Lorenzo Quintavalle – 29/01/2019

Scattering Amplitudes: A Colorful Duality

Refs.

Elvang, Huang – Scattering Amplitudes

Zvi Bern – Do I have to draw you a diagram? (YouTube)

Carrasco – Lectures on gauge and gravity amplitude relations (TASI 2014)

Bern et al. – Ultraviolet Properties of $\mathcal{N}=8$ Supergravity at five loops

Roadmap:

- Discuss gravity amplitudes: GR as an EFT
- Yang-Mills amplitude technology: Color-Kinematics Duality
- Gravity from Yang-Mills: double-copy construction
- Results in $\mathcal{N}=8$ supergravity

GR is described by the **Einstein-Hilbert action**:

$$S_{EH} = \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-g}\,R + \cancel{S_{matter}}$$

we set this to 0, we discuss **pure gravity**

Expand the metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, $h_{\mu\nu}$ is the **graviton field**; $(\kappa = \sqrt{8\pi G})$
the action can be written schematically as:

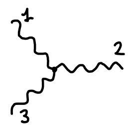
$$S_{EH} = \int d^Dx\left[h\,\partial^2 h + \kappa h^2 \partial^2 h + \kappa^2 h^3 \partial^2 h + \dots\right]$$

infinitely many terms!

Tree-level Amplitudes

@ tree level the infinite amount of interaction terms is not a big problem: for small n one could even draw the Feynman diagrams and compute the amplitudes, but we learned that there exist more efficient ways to do this.

- **3-graviton amplitudes** are constrained by special kinematics and by little group scaling (bonus material from Seva's lecture):



$$M_3^{tree}(1^-, 2^- 3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2} = A_3^{tree}[1^-, 2^-, 3^+]^2 \qquad M_3^{tree}(1^+, 2^+, 3^-) = \frac{[12]^6}{[23]^2[31]^2} = A_3^{tree}[1^+, 2^+, 3^-]^2$$

- **n-graviton amplitudes** @ tree level can be obtained using String theory's **KLT relations**, which in the infinite tension limit $\alpha' \to 0$ relate gravity scattering amplitudes M_n^{tree} to color ordered gluon amplitudes A_n^{tree}, like:

4-Point KLT relation

$$\boxed{M_4^{\text{tree}}(1,2,3,4) = -u\, A_4^{\text{tree}}[1,2,3,4]\, A_4^{\text{tree}}[1,3,2,4]} \qquad (1)$$

where one can obtain pure gravity amplitudes through the assignments

$$\text{graviton}^{\overset{\text{helicity}}{\pm 2}}(p_i) = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\pm 1}(p_i)$$

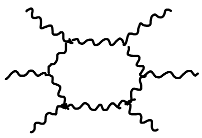
while with the other assignments one gets:

$$\text{dilaton} \oplus \text{axion} = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\mp 1}(p_i)$$

Therefore by taking the "square" of 4D Yang-Mills amplitudes one can get 4D axion-dilaton gravity at tree-level (and eventually focus on pure GR by picking the helicities which reproduce the graviton).

<u>Recap</u>: tree level GR amplitudes can be easily obtained by KLT relations like (1)

Loop-level Amplitudes



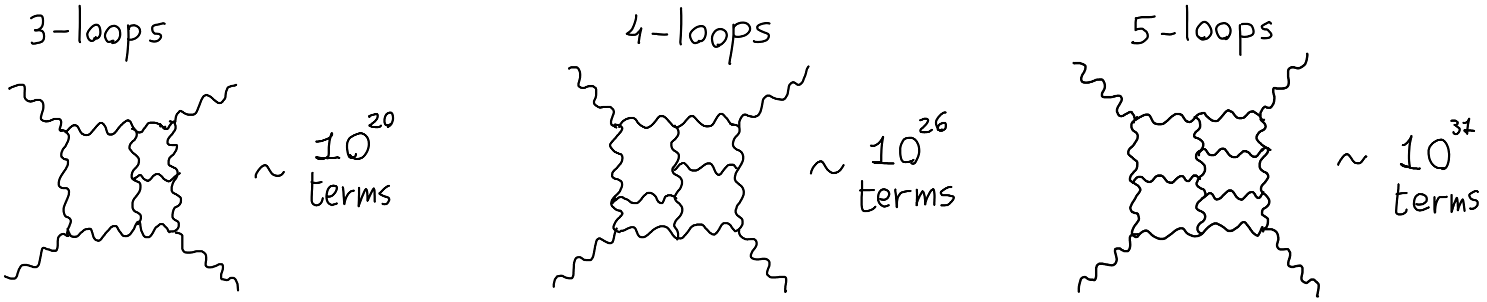
GR is notoriously problematic @ loop-level, power counting analysis suggest that the theory contains untameable divergences:

$$\text{1-loop n-graviton diagram}^* \sim \int^{\Lambda} d^4\ell\, \frac{\overbrace{(\ell^2)^n}^{\text{due to } \partial^2 \text{ in interaction terms}}}{\underbrace{(\ell^2)^n}_{\text{due to propagators}}} \sim \Lambda^4$$

* actually, one-loop pure GR is finite due to UV cancellations between diagrams. Two-loop GR has been proven to be divergent by direct computation

Despite the bad UV behaviour of the theory, we can still consider GR as an Effective Field Theory to describe the physics at low energy scales $E \ll M_{\text{Planck}} \sim 10^{19}\,\text{GeV}$.

@ loop-level, the Feynman diagram approach becomes truly problematic because of the enormous amount of terms that are there:



Is there any hope we could ever be able to compute multi-loop graviton amplitudes? KLT relations do not work @ loop level, but **Color-Kinematics duality** could light our way

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Tree-level Yang-Mills memorandum

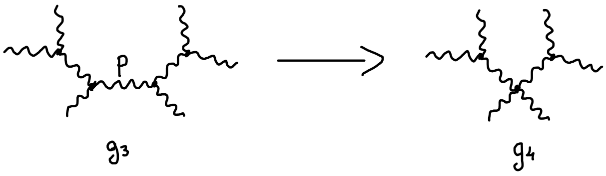
Let's leave gravity aside for the moment, and focus a bit on tree-level Yang-Mills theory.

- Every n-point amplitude can be computed starting from graphs ($\neq$ Feynman graphs) with only **trivalent vertices**

Sketch of Proof

Given a cubic graph g_3 (only trivalent vertices) and another graph g_4, almost identical to g_3 apart from two adjacent trivalent vertices which are merged into a four-point vertex

e.g.



then the denominators will be related by $d(g_3) = p^2 d(g_4)$ (g_4 is missing a propagator), and the contributions from those two graphs can be rearranged as:

$$A_m = \cdots + \frac{n(g_3)}{d(g_3)} + \frac{n(g_4)}{d(g_4)} = \cdots + \frac{n(g_3)}{d(g_3)} + \frac{n(g_4)}{d(g_3)} p^2 = \cdots + \frac{\tilde{n}(g_3)}{d(g_3)}$$

the 4-point contribution has been absorbed in the 3-point term this implies that YM amplitudes @ tree level can be written in the form

$$A_n^{\text{tree}} = \sum_{i \in \text{trivalent}} \frac{c_i n_i}{\prod_{\alpha_i} P_{\alpha_i}^2} \qquad ②$$

> c_i = color part
> n_i = Kinematic part (polariz. & momenta)

- Color factors of trivalent diagrams are related by Jacobi identities

Examples

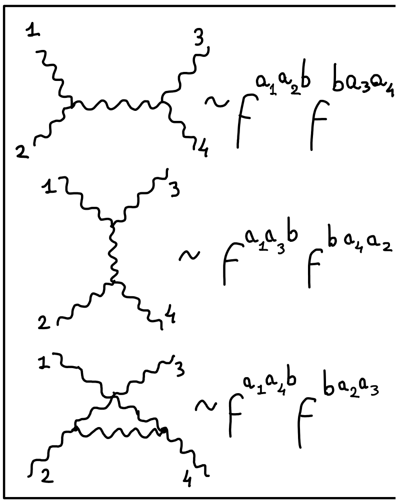
$$[[T^{a_1}, T^{a_2}], T^{a_3}] + [[T^{a_3}, T^{a_1}], T^{a_2}] + [[T^{a_2}, T^{a_3}], T^{a_1}] = 0$$

