modern cosmology

ingredient 3: statistics

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outline

structure formation equations

cosmic structure formation

structure formation is a self gravitating, fluid mechanical phenomenon

- continuity equation: evolution of the density field due to fluxes

$$\frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\vec{u}) = 0 \tag{1}$$

- Euler equation: evolution of the velocity field due to forces

$$\frac{\partial}{\partial t}\vec{u} + \vec{u}\nabla\vec{u} = -\nabla\Phi \tag{2}$$

- Poisson equation: potential sourced by density field

$$\Delta\Phi = 4\pi G\rho \tag{3}$$

- 3 quantities, 3 equations $\rightarrow$ solvable
- 2 nonlinearities: $\rho\vec{u}$ in continuity and $\vec{u}\nabla\vec{u}$ in Euler-equation

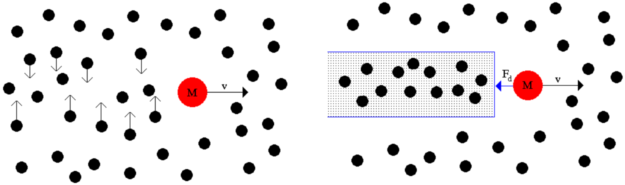
viscosity and pressure

dynamics with dark matter

dark matter is collisionless (no viscosity and pressure) and interacts gravitationally (non-saturating force)

- dark matter is collisionless → no mechanism for microscopic elastic collisions between particles, only interaction by gravity
- derivation of the fluid mechancis equation from the Boltzmann-equation: **moments method**
  - continuity equation
  - Navier-Stokes equation
  - energy equation
- system of coupled differential equations, and closure relation
- effective description of collisions: viscosity and pressure, but
  - relaxation of objects if there is no viscosity?
  - stabilisation of objects against gravity if there is no pressure?

collective dynamics: dynamical friction



source: J. Schombert

- dynamical friction emulates viscosity: there is no **microscopic model** for viscosity, but **collective processes** generate an effective viscosity
  - a particle moving through a cloud produces a wake
  - behind the particle, there is a density enhancement
  - density enhancement breaks down particle velocity
- kinetic energy of the incoming object is transformed to unordered random motion

Kelvin-Helmholtz instability

- shear flows become unstable if there are large perpendicular velocity gradients
- generation of vorticity in shear flows by the **Kelvin-Helmholtz instability**
- absent in the case of dark matter: flow is necessarily laminar

vorticity

- intuitive explanation of the nonlinearity of the Navier-Stokes eqn

$$\frac{\partial}{\partial t}\vec{u} + \vec{u}\nabla\vec{u} = \frac{\nabla p}{\rho} - \nabla\Phi + \mu\Delta\vec{u} \tag{4}$$

- vorticity equation: $\vec{\omega} \equiv \mathrm{rot}\vec{u}$

$$\underbrace{\frac{\partial\vec{\omega}}{\partial t} + \vec{u}\nabla\vec{\omega}}_{\text{material derivative}} = \underbrace{\vec{\omega}\nabla\vec{u}}_{\text{tilting}} - \underbrace{\vec{\omega}\mathrm{div}\vec{u}}_{\text{compression}} + \underbrace{\frac{1}{\rho^2}\nabla p \times \nabla\rho}_{\text{baroclinic}} + \underbrace{\mu\Delta\vec{\omega}}_{\text{diffusion}} \tag{5}$$

- generation of vorticity by
  - pressure gradients non-parallel to density gradients
  - viscous stresses

→ **not present in the case of collisionless dark matter**
→ **gravity as a conservative force is not able to induce vorticity**

- vorticity equation is a nonlinear diffusion equation, vorticity

regimes of structure formation

look at overdensity field $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$, with $\bar{\rho} = \Omega_m \rho_{crit}$

