Workshop Seminar Series - Hot Topics in QFT and String Theory

Amplitudes III - Loop Technology

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1 Introduction

In the last lecture we have seen the BCFW recursion, which allowed the reduction of tree diagrams into diagrams with fewer legs. Although this allows a huge simplification of the computations, simply by recycling results from diagrams with fewer legs to those with more, tree diagrams are not the full story yet. As taught in any introductory lecture on quantum field theory, loop diagrams contribute as higher order corrections to the tree-level amplitude and thus have to be computed as well.

In this seminar talk, we will have a look at one of the many techniques used to make the computation of loop diagrams simpler and often even possible. It turns out, that many of the appearing integrals are actually related by the so-called integration-by-parts relations and thus only few of them have to be actually computed. The plan of this talk is as follows. First of all, we will remind ourselves of the basic setup of the computation of loop integrals and work through one of the simplest examples, the bubble integral. This will demonstrate the essential techniques that are required and used to perform such integrals. We will next consider the (slightly generalized) family of such bubble integrals and derive their integration-by-parts identities, which allow us to express all these integrals in terms of a two-dimensional basis. Finally, we will discuss in which directions this technique can be applied and generalized. A good reference to get an overview of these techniques is [1].

Remark 1.1. Every loop in a Feynman diagram introduces an integration over the momentum of the internal particle. It is mostly for this reason, that loop integrals are difficult to compute. Furthermore, these integrals often contain divergent parts that have to be renormalized. We take care of these divergences by working in dimensional regularization. Let us briefly remind ourselves of this concept.

Definition 1.2. Consider a loop integral in D_0 spacetime dimensions (we will only consider $D_0 = 4$) over one loop momentum

$$I = \int \frac{d^{D_0}k}{i\pi^{D_0/2}} F(k).$$

The integrand $F(k)$ can be determined by the corresponding Feynman diagram and contains the propagators of the loop. The dimensionally regularized integral is obtained by considering the same integral instead in $D = D_0 - 2\epsilon$ dimensions for some small ϵ. More precisely, we analytically continue the integral as a function of the integer dimension to arbitrary (even complex) dimensions. We can then consider the Laurent expansion of the integral around $\epsilon = 0$, which in general will be given as

$$I = \sum_{k \geq k_0} I_k \epsilon^k \tag{1.1}$$

for some $k_0 \in \mathbb{Z}$. We can now identify the divergent parts as those where ϵ appears with a negative power and deal with them in a suitable way, eg. by introducing counter-terms.

2 Essential computation techniques: the bubble integral

We now consider a simple loop integral as an example of the usual integration techniques required in order to solve such integrals. Although the actual computation in our example will not be too hard, it already requires several non-trivial steps and highlights the difficulties one encounters when computing more difficult loop integrals.

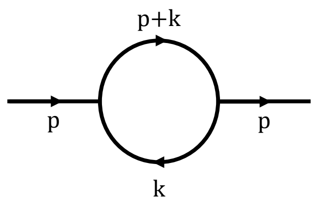

Figure 1. The bubble diagram, a one-loop diagram with a particle of momentum p entering and exiting the loop.

Definition 2.1. We consider the bubble diagram as depicted in figure 1. From the diagram we can read off the so-called bubble integral, assuming we have a theory with the usual scalar propagators and the particles involved are of mass m, which is given by

$$B(p^2, m^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(-k^2 + m^2)(-(k+p)^2 + m^2)},$$

where we work in dimensional regularization $D = 4 - 2\epsilon$. The additional factor $e^{\gamma_E \epsilon}$, where γ_E is the Euler-Mascheroni constant, is chosen for later convenience. Note that we have suppressed the Feynman prescription, where $(i\delta)$ for an infinitesimal δ is added in each propagator to deal with its poles. This is equivalent to specifying the contour of integration. We chose the usual causal contour, where we extend the integration along the real part of k_0 into the lower half of the complex plane for negative values and into the upper half for positive values.

