amp}(z)}_{\text{holomorphic}}$ with original amplitude $= \text{Amp}(0)$

- Analytic properties of $\text{Amp}(z)$:
  - rational function of $\{|i\rangle, [i|, z\}$
  - $\text{Amp}(0)$ has poles where denom. of Feynman prop. $\to 0$, color ordering implies denom. $\frac{1}{(p_i + p_{i+1} + \ldots + p_j)^2}$ sum of adjacent



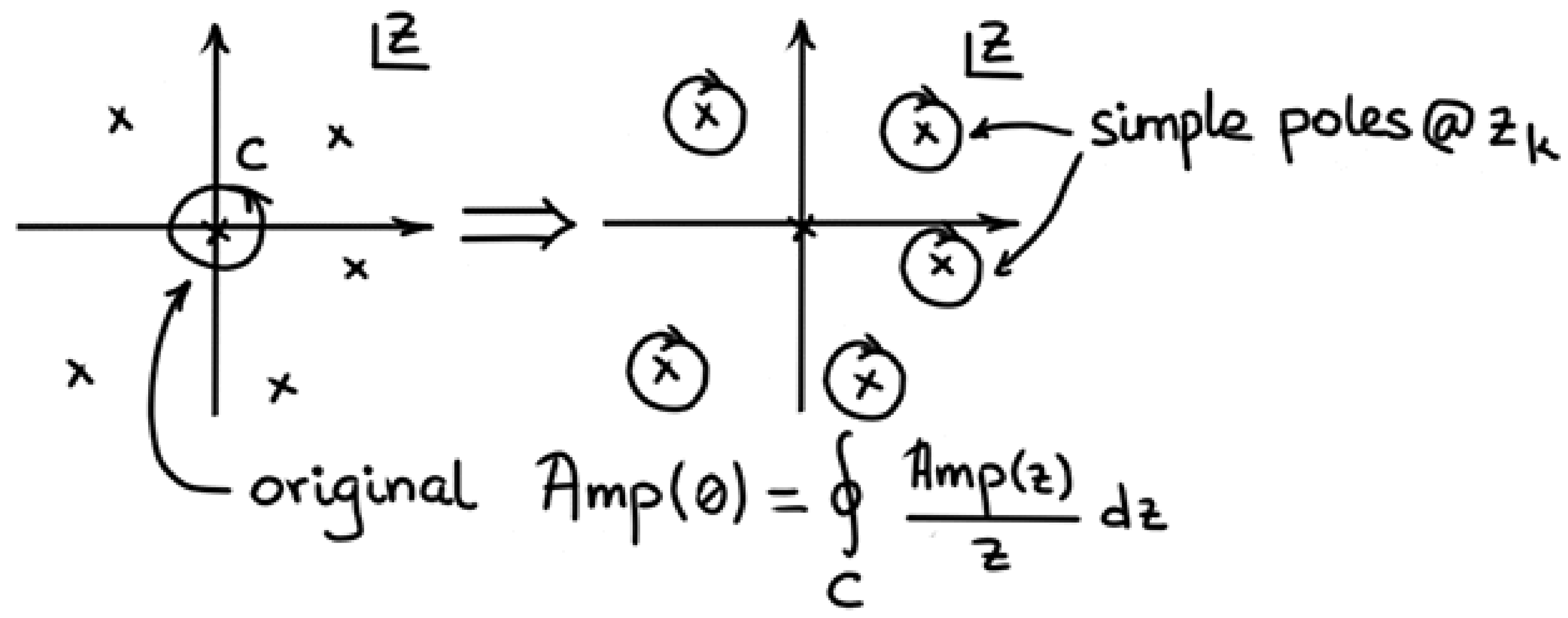
  - $\text{Amp}(z)$ has only simple poles (Feynman gauge) in z of form

$$\frac{1}{\hat{P}_k^2(z)} = \frac{1}{(p_k + \ldots + p_n + \hat{p}_1(z))^2} = \frac{1}{P_k^2 - 2z\, r \cdot P_k} \qquad P_k = p_k + \ldots + p_1$$

- Now consider holomorphic fun. $\frac{\text{Amp}(z)}{z}$ & use Cauchy's theorem:

$$\text{Amp}(0) = -\sum_{z_k} \underset{z = z_k}{\text{Res}} \frac{\text{Amp}(z)}{z} + \text{Boundary}$$

possible pole @ ∞ (see page 4)