- analytical calculations are possible in the regime of linear structure formation, $\delta \ll 1$
  $\rightarrow$ homogeneous growth, dependence on dark energy, number density of objects
- transition to non-linear structure growth can be treated in perturbation theory (difficult!), $\delta \sim 1$
  $\rightarrow$ first numerical approaches (Zel'dovich approximation), directly solvable for geometrically simple cases (spherical collapse)
- non-linear structure formation at late times, $\delta > 1$
  $\rightarrow$ higher order perturbation theory (even more difficult), ultimately: direct simulation with n-body codes

linearisation: perturbation theory for $\delta \ll 1$

- move from physical to comoving frame, related by scale-factor a
- use density $\delta = \Delta\rho/\rho$ and comoving velocity $\vec{u} = \vec{\upsilon}/a$
  - **linearised continuity equation**:
$$\frac{\partial}{\partial t}\delta + \mathrm{div}\vec{u} = 0$$
  - **linearised Euler equation**: evolve momentum
$$\frac{\partial}{\partial t}\vec{u} + 2H(a)\vec{u} = -\frac{\nabla\Phi}{a^2}$$
  - **Poisson equation**: generate potential
$$\Delta\Phi = 4\pi G\rho_0 a^2\delta$$

question

derive the linearised equations by subsituting a perturbative series $\rho = \rho_0(1+\delta)$ for all quantities, in the comoving frame

growth equation

- structure formation is homogeneous in the linear regime, all spatial derivatives drop out
- combine continuity, Jeans- and Poisson-eqn. for differential equation for the temporal evolution of δ:

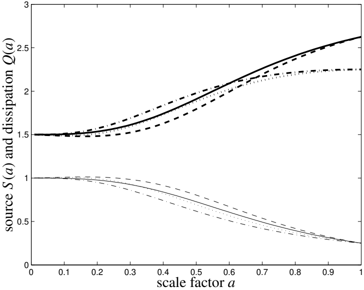
$$\frac{d^2\delta}{da^2} + \frac{1}{a}\left(3 + \frac{d\ln H}{d\ln a}\right)\frac{d\delta}{da} = \frac{3\Omega_M(a)}{2a^2}\delta \qquad (6)$$

- growth function $D_+(a) \equiv \delta(a)/\delta(a=1)$ (growing mode)
  - position and time dependence separated: $\delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x})$
  - in Fourier-space modes grows independently: $\delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k})$
- for standard gravity, the growth function is determined by $H(a)$

question

derive $H(a)$ as a function of $D_+(a)$

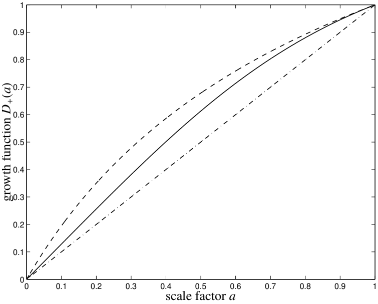
terms in the growth equation



source (thin line) and dissipation (thick line)

- two terms in growth equation:
  - source $Q(a) = \Omega_m(a)$: large $\Omega_m(a)$ make the grav. fields strong
  - dissipation $S(a) = 3 + d\ln H/d\ln a$: structures grow if their dynamical time scale is smaller than the Hubble time scale $1/H(a)$

growth function



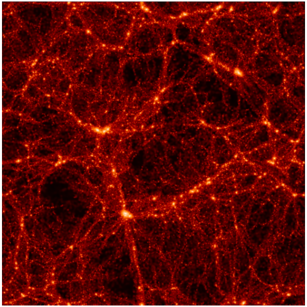
$D_+(a)$ **for** $\Omega_m = 1$ **(dash-dotted), for** $\Omega_\Lambda = 0.7$ **(solid) and** $\Omega_k = 0.7$ **(dashed)**

- density field grows $\propto a$ in $\Omega_m = 1$ universes, faster if $w < 0$

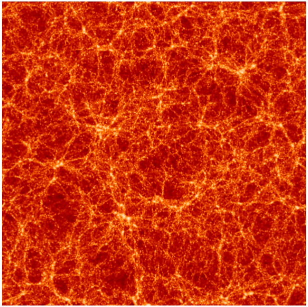
question

derive growth equation, use scale-factor a as time variable, and show that $D_+(a) = a$ is a solution for $\Omega_m = 1$

nonlinear density fields



ΛCDM



SCDM ($\Omega_m = 1$)

source: Virgo consortium

- dark energy influences nonlinear structure formation
- how does nonlinear structure formation change the statistics of the density field?

mode coupling

- linear regime structure formation: homogeneous growth

$$\delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x}) \rightarrow \delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k}) \tag{7}$$

- separation fails if the growth is nonlinear, because a void can't get more empty than $\delta = -1$, but a cluster can grow to $\delta \simeq 200$

$$\delta(\vec{x}, a) = D_+(a, \vec{x})\delta_0(\vec{x}) \tag{8}$$

- product of two $\vec{x}$-dependent quantities in real space $\rightarrow$ convolution in Fourier space:

$$\delta(\vec{k}, a) = \int d^3k' D_+(a, \vec{k} - \vec{k}')\delta_0(\vec{k}') \tag{9}$$