Proposition 2.2. The bubble integral can be computed to give

$$B(p^2, m^2) = e^{\gamma_E \epsilon} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \left[x(x-1)p^2 + m^2\right]^{\frac{D}{2}-2}.$$

Remark 2.3. Before we turn to the proof of proposition 2.2, let us remark that so far we have not made use of dimensional regularization. In fact, the result of proposition 2.2 can be used to demonstrate how precisely we analytically continue to arbitrary dimension. In the original integral, the dimension appears directly only in the measure, whereas now the dimension appears in the arguments of meromorphic functions only, which can easily be continued to complex arguments.

One crucial step required to compute the bubble integral is to rewrite the integral in the so-called Feynman parametrization. Let us recall the basic result before we begin the actual computation. The reader is invited to prove it, which can be done by rewriting the fraction in terms of an integral over the complex plane.

Lemma 2.4. For complex numbers A and B, such that the line segment between them does not contain the origin, we can write

$$\frac{1}{AB} = \int_0^1 \frac{dx}{\left[xA + (1-x)B\right]^2}.$$

Proof of prop. (2.2). The guiding principle for the first part of the proof will be to rewrite the integral such that we only integrate over k^2. Making the spherical invariance of the integrand obvious, this will allow us to perform the spherical part of the momentum integration leaving us with the radial part for the remainder of the proof. We thus first apply the Feynman parametrization and obtain, for now absorbing the numerical constants into the prefactor C, the integral

$$B(p^2, m^2) = C \int d^D k \int_0^1 dx \, \left[x(-k^2 + m^2) + (1-x)\left(-(k+p)^2 + m^2\right)\right]^{-2}.$$

We can now complete the square into k and make use of the translation invariance of the measure. Thus, only considering the square root of the inverse integrand I, we compute

$$\begin{aligned} I &= -xk^2 + xm^2 + (1-x)m^2 - (1-x)(k+p)^2 \\ &= -xk^2 + m^2 - (1-x)(k^2 + 2kp + p^2) \\ &= -k^2 - 2(1-x)kp + m^2 + (x-1)p^2 \\ &= -\left(k + (x-1)p\right)^2 + (x-1)^2 p^2 + (x-1)p^2 + m^2 \\ &= -\left(k + (x-1)p\right)^2 + x(x-1)p^2 + m^2. \end{aligned}$$

Plugging this back into the integral and shifting the momentum integration, we obtain

$$B(p^2, m^2) = C \int d^D k \int_0^1 dx \, \left[-k^2 + x(x-1)p^2 + m^2\right]^{-2}.$$

Note that until now working in Minkowski space, we have $k^2 = k_0^2 - \boldsymbol{k}^2$. We now perform a Wick rotation, that is we effectively substitute $k_0 \to \tilde{k}_0 = -ik_0$. Note that this is possible due to the choice of integration contour. The measure thus picks up a factor of i and k^2 becomes $-k_E^2 = -\tilde{k}_0^2 - \boldsymbol{k}^2$, whereas the subscript E denotes that we now mean the Euclidean norm. After this Wick rotation, the integral becomes

$$B(p^2, m^2) = iC \int d^D k \int_0^1 dx \, \left[k_E^2 + x(x-1)p^2 + m^2\right]^{-2},$$

whereas the momentum integration is now over $\mathbb{R}^D$, ie the Euclidean space. As we are from now on only working in Euclidean space in this proof, we will drop the E and work with k as a usual Euclidean vector. We now directly see that the integrand is spherically invariant. We thus split the integration into the radial and spherical parts, that is

$$d^D k = |k|^{D-1} \, d\,|k| \, d\Omega_D,$$

whereas $d\Omega_D$ denotes the integration over the spherical part, that is over the unit $(D-1)$-sphere. As the integrand is independent of this spherical part, we can perform this integration directly by remembering that