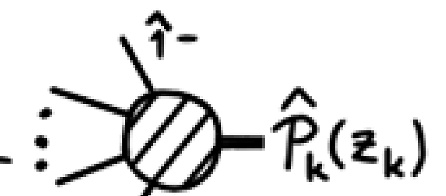


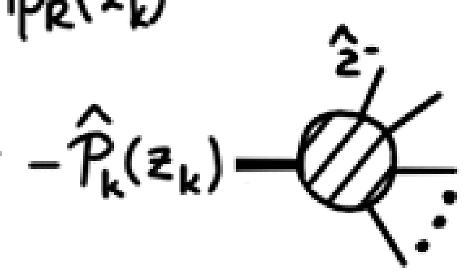
where poles z_k are located at:

$$z_k = \frac{P_k^2}{2 r \cdot P_k}, \quad k \in \{4, \ldots, n\}$$

- Residues @ z_k:

$$\text{Amp}(z) \xrightarrow[z \to z_k]{} -\underbrace{\frac{z_k}{z - z_k}}_{\text{pole}} \text{Amp}_L(z_k) \cdot \frac{1}{P_k^2} \cdot \text{Amp}_R(z_k)$$

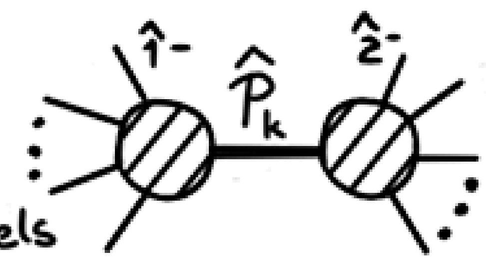




- So we get the BCFW recursion relation:

amps with fewer legs

$$\boxed{\text{Amp}(0) = \sum_{\substack{\text{channels} \\ k}} \text{Amp}_L(z_k) \cdot \frac{1}{P_k^2} \cdot \text{Amp}_R(z_k)} = \sum_{\substack{\text{channels} \\ k}}$$



sum over all factorization channels that separate $\hat{1}$ and $\hat{2}$

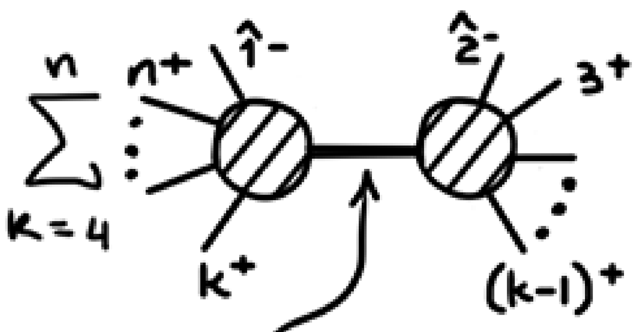
* Discussion: (bonus material)

- we assumed that there is no Boundary terms from pole @ ∞. It holds (page 4) for some helicity config's.
- also can consider more general 2 lines shifts that lead to similar formulas $|\hat{i}] = |i] + z|j]$ $|\hat{i}\rangle = |i\rangle$, $|\hat{j}] = |j]$ $|\hat{j}\rangle = |j\rangle - z|i\rangle$
- implicit: summation over all on-shell states. For gluons it means $\sum_{\text{helicity}}$

BCFW

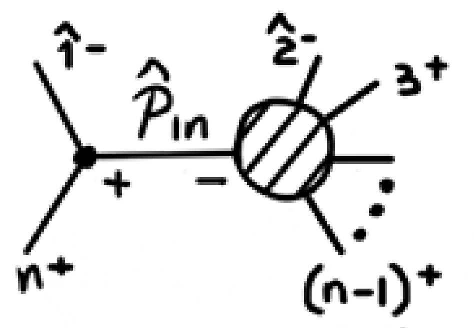
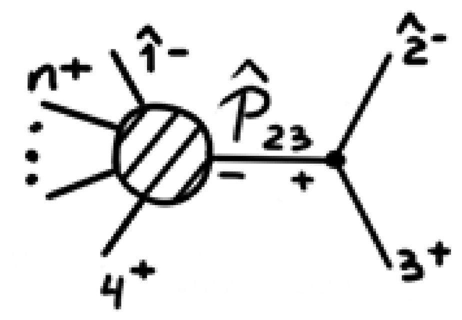
* Proof of Parke-Taylor formula:

- only consider case of $(1^- 2^- 3^+ \ldots n^+)$
- by induction in #gluons $= n$
- BCFW recursion for $i = 1^-, j = 2^-$ shift
- $\mathrm{Amp}(1^- 2^+ \ldots n^+) = 0$ for $n > 3$

- $\mathrm{Amp}(1^- 2^- 3^+ \ldots n^+) = \sum_{k=4}^{n}$  $= \sum_{k=4}^{n} \sum_{h_k = \pm} \mathrm{Amp}(\underbrace{\hat{1}^-, \hat{P}_k^{h_k}, k^+ \ldots n^+}_{n-k+3}) \times \frac{1}{P_k^2} \times \mathrm{Amp}(\underbrace{-\hat{P}_k^{-h_k}, \hat{2}^-, 3^+ \ldots (k-1)^+}_{k-1})$

$$\begin{cases} \hat{P}_k = \hat{p}_2(z_k) + p_3 + \ldots + p_{k-1} \\ \hat{P}_k^2 = 0 \end{cases}$$

$P_k = p_2 + p_3 + \ldots + p_{k-1} = \hat{P}_k(0)$

- $\mathrm{Amp}(1^- 2^- 3^+ \ldots n^+) =$  $+$  only 2 terms!

$\hat{P}_{1n} = \hat{p}_1 + p_n \longleftarrow$ must be null

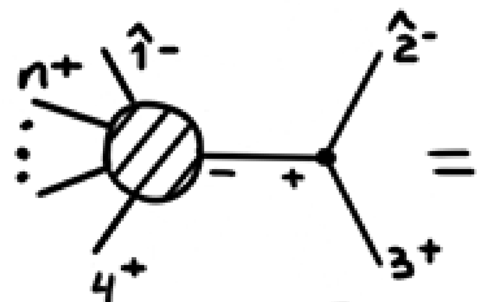
$\Downarrow$

$\hat{P}_{1n}^2 = \langle 1 n \rangle [\hat{1} n]$ vanishes

- $\mathrm{Amp}(\hat{1}^-, -\hat{P}_{1n}^+, n^+) = \frac{[\hat{P}_{1n} n]^3}{[n \hat{1}][\hat{1} \hat{P}_{1n}]} \overset{?}{=} 0 \longleftarrow$ to show that we observe that all $[\ldots] \propto [\hat{1} n]$

  - $|\hat{P}_{1n}\rangle [\hat{P}_{1n}| = |\hat{1}\rangle [\hat{1}| + |n\rangle [n|$
  - $|\hat{P}_{1n}\rangle [\hat{P}_{1n} n] = |\hat{1}\rangle [\hat{1} n] \longrightarrow 0$
  - $|\hat{P}_{1n}\rangle [\hat{P}_{1n} \hat{1}] = -|\hat{P}_{1n}\rangle [\hat{1} \hat{P}_{1n}] = |n\rangle [n \hat{1}] \longrightarrow 0$

(bonus material)

- $\mathrm{Amp}(1^- 2^- 3^+ \ldots n^+) =$  $= \mathrm{Amp}(\hat{1}^-, \hat{P}_{23}^-, 4^+ \ldots n^+) \times \frac{1}{P_{23}^2} \times \mathrm{Amp}(-\hat{P}_{23}^+, \hat{2}^-, 3^+) \longleftarrow$ only 1 term!

induction hypoth.

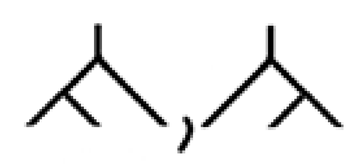
$$= \frac{\langle \hat{1} \hat{P}_{23} \rangle^4}{\langle \hat{1} \hat{P}_{23} \rangle \langle \hat{P}_{23} 4 \rangle \ldots \langle n \hat{1} \rangle} \times \frac{1}{\langle 23 \rangle [23]} \times \frac{[3 \hat{P}_{23}]^3}{[\hat{P}_{23} \hat{2}][\hat{2} 3]}$$

  - $\langle \hat{1} \hat{P}_{23} \rangle [3 \hat{P}_{23}] = \langle \hat{1} | \hat{p}_2 + p_3 | 3] = \langle \hat{1} \hat{2} \rangle [\hat{2} 3] = -\langle 12 \rangle [23]$
  - $\langle \hat{P}_{23} 4 \rangle [\hat{P}_{23} \hat{2}] = \langle 4 | 3 | 2] = -\langle 34 \rangle [23]$