- k-modes do not evolve independently: **mode coupling**
- correlation produces a non-Gaussian field (central limit theorem)

perturbation theory

- perturbative series in density field:

$$\delta(\vec{x},a) = D_+(a)\delta^{(1)}(\vec{x}) + D_+^2(a)\delta^{(2)}(\vec{x}) + D_+^3(a)\delta^{(3)}(\vec{x}) + \ldots \qquad (10)$$

- lowest order:

$$\delta^{(2)}(\vec{k}) = \int \frac{d^3p}{(2\pi)^3} M_2(\vec{k}-\vec{p},\vec{p})\delta(\vec{p})\delta(|\vec{k}-\vec{p}|) \qquad (11)$$

- with mode coupling

$$M_2(\vec{p},\vec{q}) = \frac{10}{7} + \frac{\vec{p}\vec{q}}{pq}\left(\frac{p}{q}+\frac{q}{p}\right) + \frac{4}{7}\left(\frac{\vec{p}\vec{q}}{pq}\right)^2 \qquad (12)$$

- properties:
  - time-independent, no scale $\vec{p}_0$
  - strongest coupling if $\vec{p} = \vec{q}$
  - some coupling of modes $\vec{p} \perp \vec{q}$
  - no coupling if $\vec{p} = -\vec{q}$

homogeneity, linearity and Gaussianity

homogeneity, linearity and Gaussianity

...almost the same thing in structure formation!

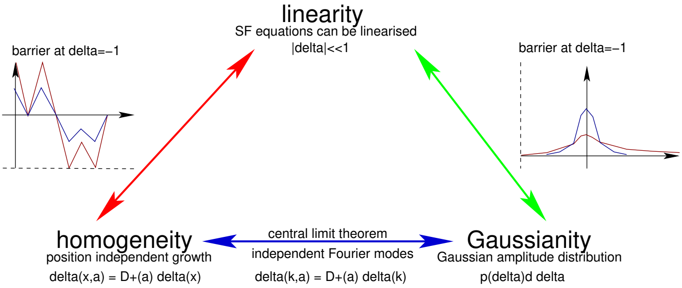
- linearity
  - eqns can be linearised: $|\delta| \ll 1$
  - linearisation fails: $|\delta| \simeq 1$
- homogeneity
  - homogeneous: $\delta(\vec{x}, a) = D_+(a)\delta(\vec{x}, a = 1)$
  - inhomogeneous: $\delta(\vec{x}, a) = D_+(\vec{x}, a)\delta(\vec{x}, a = 1)$
- Gaussianity (with central limit theorem)
  - Gaussian amplitude distribution $p(\delta)d\delta$
  - non-Gaussian (lognormal) distribution $p(\delta)d\delta$

mode coupling

easiest way to visualise: resonance phenomenon

nonlinearity triangle

- linearity, homogeneity and Gaussianity imply each other
- nonlinear structure formation breaks homogeneity and produces non-Gaussian statistics
- mode coupling - can be described in perturbation theory

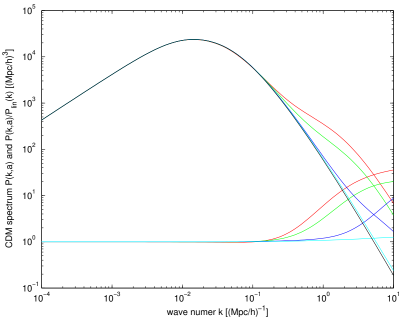


link between dynamics and statistics

- nonlinear structure formation couples modes
- superposition of various k-modes (not independent anymore) generate a non-Gaussian density field
- non-Gaussian density field:
  - odd moments are not necessarily zero
  - even moments are not powers of the variance
- finite correlation length: n-point correlation functions
  - 3-point-function: bispectrum
  - 4-point-function: trispectrum