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}.$$

Note that we now see the reason for including the normalisation of $i\pi^{\frac{D}{2}}$ in the measure, which is introduced in order to cancel the i from the Wick rotation and the volume of the sphere. For $D = 4$, the gamma function appearing in this volume equals one, we will, however, keep it for now. Renaming the integration variable for convenience, we have

$$B(p^2, m^2) = \frac{e^{\gamma_E \epsilon}}{\Gamma\left(\frac{D}{2}\right)} \int_0^1 dx \int_0^\infty \frac{2y^{D-1} dy}{\left[y^2 + x(x-1)p^2 + m^2\right]^2}.$$

We define $K = x(x-1)p^2 + m^2$ and substitute the integration variable from y to $u = y^2$ and then from u to $Kv = u$. The measure thus transforms as

$$2y^{D-1} dy = u^{\left(\frac{D}{2}-1\right)} du = K^{\frac{D}{2}} dv$$

and the integrand becomes $\left[y^2 + K\right]^{-2} = K^{-2} \left[v+1\right]^{-2}$, such that the integral becomes

$$B(p^2, m^2) = \frac{e^{\gamma_E \epsilon}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty dv \, v^{\left(\frac{D}{2}-1\right)} \left(1+v\right)^{-2} \int_0^1 dx \; K^{\left(\frac{D}{2}-2\right)},$$

whereas the integrals now factor, as all x-dependence is in K. The integration over v can be recognized to be one of the definitions of the beta function

$$B(\alpha, \beta) = \int_0^\infty dt \, t^{\alpha-1} \left(1+t\right)^{-(\alpha+\beta)},$$

which can be evaluated in terms of the more usually known gamma function by

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

We thus see that the integral over v is given by a beta function with $\alpha=\frac{D}{2}$ and $\beta=2-\frac{D}{2}$, such that, using its representation in terms of gamma functions, we get

$$\begin{aligned}
B(p^2,m^2)&=\frac{e^{\gamma_E\epsilon}}{\Gamma\left(\frac{D}{2}\right)}\frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(2\right)}\int_0^1 dx\ K^{\left(\frac{D}{2}-2\right)},\\
&=e^{\gamma_E\epsilon}\Gamma\left(2-\frac{D}{2}\right)\int_0^1 dx\left[x(x-1)p^2+m^2\right]^{\frac{D}{2}-2},
\end{aligned}$$

using that $\Gamma(2)=1!=1$. □

Having done most of the integration work, we can now do the dimensional regularization and expand the result into a power series in terms of ϵ. In order to do so, let us first collect a few results required for this expansion.

Lemma 2.5. Working explicitly in $D_0=4$, we get

$$\begin{aligned}
A^{\frac{D}{2}-2}&=A^{-\epsilon}=1-\epsilon\log A+\frac{1}{2}\epsilon^2\log^2(A)+\mathcal{O}(\epsilon^3),\\
\Gamma\left(2-\frac{D}{2}\right)&=\Gamma\left(\epsilon\right)=\frac{1}{\epsilon}e^{-\gamma_E\epsilon}\exp\left(\sum_{k=2}^{\infty}\epsilon^k\frac{(-1)^k}{k}\zeta_k\right)=e^{-\gamma_E\epsilon}\left(\frac{1}{\epsilon}+\frac{\zeta_2}{2}\epsilon+\mathcal{O}(\epsilon^2)\right),
\end{aligned}$$

where $\zeta_k=\zeta(k)$ are the integer values of the Riemann zeta function.

Remark 2.6. Note that the reason we included the exponential normalisation in the integral in definition 2.1 is the inverse factor coming from the expansion of the gamma function.

These results allow us to expand the integrand and the gamma function in front of the remaining integral. We still, however, have to perform the integration over x, which can now be done term by term. Let us state and prove this for the somewhat simpler case of $m=0$.