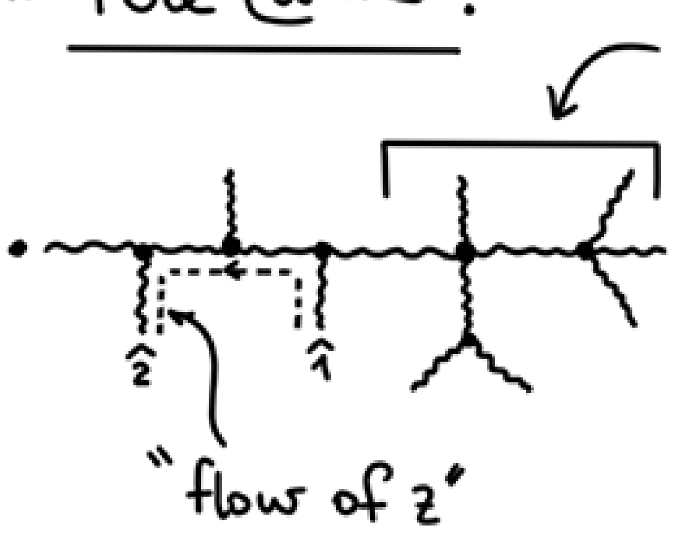
(bonus material)

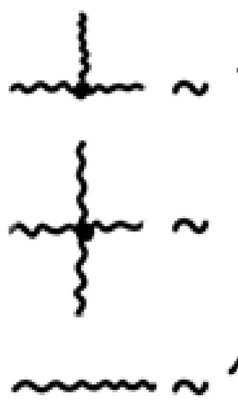
- $$\mathrm{Amp}(1^- 2^- 3^+ \ldots n^+) = \frac{-\langle 12 \rangle^3 [23]^3}{(-\langle 34 \rangle [23]) \langle 45 \rangle \ldots \langle n1 \rangle \langle 23 \rangle [23]^2} = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \ldots \langle n1 \rangle}$$ ▨

* Discussion: (bonus material)

- Similarly one can derive the general Parke-Taylor formula for $\mathrm{Amp}(1^+ \ldots i^- \ldots j^- \ldots n^+)$
- How many Feynman dias are resummed? Consider a lower bound: trivalent color ordered graphs
  - trivalent with 4 legs: $\{$  $\}$
  - trivalent with n legs: counted by Catalan numbers $C_{n-2} = \frac{1}{n}\binom{2(n-2)}{n-2}$
  - grow as $\sim \frac{4^n}{n^{3/2}\sqrt{n}}$

* Pole @∞:





$$\epsilon_1^{+\dot\alpha\alpha} = \sqrt{2}\,\frac{|q_1\rangle^{\dot\alpha}[\hat1|^\alpha}{\langle \hat1 q_1\rangle} \sim z \qquad |\hat1] = |1] + z|2]$$

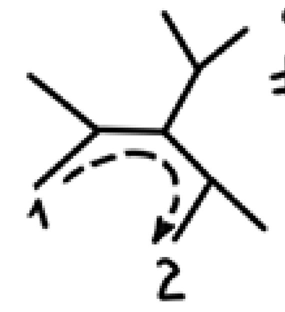
$$\epsilon_1^{-\dot\alpha\alpha} = \sqrt{2}\,\frac{|\hat1\rangle^{\dot\alpha}[q_1|^\alpha}{[\hat1 q_1]} \sim \frac{1}{z} \qquad |\hat2] = |2]$$

$$|\hat1\rangle = |1\rangle \qquad |\hat2\rangle = |2\rangle - z|1\rangle$$

$$(\epsilon_+(p;q))^{\dot\alpha\beta} = \sqrt{2}\,\frac{|q\rangle^{\dot\alpha}[p|^\beta}{\langle qp\rangle} \qquad \epsilon_2^{+\dot\alpha\alpha} = \sqrt{2}\,\frac{|q_2\rangle^{\dot\alpha}[\hat2|^\alpha}{\langle \hat2 q_2\rangle} \sim \frac{1}{z}$$

$$(\epsilon_-(p;q))^{\dot\alpha\beta} = \sqrt{2}\,\frac{|p\rangle^{\dot\alpha}[q|^\beta}{[qp]} \qquad \epsilon_2^{-\dot\alpha\alpha} = \sqrt{2}\,\frac{|\hat2\rangle^{\dot\alpha}[q_2|^\alpha}{[\hat2 q_2]} \sim z$$