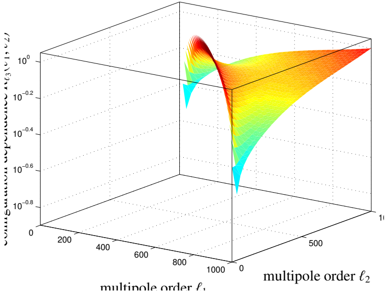
higher order correlations quickly become unpractical, and are really difficult to determine

nonlinear CDM spectrum $P(k)$



- fit to numerical data, $z = 9, 4, 1, 0$, normalised on large scales
- extra power on large scales, time dependent, saturates
- on top of scaling $P(k, a) \propto D_+^2(a)$

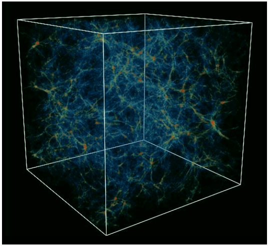
quantification of non-Gaussianities: bispectrum



- bispectrum (3-point function) quantifies nonlinearity to lowest order
- configuration dependence: compare arbitrary triangle to equilateral triangle, keeping base fixed:

$$R_{\ell_3}(\ell_1,\ell_2) = \frac{\ell_1\ell_2}{\ell_3^2}\sqrt{\left|\frac{B(\ell_1,\ell_2,\ell_3)}{B(\ell_3,\ell_3,\ell_3)}\right|} \quad (13)$$

n-body simulations of structure formation



- basic issue: gravity is long-ranged, for each particle the gravitational force of all other particle needs to be summed up, **complexity** n^2
- algorithmic challenge to break down n^2-scaling
  - **particle-mesh**
  - **particle^3-mesh**
  - **tree**-codes
  - **tree-particle mesh**

Zel'dovich-approximation: idea

density $\delta(\vec{r})$	density $\delta_0(\vec{r})$
↓ $\Phi(\vec{r})=-G\int d^3r'\,\frac{\delta(\vec{r}')}{\vert\vec{r}-\vec{r}'\vert}$	↓ $\Psi_0(\vec{r})=-G\int d^3r'\,\frac{\delta_0(\vec{r}')}{\vert\vec{r}-\vec{r}'\vert}$
potential $\Psi(\vec{r})$	potential $\Psi_0(\vec{r})$
↓ $\vec{\upsilon}\propto-\nabla\Psi$	↓ $\vec{\upsilon}\propto D_+\nabla\vec{\Psi}_0$
velocity $\vec{\upsilon}(\vec{r})$	velocity $\vec{\upsilon}(\vec{r})$
↓ $\vec{x}=\vec{q}-\nabla\Psi\Delta t$	↓ $\vec{x}=\vec{q}-D_+(a)\nabla\Psi_0$
density $\delta(\vec{r})$	density $\delta(\vec{r})$

- probe into nonlinear structure formation
- avoid full nonlinear dynamics, use clever approximation

Zel'dovich-approximation

- evolution of perturbation in the translinear regime
- idea: follow trajectories of particles that accumulate in a region and produce a density fluctuation
- physical position $\vec{r}$ (Euler) can be related to initial position $\vec{q}$ (Lagrange)

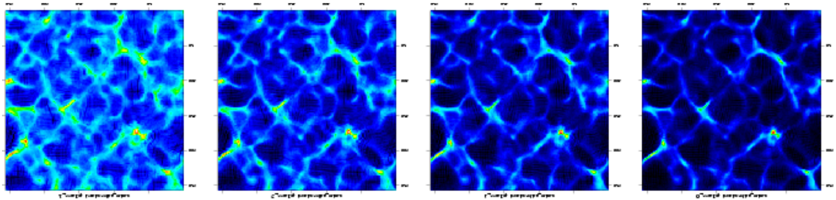
$$\vec{x} = \frac{\vec{r}(t)}{a} = \vec{q} + D_+(t)\nabla\Psi(\vec{q}) \qquad (14)$$

- two contributions: Hubble-flow and local deviation, expressed by displacement field $\Psi(\vec{q})$
- displacement field Ψ is a solution to Poisson eqn. $\Delta\Psi = \delta$
- evolution dominated by overall potential, not by self-gravity

question

can δ become infinite in the Zel'dovich-approximation? what happens in Nature?