Corollary 2.7. The expansion of the massless bubble integral into powers of ϵ is given for the first few terms by

$$B(p^2,0)=\frac{1}{\epsilon}+2-\log\left(-p^2\right)+\epsilon\left[\frac{1}{2}\log^2\left(-p^2\right)-2\log\left(-p^2\right)-\frac{1}{2}\zeta_2+4\right]+\mathcal{O}\left(\epsilon^2\right).$$

Proof. We begin with the remaining integration, which for $m=0$ and using $D=4-2\epsilon$ can be written as

$$\int_0^1 dx\left[x(x-1)p^2+m^2\right]^{\frac{D}{2}-2}=\left(-p^2\right)^{-\epsilon}\int_0^1 dx\left[x(1-x)\right]^{-\epsilon}.$$

Using the expansion as stated in lemma (2.5), we can rewrite the integration as

$$\begin{aligned}\int_0^1 dx\, [x(1-x)]^{-\epsilon} &= \int_0^1 dx \left(1 - \epsilon \log\left(x(1-x)\right) + \frac{1}{2}\log^2\left(x(1-x)\right) + \mathcal{O}(\epsilon^3)\right) \\ &= 1 + 2\epsilon + \left(4 - \frac{\pi^2}{6}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) = 1 + 2\epsilon + (4 - \zeta_2)\,\epsilon^2 + \mathcal{O}(\epsilon^3),\end{aligned}$$

whereas we have evaluated the integrations termwise and used the well-known identity $\zeta_2 = \frac{\pi^2}{6}$. We now have all the pieces required for the expansion and can put them together, also using lemma (2.5) for the expansion of $\left(-p^2\right)^{-\epsilon}$, to get

$$\begin{aligned}B(p^2, 0) = &\left(\frac{1}{\epsilon} + \frac{\zeta_2}{2}\epsilon + \mathcal{O}(\epsilon^2)\right)\left(1 - \epsilon \log\left(-p^2\right) + \frac{1}{2}\epsilon^2 \log^2\left(-p^2\right) + \mathcal{O}(\epsilon^3)\right) \cdot \\ &\cdot \left(1 + 2\epsilon + (4 - \zeta_2)\,\epsilon^2 + \mathcal{O}(\epsilon^3)\right) \\ =&\frac{1}{\epsilon} + \frac{1}{\epsilon}\left(2\epsilon - \epsilon \log\left(-p^2\right)\right) + \frac{1}{\epsilon}\left(\frac{1}{2}\epsilon^2 \log^2\left(-p^2\right) + (4 - \zeta_2)\,\epsilon^2 - 2\epsilon^2 \log\left(-p^2\right)\right) \\ &+ \frac{\zeta_2}{2}\epsilon + \mathcal{O}\left(\epsilon^2\right),\end{aligned}$$

which, after rearranging some terms, is exactly the stated result. □

Remark 2.8. Before we go on with the next section, let us remark on the result. We see that the coefficients of ϵ are given by logarithms and zeta values. This is a common feature for a large class of amplitudes and has connections to pure mathematics. It is established that the coefficients of these power series of Feynman integrals are from a certain class of numbers (the so-called periods, to which the logarithms and zeta-values belong) and in many cases of $\mathcal{N} = 4$ Super Yang Mills amplitudes only (generalizations of) the logarithms and zeta-values appear. These have a very rich mathematical structure, which allows the computation of such amplitudes without ever having to refer to Feynman diagrams. A good introduction to this fascinating area of amplitudes research is [2].

3 Integration-by-parts identities

In the previous section we have seen how to compute loop integrals by considering the bubble integral as an example. Despite being one of the simplest loop integrals, solving it required a considerable amount of non-trivial steps. For more complicated diagrams, many more such diagrams appear, most of which are extremely difficult to solve just using the tools from the example. We thus require some more techniques that allow us to deal with these quantities. Following a core principle of the amplitudes field, we will manipulate the loop integrals such that we can reduce the required amount of computation by relating the integrals to each other thus allowing us to recycle results.

Definition 3.1. Closely following the loop integral example, we now consider the (generalized) family of bubble integrals of the form

$$J(a_1, a_2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(-k^2 + m^2)^{a_1}(-(k+p)^2 + m^2)^{a_2}},$$

for integer a and b. Note that $J(a_1, a_2) = J(a_2, a_1)$ due to the translation invariance of the measure.