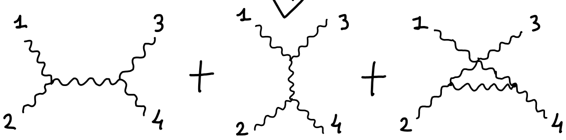
$$\Downarrow$$

$$\underbrace{f^{a_1a_2b} f^{ba_3a_4}}_{c_s} + \underbrace{f^{a_3a_1b} f^{ba_2a_4}}_{c_t} + \underbrace{f^{a_2a_3b} f^{ba_1a_4}}_{c_u} = 0$$

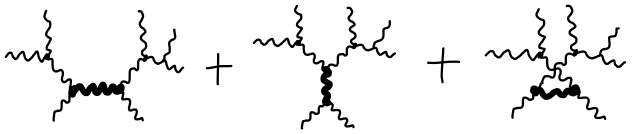
$$c_s + c_t + c_u = 0$$

$$\Downarrow$$





$= 0$



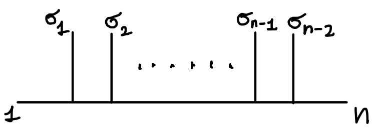
$= 0$

$$c_i + c_j + c_k = 0$$

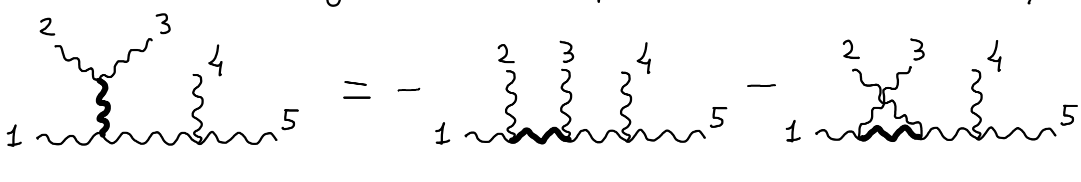
- There are $(n-2)!$ n-point tree level trivalent diagrams whose color factors are independent under Jacobi relations.

Sketch of Proof

Choosing 2 legs to be fixed (e.g. 1 and n), every diagram which contributes to an n-point amplitude can be written as a linear combination of diagrams of the following form:



this can be done through the use of Jacobi identities, e.g:



Since there are $(n-2)!$ ways to rearrange the σ_i lines we conclude that the number of independent diagrams that can be used to represent an n-point amplitude in YM is $(n-2)!$.
We can write these amplitudes in the color basis as:

$$A_n^{\text{full, tree}} = \sum_{\sigma_i \in S_{n-2}} f^{a_1 a_{\sigma_1} b_1} f^{b_1 a_{\sigma_2} b_2} \cdots f^{b_{n-3} a_{\sigma_{n-2}} a_n} A_n(1, \sigma_1, \dots, \sigma_{n-2}, n)$$

Color-Kinematics Duality

We now know that we can write a (super) Yang-Mills amplitude in the form 2 using color factors c_i and kinematic numerators n_i; we also know that there are $(n-2)!$ independent color factors and $(n-2)!$ independent amplitudes in the color basis.

Color-Kinematics Duality states that there exists a representation in which the numerators $\mathbf{n_i}$ have <u>the same algebraic properties</u> of the corresponding color factors $\mathbf{c_i}$.

$$c_i = -c_j \iff n_i = -n_j$$

$$c_i + c_j + c_k = 0 \iff n_i + n_j + n_k = 0$$

This means that there are $(n-2)!$ independent numerators $\hat{n}_i$. We can expand the linearly independent partial amplitudes in terms of the numerators:

$$\boxed{A_{(i)} = \sum_{j=1}^{(n-2)!} \Theta_{ij}\, \hat{n}_j} \qquad (3)$$

where Θ_{ij} is a $(n-2)! \times (n-2)!$ matrix made of only massless scalar propagators.
It can be proven that in general Θ_{ij} has rank $(n-3)!$. This means that there are $(n-2)! - (n-3)!$ linear relations among the

partial amplitudes: these are called **BCJ relations.**

Let's see an example on how these relations can be obtained.