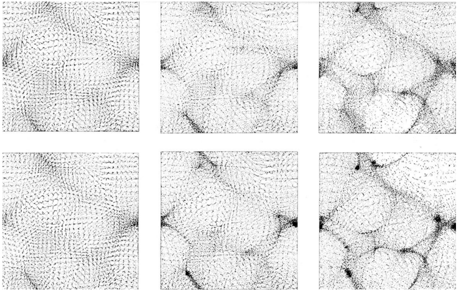
Zel'dovich-approximation: quick realisation



time sequence of structure formation in a dark energy cosmology

- formation of sheets and filaments
- very fast computational scheme (above pic: seconds!!)
- can't use Zel'dovich approximation, if trajectories cross
- no relaxation (collapsing sphere would reexpand to orginial radius)

Zel'dovich: comparision to exact solution



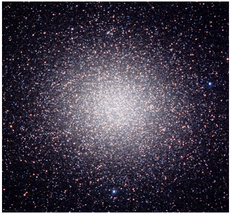
comparison between Zeldovich and exact solution, source: N. Wright

- reexpanding structures, no dissipation, no formation of objects
- qualitative agreement on large scales, small densities

gravothermal instability: thermal energy

- consider self-gravitating system, exchanging (thermal) energy with environment, Lynden-Bell & Wood (1968)
- example: cluster of galaxies loses energy in the form of thermal X-ray radiation, Coma: few 10^{44} erg/sec
  1. energy is removed from a self-gravitating object, on a time-scale $t_{remove} \gg$ dynamical time-scale t_{dyn}
  2. system assumes a new equilibrium state deeper inside its own potential well (quasi-stationary, no relaxation)
  3. release of gravitational binding energy, particles speed up
  4. velocity dispersion (temperature) rises
- reacts on removal of thermal energy by heating up!
- self-gravitating systems have a **negative specific heat** c
- systems cool, if $t_{remove} \ll t_{dyn}$, in this case $c > 0$
- stability of self-gravitating non-isolated systems?

gravothermal instability: particles



globular cluster Omega Centauri, source: Loke Kun Tan

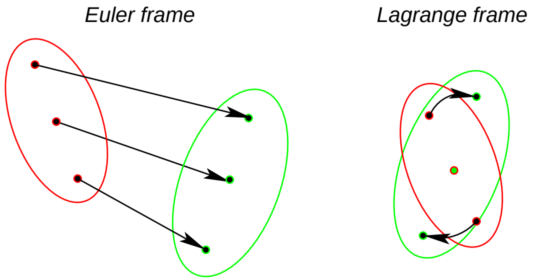
- stars continuously reshuffle their kinetic energy in a globular cluster
- kinetic energy of a star fluctuates, can get gravitationally unbound
- star leaves cluster on parabolic orbit, does not take away energy
- gravitational binding energy distributed among fewer stars
- **system heats up by evaporating stars**, eventually **disintegrates**

angular momentum of galaxies

galaxy M81, HST image

- vorticity can't be generated in inviscid fluids
- flow is laminar
- initial vorticity decreases $\propto 1/a$

angular momentum: tidal shearing



- non-constant displacement mapping across protogalactic cloud
- tidal forces $\partial_i\partial_j\Psi$ set protogalactic cloud into rotation
- in addition: anisotropic deformation (not drawn!)
- gravitational collapse: non-simply connected fields

tidal shearing in Zel'dovich-approximation

- current paradigm: galactic haloes acquire angular momentum by tidal shearing (White 1984)

$$\vec{L} \simeq \rho_0 a^5 \int_{V_L} d^3q (\vec{q} - \bar{q}) \times \dot{\vec{x}} \quad (15)$$

- tidal shearing can be described in Zel'dovich approximation

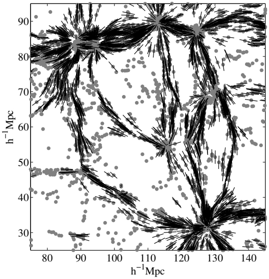
$$\vec{x}(\vec{q}, t) = \vec{q} - D_+(t) \nabla\Psi(\vec{q}) \rightarrow \dot{\vec{x}} = -\dot{D}_+ \nabla\Psi \quad (16)$$

- 2 relevant quantities: inertia $I_{\alpha\beta}$ and shear $\Psi_{\alpha\beta}$

$$L_\alpha = a^2 \dot{D}_+ \epsilon_{\alpha\beta\gamma} I_{\beta\sigma} \Psi_{\sigma\gamma} \quad (17)$$

- tidal shear $\Psi_{\alpha\beta} = \partial_\alpha \partial_\beta \Psi$, derived from Zel'dovich displacement field $\Psi \propto \Phi$, solution to $\Delta\Psi = \delta$