Remark 3.2. Note that we computed $J(1,1) = B(p^2, m^2)$ in the previous section.

Proposition 3.3. The bubble integrals satisfy the integration-by-parts (IBP) identities

$$\begin{aligned} 0 = &(D - 2a_1 - a_2)\, J(a_1, a_2) - a_2 J(a_1 - 1, a_2 + 1) \\ &+ 2a_1 m^2 J(a_1 + 1, a_2) + a_2(2m^2 - p^2) J(a_1, a_2 + 1). \end{aligned}$$

Proof. As the name suggests[1], we can derive the IBP identities by noticing that we have

$$0 = \int d^D k\, \partial_\mu \left(k^\mu \frac{1}{(-k^2 + m^2)^{a_1}(-(k+p)^2 + m^2)^{a_2}} \right),$$

whereas the boundary term vanishes for $a + b \geq 1$. In order to keep the computations organized, let us define $A(q) = -(k+q)^2 + m^2$ thus allowing us to write the integrand (of the original bubble integral) as

$$I(a_1, a_2) = \frac{1}{(-k^2 + m^2)^{a_1}(-(k+p)^2 + m^2)^{a_2}} = \frac{1}{A(0)^{a_1} A(p)^{a_2}}.$$

From this definition we can straightforwardly read off the relations

$$J(a_1, a_2) = \int d^D k\, I(a_1, a_2), \quad A(0)^x A(p)^y I(a_1, a_2) = I(a_1 - x, a_2 - y),$$

which allow us to keep the computation concise. We can now perform the differentiation to get

$$0 = \int d^D k\, \partial_\mu \left(k^\mu I(a_1, a_2) \right) = D J(a_1, a_2) + \int d^D k\, k^\mu \partial_\mu I(a_1, a_2), \tag{3.1}$$

whereas we have used $\partial_\mu k^\mu = D$, allowing us to pull this factor out of the integral thus leading to the first term. Continuing with the differentiation, we get

$$\begin{aligned} k^\mu \partial_\mu I\,(a_1, a_2) &= k^\mu \partial_\mu \left(A(0)^{-a_1} A(p)^{-a_2} \right) \\ &= k^\mu \partial_\mu \left(A(0)^{-a_1} \right) A(p)^{-a_2} + A(0)^{-a_1} k^\mu \partial_\mu \left(A(p)^{-a_2} \right) \\ &= -a_1 A(0)^{-a_1 - 1} A(p)^{-a_2} k^\mu \partial_\mu A(0) - a_2 A(0)^{-a_1} A(p)^{-a_2 - 1} k^\mu \partial_\mu A(p) \\ &= -a_1 I\,(a_1 + 1, a_2)\, k^\mu \partial_\mu A(0) - a_2 I\,(a_1, a_2 + 1)\, k^\mu \partial_\mu A(p). \end{aligned}$$

As we differentiate with respect to k we only need to compute $k^\mu \partial_\mu A(q)$ to get all the missing terms in this computation. We thus continue with

$$\begin{aligned} k^\mu \partial_\mu A(q) &= -k^\mu \partial_\mu (k+q)^2 = -2\,(k+q)_\nu\, k^\mu \partial_\mu (k+q)^\nu \\ &= -2\,(k+p)_\nu\, k^\nu = -2\left(k^2 + kq\right). \end{aligned}$$

[1] This is because we can rewrite integration by part as a consequence of Stokes' theorem

$$\int_M dx\, \frac{df(x)}{dx} g(x) = [f(x)g(x)]_{\partial M} - \int_M dx\, f(x) \frac{dg(x)}{dx} \iff 0 = -\,[f(x)g(x)]_{\partial M} + \int_M dx\, \frac{d}{dx}\left(f(x)g(x)\right).$$

We want to write this in terms of $A(q)$ again and thus compute

$$
\begin{aligned}
k^2 + kq &= \frac{1}{2}(k+q)^2 + \frac{1}{2}\left(k^2 - q^2\right) \\
&= \frac{1}{2}\left[(k+q)^2 - m^2\right] + \frac{1}{2}\left[k^2 - m^2\right] - \frac{1}{2}\left(q^2 - 2m^2\right) \\
&= -\frac{1}{2}A(q) - \frac{1}{2}A(0) - \frac{1}{2}\left(q^2 - 2m^2\right).
\end{aligned}
$$