$$\boxed{\text{Example: BCJ relations in 4-gluon scattering}}$$

The full 4-point gluon amplitude can be written in terms of the color factors of channel s,t,u as:

$$A_4^{\text{full, tree}} = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

Using the tools introduced in Philipp's lecture, we can reorganize the color factors into a sum of traces like

$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4} = \left\{ \begin{array}{l} f^{abc} = -i\left[\mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^a T^c T^b\right)\right] \\ \left(T^a\right)_i^{\ j} \left(T^a\right)_k^{\ l} = \delta_i^{\ l} \delta_k^{\ j} - \frac{1}{N_c} \delta_i^{\ j} \delta_k^{\ l} \end{array} \right\} =$$

$$\propto \mathrm{Tr}\left(T^{a_1} T^{a_2} T^{a_3} T^{a_4}\right) - \mathrm{Tr}\left(T^{a_1} T^{a_2} T^{a_4} T^{a_3}\right) - \mathrm{Tr}\left(T^{a_1} T^{a_3} T^{a_4} T^{a_2}\right) + \mathrm{Tr}\left(T^{a_1} T^{a_4} T^{a_3} T^{a_2}\right)$$

and then isolate the color ordered amplitudes in the full amplitude by collecting the terms with the same trace factors.

Since there are $(n-2)! = 2$ independent partial amplitudes, we can focus on only two of them, e.g.

$$A_4\left[1,2,3,4\right] = -\frac{n_s}{s} + \frac{n_u}{u} \qquad A_4\left[1,3,2,4\right] = -\frac{n_u}{u} + \frac{n_t}{t}$$

we know that only two numerators are independent, we can then pick $(\hat{n}_1, \hat{n}_2) = (n_s, n_u)$ and use the Jacobi identity to express $n_t = -n_s - n_u$.

We can then write ③ in our case to identify the matrix Θ_{ij}:

$$\begin{pmatrix} A_4[1,2,3,4] \\ A_4[1,3,2,4] \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{s} & \frac{1}{u} \\ -\frac{1}{t} & -\frac{1}{u}-\frac{1}{t} \end{pmatrix}}_{\Theta_{ij}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix} \qquad (4)$$

As expected, Θ_{ij} has rank $(n-3)! = 1$, in fact:

$$\frac{\Theta_{22}}{\Theta_{12}} = \frac{-t-u}{\cancel{u}\,t}\cdot\cancel{u} = \left\{ s+t+u = m^2 = 0 \right\} = \frac{s}{t} = \frac{\Theta_{21}}{\Theta_{11}}$$

We then expect to have relations between the partial amplitudes. From the first row of (4) we can write

$$\hat{n}_1 = -s\,A_4[1,2,3,4] + \frac{s}{u}\,\hat{n}_2 \qquad (5)$$

which, once substituted in the second row gives:

$$A_4[1,3,2,4] = \frac{s}{t}\,A_4[1,2,3,4] - \cancel{\left(\frac{s+t+u}{ut}\right)}\hat{n}_2$$

$\hat{n}_2$ disappeared from the equations, we can set its value to be whatever we want (pure gauge)

$$\Downarrow$$

(an example of) **BCJ relation**

$$\boxed{t\,A_4[1,3,2,4] = s\,A_4[1,2,3,4]} \qquad (6)$$

Back to gravity: Double Copy

Let's recall the KLT relations (1). Their form suggest the existence of a deep relation between gravity amplitudes and Yang-Mills amplitudes.
Turns out that, given a (super) Yang-Mills theory, once you have found a set of numerators $\hat{n}_i$ which satisfy the Color-Kinematics

duality, then you can obtain the n-point tree amplitude M_n of the (super) gravity theory whose spectrum is the square of the (S)YM under consideration just by the simple formula:

BCJ double-copy relation

$$M_n = \sum_{i \in \text{cubic}} \frac{n_i^2}{\prod_{\alpha_i} p_{\alpha_i}^2} \qquad (7)$$