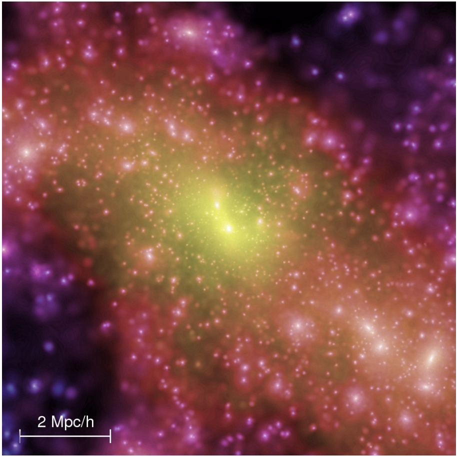
tidal interaction with the large-scale structure



alignment of haloes with the tidal field, source: O. Hahn

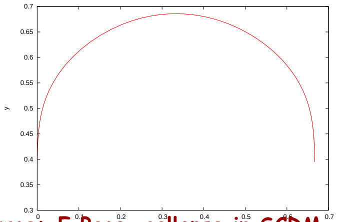
- haloes interact with the large-scale structure with tidal forces
- decomposition $I\Psi = \frac{1}{2}[I, \Psi] + \frac{1}{2}\{I, \Psi\}$
  - commutator $[I, \Psi]$: angular momentum generation
  - anticommutator $\{I, \Psi\}$: anisotropic deformation

nonlinearly evolved density field



source: V.Springel, Millenium simulation

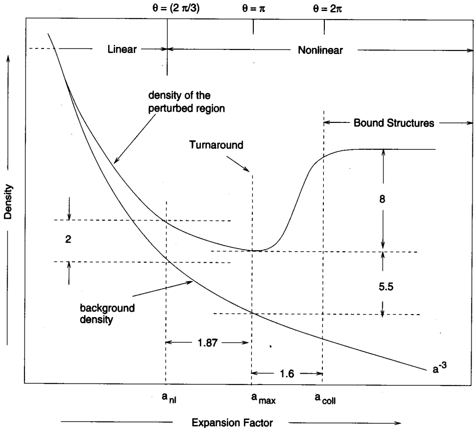
spherical collapse



source: F.Pace, collapse in SCDM

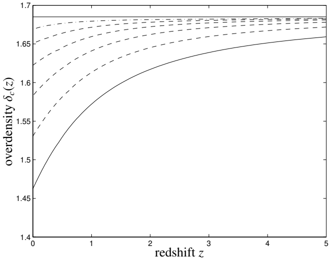
- formation of a bound dark matter object: gravitational collapse
- three phase process:
  1. perturbation expands with Hubble expansion, but at a lower rate
  2. perturbation decouples from Hubble expansion → turn around
  3. perturbation collapses under its own gravity

density evolution in a collapsing halo



source: Padmanabhan, theoretical astrophysics

collapse overdensity in different cosmologies



overdensity needed for a perturbation to collapse at redshift z

- SCDM: collapse overdensity of $\delta_c = 1.686$, very similar in ΛCDM
- dark energy cosmologies require **smaller** collapse overdensities
- sensitivity towards dark energy parameters

relaxation

- in the dynamical evolution, systems tend towards a final state which is not very sensitive on the initial conditions → **relaxation**
- usually, this is accompanied by generation of entropy, which defines an arrow of time
- in cosmology, galaxies with very similar properties form from a Gaussian fluctuation in the matter distribution
- but: dark matter is a collisionless fluid!
  - no viscosity in Euler-eqn. which can dissipate velocities
  - transformation from kinetic energy to heat is not possible
  - no Kelvin-Helmholtz instability and Kolmogorov cascading
  - Euler-equation is time-reversible and no entropy is generated
  - relaxation does not take place

question

show that the Euler-eqn. and the vorticity eqn. are time-reversible

relaxation: 1. two-body relaxation

two-body relaxation

relaxation with **Keplerian** (time-reversible) orbits in a succession of two-body encounters

- consider a system with N stars of size R, density of stars is $n \sim N/R^3$, total mass $M = Nm$
- shoot a single star into the cloud an track its transverse velocity
- in a single encounter the velocity changes