Using this result, we get

$$
k^\mu \partial_\mu A(q) = A(q) + A(0) + q^2 - 2m^2
$$

and can thus continue with the actual computation to get

$$
\begin{aligned}
k^\mu \partial_\mu I(a_1, a_2) &= -a_1 I(a_1+1, a_2)\left(2A(0) - 2m^2\right) \\
&\quad - a_2 I(a_1, a_2+1)\left(A(q) + A(0) + q^2 - 2m^2\right) \\
&= -2a_1 I(a_1, a_2) + 2a_1 m^2 I(a_1+1, a_2) - a_2 I(a_1, a_2) \\
&\quad - a_2 I(a_1-1, a_2+1) + a_2\left(2m^2 - q^2\right) I(a_1, a_2+1).
\end{aligned}
$$

Plugging all these results into equation (3.1), we get

$$
\begin{aligned}
0 = &(D - 2a_1 - a_2) J(a_1, a_2) - a_2 J(a_1 - 1, a_2 + 1) \\
&+ 2a_1 m^2 J(a_1 + 1, a_2) + a_2(2m^2 - p^2) J(a_1, a_2 + 1),
\end{aligned}
$$

which is the integration-by-part identity that we stated. □

Remark 3.4. Using the symmetry $J(a_1, a_2) = J(a_2, a_1)$, we obtain a very similar equation and thus get two equations for the two integrals $J(a_1 + 1, a_2)$ and $J(a_1, a_2 + 1)$. If both a_1 and a_2 are non-zero, we can solve these equations and thus express these two integrals in terms of integrals $J(b_1, b_2)$ with $b_1 + b_2 = a_1 + a_2 - 1$.

Remark 3.5. For the case that one of these integers is zero, eg. $a_2 = 0$, we can also express the integral $J(a_1 + 1, 0)$ in terms of integrals whose sum of indices is one less. In particular, we get

$$
J(a_1 + 1, 0) = \frac{2a_1 - D}{2a_1 m^2} J(a_1, 0), \tag{3.2}
$$

which can be immediately read off from the IBP identity. Using the symmetry of J this also relates $J(0, a_2 + 1)$ to $J(0, a_2)$. Note that the integral $J(a, 0)$ corresponds to the tadpole diagram. In this way, we can consider the simpler tadpole diagram as a part of the more complicated bubble diagram. The same view can be taken for even more complicated loop diagrams, we could, for example, consider the tadpole, bubble and triangle diagram to be part of the box diagram.

Corollary 3.6. All integrals of the generalized bubble integral family can be expressed in terms of a basis of two appropriate such integrals, the so-called master integrals. Thus, the space of these integrals is two-dimensional with a basis given for example by $J(1, 0)$ and $J(1, 1)$.

Proof. Using the arguments made in remarks 3.4 and 3.5 we can decompose any bubble integral with positive a_1 and a_2 into integrals whose sum of indices is one less than $a_1 + a_2$. We can repeat this process, until we obtain integrals whose sum of indices is 1 or 2, that is until we have decomposed the original integral into $J(1,0)$, $J(0,1)$ and $J(1,1)$[2]. These integrals cannot be further decomposed, as the derived equations only relate $J(a_1+1, a_2)$ and $J(a_1, a_2+1)$ to integrals with sum of indices equal to $a_1 + a_2$ if both a_1 and a_2 are non-zero or using the special case $J(a_1+1, 0)$ to $J(a_1, 0)$ if a_1 is non-zero and thus does not allow any further decomposition. Using that $J(0,1) = J(1,0)$, the statement is proven. □

4 Outlook

We have seen in this seminar talk how we can use classical techniques to compute loop integrals, at least for examples where this can be done with a moderate amount of work. In order to reduce the required work by as much as possible, we then proceeded to derive the integration-by-parts identities for the example we considered. It turns out that these identities allow us to reduce the (infinite) family of integrals to a basis of two master-integrals. Having computed only these two integrals, we can express all the other integrals in terms of these results.