- Consider the "worst" case: trivalent graphs
  - without ϵ's these graphs scale as $\sim z$
  - with ϵ's:



$\hat1^+\ \hat2^-$	z^3 ← naively
$\hat1^-\ \hat2^+$	$1/z$
$\hat1^+\ \hat2^+$	z
$\hat1^-\ \hat2^-$	z

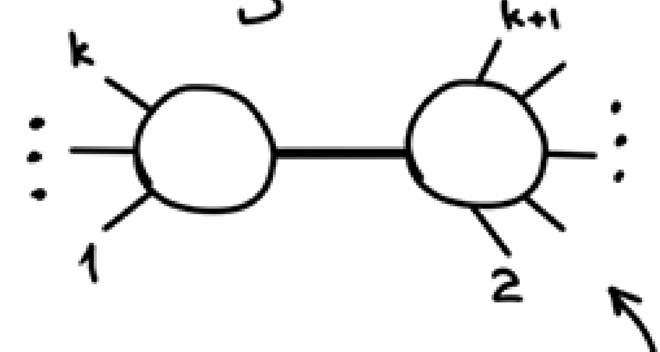
can show that these also $\sim 1/z$

- For the proof of Parke-Taylor we used $\hat1^-\ \hat2^-$

* Spurious poles:

locality

- color ordered tree amps: phys. poles @ adjacent external lines go on-shell
- MHV: don't have multi-particle poles (see Parke-Taylor)
  $(++-\ldots-)$



- NMHV: have both 2-particle $\langle i, i+1\rangle$ & 3-particle $P_{126}^2 = (p_6+p_1+p_2)^2$
  $(+++-\ldots-)$ but also spurious poles @ $\langle 5|1+6|2]$ doesn't correspond to a phys. pole
  Residues of such poles should cancel

$$\mathrm{Amp}(1^-2^-3^-4^+5^+6^+) = \frac{\langle 3|1+2|6]^3}{P_{126}^2\,[21][16]\langle34\rangle\langle45\rangle\langle5|1+6|2]} + \frac{\langle 1|5+6|4]^3}{P_{156}^2\,[23][34]\langle56\rangle\langle61\rangle\langle5|1+6|2]}$$

- Typically BCFW makes expressions compact at cost of introducing spurious poles
  $\Rightarrow$ bluring the locality

* When does it work?

- Yang-Mills theory & gluon scattering
- Scalar QED
- Scalar theory
- $N=4$ super Yang-Mills theory
- Gravity

* 3 gluon amps: (bonus material) derive 3pt. amps from little group scaling

leave on-shell mom. inv.

- on-shell mom. $p^{\dot\alpha\alpha} = -|p\rangle^{\dot\alpha}[p|^\alpha$ is inv. under rescaling $|p\rangle \mapsto t|p\rangle$, $|p] \mapsto \frac{1}{t}|p]$
- in amps. only external lines scale under (not verticies or props) $\Rightarrow$ for gluons $\epsilon^h \mapsto t^{-2h}\epsilon^h$ which implies

$$\mathrm{Amp}(\ldots\{t_i|i\rangle, \tfrac{1}{t_i}|i], h_i\}\ldots) = t_i^{-2h_i}\,\mathrm{Amp}(\ldots\{|i\rangle, |i], h_i\}\ldots)$$

- consider ansatz $\mathrm{Amp}(1^{h_1}2^{h_2}3^{h_3}) = c\,\langle12\rangle^{x_{12}}\langle13\rangle^{x_{13}}\langle23\rangle^{x_{23}} \Rightarrow -2h_1 = x_{12}+x_{13},\ -2h_2 = x_{12}+x_{23},\ -2h_3 = x_{13}+x_{23}$
- so we see that 3 gluon amp. is uniquely fixed $\mathrm{Amp}(1^{h_1}2^{h_2}3^{h_3}) = c\,\langle12\rangle^{h_3-h_1-h_2}\langle13\rangle^{h_2-h_1-h_3}\langle23\rangle^{h_1-h_2-h_3}$