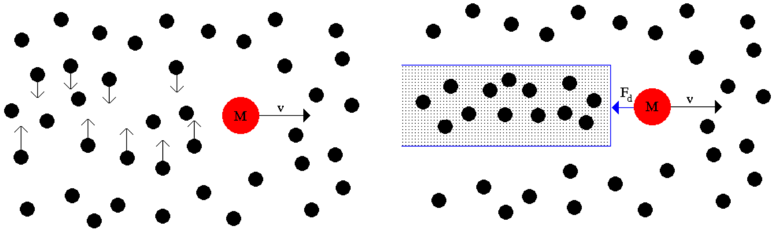
$$\delta\upsilon_\perp(\text{single}) \sim \frac{Gm}{b^2}\frac{2b}{\upsilon} \sim \frac{2Gm}{b\upsilon} \tag{18}$$

with impact parameter b, using Born-approx. with $\delta t = 2b/\upsilon$

- multiple encounters: add random kicks, so variance $\delta\upsilon_\perp^2$ grows

$$\frac{d}{dt}\delta\upsilon_\perp^2 \sim 2\pi\int bdb\,\delta\upsilon_\perp(\text{single})\,n\upsilon = \frac{8\pi G^2m^2n}{\upsilon}\ln\left(\frac{b_{max}}{b_{min}}\right) \tag{19}$$

relaxation: 2. dynamical friction



source: J. Schombert

- system of reference with moving particle
- all other particle zoom past on hyperbolic orbits, orbit/gravitational scattering depends sensitively on the impact parameter
- directed, ordered velocities → random transverse velocities

relaxation: 3. violent relaxation

- proposed by Lynden-Bell for explaining the brightness profile of elliptical galaxie, wipes out structure of spiral galaxies in the merging
- each particle sees a rapidly fluctuating potential generated by all particles

$$\frac{dE}{dt} = \frac{m}{2}\frac{du^2}{dt} + \frac{\partial\Phi}{\partial t} + \vec{u}\nabla\Phi \tag{20}$$

- dynamic kind of scattering mediated by grav. field

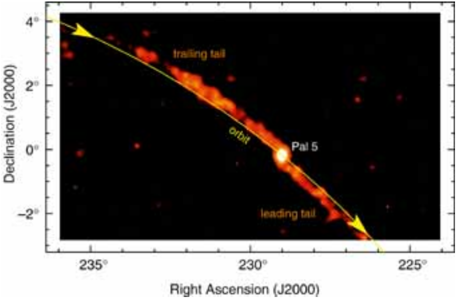
$$\text{with} \quad \frac{du^2}{dt} = 2\vec{u}\frac{d\vec{u}}{dt} = -\frac{2}{m}\vec{u}\nabla\Phi \quad \rightarrow \quad \frac{dE}{dt} = \frac{\partial\Phi}{\partial t} \tag{21}$$

- even particles with initially similar trajectories get separated

violent relaxation

important relaxation mechanism, due to long-reaching gravity

relaxation: 4. phase space mixing



globular cluster Palomar-5, source: J. Staude

- time evolution of a globular cluster orbiting the Milky Way:
  - stars closer to Galactic centre move faster
  - stars further away move slower
- with time, the streams get more elongated and eventually form a tightly wound spiral

relaxation: 4. phase space mixing

- naive interpretation:
  system produces structure on smaller and smaller scales (spiral winds up), eventually crosses thermodynamic scale λ
- but: the system is time-reversible and does conserve full phase space information
- relaxation does not take place, the system remembers its initial conditions
- thermodynamic scale is not well defined, gravity is a power law!
- solution: no matter how small the thermodynamic scale is chosen, the system will always wipe out structures above this scale with time $\rightarrow$ **coarse-graining**

generation of entropy

phase space density f measured above this scale decreases, and entropy $S \propto -\int d^3p d^3q\, f \ln f$ increases

final state: virialisation

final state

relaxation mechanisms generate a final state which does not depend on the initial conditions, e.g. a stable galaxy from some random flucutation in the Gaussian density field