Example

Let us use (7) in the 4-particle case. We have:

$$M_4 = \frac{n_s^2}{s} + \frac{n_u^2}{u} + \frac{n_t^2}{t}$$

But these numerators satisfy the duality with $(\hat{n}_1, \hat{n}_2) = (n_s, n_u)$ and $n_t = -n_s - n_u$, so:

$$M_4 = \frac{\hat{n}_1^2}{s} + \frac{\hat{n}_2^2}{u} + \frac{(\hat{n}_1 + \hat{n}_2)^2}{t}$$

but $\hat{n}_2$ disappeared completely in our calculation for the BCJ relation (6), we can set its value to be zero and use (5) to express $\hat{n}_1$. We get

$$M_4 = s\left(1 + \frac{s}{t}\right) A_4[1,2,3,4]^2 = -\frac{su}{t} A_4[1,2,3,4]^2$$

using the BCJ relation (6), we can write the amplitude in the form:

$$\boxed{M_4 = -u\, A_4[1,2,3,4]\, A_4[1,3,2,4]}$$

which is exacty the KLT relation (1) which describes the scattering of four gravitons @ tree-level.

The double copy construction 7 makes the computation of gravity scattering amplitudes even easier than using the KLT approach. Moreover, as long as one can found numerators satisfying the color-kinematics duality, the double copy construction can be used also @ loop level, where once written the full L-loop Yang-Mills amplitude

$$A_n^{L\text{-loop}} = \sum_{j \in \text{cubic}} \int \left(\prod_{K=1}^{L} \frac{d^D \ell_K}{(2\pi)^D} \right) \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

symmetry factor of the diagram

and once obtained a set of duality-satisfying numerators $\hat{n}_j$, we can use the extended double-copy formula:

$$M_n^{L\text{-loop}} = \sum_{j \in \text{cubic}} \int \left(\prod_{K=1}^{L} \frac{d^D \ell_K}{(2\pi)^D} \right) \frac{1}{S_j} \frac{n_j \hat{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

A UV-finite theory of gravity?

The double-copy construction can be used (with some complications) to compute GR amplitudes, and it is currently being applied to gravitational wave physics in the attempt of improving the results one can obtain (see e.g. 1901.04424).
A remarkable application of double-copy has been done in the context of supersymmetric theories.

As we have seen, GR contains untameable divergences. An attempt that has been done to build a renormalizable theory of gravity is the introduction of **supersymmetries** (SUSY's), special symmetries that relate fermions and bosons

$$\underset{\text{graviton, helicity}=-2}{h^-} \xrightarrow{\tilde{Q}} \underset{\text{gravitino, helicity}=-\frac{3}{2}}{\psi^-}$$

supersymmetric theories have a better UV behaviour than non-SUSY ones. The maximal amount of SUSY's a theory can have is 8 (there are 8 half-steps between h^- and h^+; no spin >2). The theory with maximal SUSY is **$\mathcal{N}=8$ supergravity** ($\mathcal{N}=8$ SUGRA):

$$h^- \longrightarrow \psi^- \longrightarrow v^- \longrightarrow \chi^- \longrightarrow S \longrightarrow \chi^+ \longrightarrow v^+ \longrightarrow \psi^+ \longrightarrow h^+$$

gravi-photons   scalars   gravi-photinos

There has been quite an active debate on whether $\mathcal{N}=8$ SUGRA in four dimensions is UV finite or not, and it is not yet settled!
Back in the '80s many people thought $\mathcal{N}=8$ SUGRA had divergences @ 3-loops, and it was given up for dead. This until 2007, when Bern et al. computed 3-loops amplitudes and showed that these are finite.
At every loop order can be associated a **critical dimension D_c**, over which the theory becomes divergent.

First argument predicted: $\quad D_c(L) = \frac{6}{L} + 4 \quad , L > 1$

Refined argument (symmetries): $\quad D_c(5\text{-loops}) = \frac{24}{5} \quad , \quad D_c(7\text{-loops}) < 4$

Modern approaches to scattering amplitudes, comprising the double-copy construction of $\mathcal{N}=8 \text{ SUGRA} = (\mathcal{N}=4 \text{ SYM})^2$, allowed Bern et al. to investigate the theory up to 5-loops, and the results are in agreement with the predictions. This, however, has not settled the debate: the same people who are computing these amplitudes argue there could be **enhanced UV cancellations** @ 7-loops.

Bottles of wine are at stake:

David Gross "it will diverge"	vs.	Zvi Bern "it won't diverge"

Who is going to win?