We have only considered one-loop integrals. In more general cases, we can generate IBP identities for L-loop integrals by considering similarly that

$$0 = \int \prod_{i=1}^{L} \left(\frac{d^D k_i}{i\pi^{D/2}} \right) \sum_{j=1}^{L} \frac{\partial}{\partial k_j^\mu} \frac{v_j^\mu}{D_1^{a_1} \dots D_k^{a_k}}, \tag{4.1}$$

whereas v_j^μ is a polynomial in the internal and external momenta and D_k denotes the inverse propagator. By choosing appropriate polynomials v_j^μ we can generate many, more or less, useful integration-by-parts identities. There are by now many implementations that do the reduction by these IBP identities automatically. One possible realization is to recast the relation (4.1) into a problem of algebraic geometry, such that the well-established computational techniques in this area can be used (see eg. [3]).

Being able to express Feynman integrals in terms of master-integrals also allows us to determine the latter by differential equations. Let us briefly see this in our bubble integral example. Instead of the basis $J(1,0)$ and $J(1,1)$ we will now use $J(3,0)$ and $J(2,1)$, which have the advantage of being finite as $\epsilon \to 0$. The first of these integrals can be computed by the techniques used to compute $J(1,1)$ to give

$$J(3,0) = \frac{\Gamma\left(3 - \frac{D}{2}\right)}{\Gamma(3)} \frac{1}{(m^2)^{3-D/2}}. \tag{4.2}$$

[2]The other integrals with sum of indices equal to 2, that is $J(2,0)$ and $J(0,2)$, are related to $J(1,0)$ and $J(0,1)$ by equation (3.2) and the symmetry of J.

Now consider the other basis integral $J(2,1)$, and compute its derivative with respect to m^2. This leads to

$$\begin{aligned}
\partial_{m^2} J\,(2,1) &= C \int d^D k\, \partial_{m^2} \frac{1}{(-k^2+m^2)^2 \left(-(k+p)^2+m^2\right)} \\
&= C \int d^D k \left[\frac{-2\, \partial_{m^2} \left(-k^2+m^2\right)}{(-k^2+m^2)^3 \left(-(k+p)^2+m^2\right)} + \frac{-1\, \partial_{m^2} \left(-(k+p)^2+m^2\right)}{(-k^2+m^2)^2 \left(-(k+p)^2+m^2\right)^2} \right] \\
&= -2J\,(3,1) - J\,(2,2)\,,
\end{aligned}$$

and thus gives a first-order differential equation for $J\,(2,1)$ in terms of two other integrals. But we can relate these by the IBP reduction to $J(2,1)$ and $J(3,0)$ and thus obtain the differential equation

$$\partial_{m^2} J\,(2,1) = \frac{2}{4m^2-p^2} \left[(D-5)\, J\,(2,1) - J\,(3,0)\right].$$

Now we have a linear, first-order differential equation for $J\,(2,1)$ with an inhomogeneous part given by $J\,(3,0)$, which is known by equation (4.2). With a little more work, this differential equation can be solved, such that we can completely avoid explicitly performing the loop integral.

If one carefully chooses a basis $\boldsymbol{f}$ of Feynman integrals f_i, it is possible to rearrange the differential equations they satisfy into the form

$$\partial \boldsymbol{f} = \epsilon A \boldsymbol{f}, \tag{4.3}$$

for some coefficient matrix A, which does not depend on ϵ. This implies that the right hand side of the differential equation is proportional to ϵ. When solving for the basis integrals, we want to express the solution as a power series in ϵ as in equation (1.1). Making such a general ansatz, the differential equation (4.3) decouples and can thus be solved order by order in ϵ. Much more can be done along this line and we refer to chapter 3.8 of [1], where this is and further directions are discussed in some more detail.

References

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