- a virialised object does not evolve anymore and is characterised by a symmetric phase space distribution → **equipartition**, and a velocity distribution which depends only on constants of motion
- systems are stabilised against gravity by their particle motion, despite the lack of a microscopic collision mechanism which provides pressure
- virial relation $2\langle T\rangle = -\langle V\rangle$ between mass, size and temperature

$$\langle \upsilon^2\rangle = 3\sigma_\upsilon^2 = \frac{GM}{R} \rightarrow M \simeq \frac{3R\sigma_\upsilon^2}{G} = 10^{15}M_\odot/h\left(\frac{R}{1.5\mathrm{Mpc}/h}\right)\left(\frac{\sigma_\upsilon}{1000\mathrm{km/s}}\right)^2 \quad (22)$$

stability: density profiles of dark matter objects

- does a final state exist? needs to maximise entropy...
- use phase space density f for describing the steady-state distribution of particles in a dark matter halo
- solution need to be a solution of the collisionless steady-state ($\partial f/\partial t = 0$) Boltzmann-eqn.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{\upsilon}\nabla_x f - \nabla\Phi\nabla_\upsilon f = 0 \tag{23}$$

- and they need to be self consistent: the mass distribution generates its own potential

$$\Delta\Phi = 4\pi G\rho \text{ with } \rho = m\int d^3\upsilon\, f(\vec{x},\vec{\upsilon}) \tag{24}$$

- originally for galactic dynamics, applies for dark matter as well (collisionlessness)

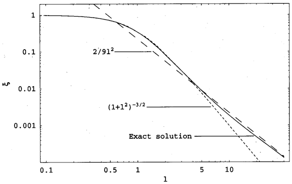
self-consistent solutions of dark matter objects

- Ansatz for phase space density f: should depend on the integrals of motion C, because then f satisfies the steady-state Boltzmann-equation: $df/dt = \partial f/\partial C \times \partial C/\partial t$
- shift potential Φ: $\psi = -\Phi + \Phi_0$, with constant Φ_0 (make ψ vanish at boundary)
- simple approach: phase space density $f(\vec{x}, \vec{\upsilon})$ depends only on $\varepsilon = \psi - \upsilon^2/2$, assumption of spherical symmetry
- matter density ρ for a model follows from

$$\rho(\vec{x}) = \int_0^\psi d\varepsilon\, 4\pi f(\varepsilon) \sqrt{2(\psi - \varepsilon)} \tag{25}$$

- substitute ρ in Poisson equation: $\Delta\psi = -4\pi G\rho$, solve for ψ as a function of ε, boundary conditions on $\psi(0) = \psi_0$ and $\psi'(0) = 0$

singular isothermal sphere



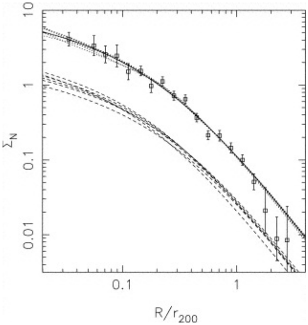
credit: Padmanabhan, theoretical astrophysics

- distribution function, motivated by Boltzmann statistics

$$f(\varepsilon) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} \exp\left(\frac{\varepsilon}{\sigma^2}\right) \quad (26)$$

- properties:
  - constant velocity dispersion inside object, $\sigma^2 = 3\langle\upsilon^2\rangle$
  - temperature assignment $k_B T \propto \sigma^2$
  - numerical solution to Boltzmann-problem exists, finite core density
  - at large radii, $\rho \propto r^{-2} \rightarrow$ flat rotation curve

Navarro-Frenk-White profile



question

construct a possible fitting formula for the NFW-profile!

Navarro-Frenk-White profile

- Navarro, Frenk + White: haloes in n-body simulation show a profile:

$$\rho \propto \frac{1}{x(1+x^2)} \quad \text{with} \quad x \equiv \frac{r}{r_c} \quad \text{and} \quad r_c = c r_{vir} \tag{27}$$

- universal density profile, applicable to haloes of all masses
- fitting formula breaks down:
  - infinite core density
  - total mass diverges logarithmically
- very long lived transitional state (gravothermal instability)
- scale radius r_s is related to virial radius by concentration parameter c
- c has a weak dependence on mass in dark energy models

question

show that the NFW-profile allows flat rotation curves!

what's the size of the galactic disk? what happens if the

number density of collapsed objects

halo formation

haloes form at peaks in the density field → reflect the fluctuations statistics in the high-δ tail of the probability density

- valuable source of information on Ω_m, σ_8, w and h
- prediction of the number density of haloes from the spectrum $P(k)$ → **Press-Schechter formalism**
- relate mass M to a length scale R

$$M = \frac{4\pi}{3}\Omega_m \rho_{crit} R^3 \tag{28}$$

- how often does the density field try to exceed some threshold δ_c on the mass scale M?

Press-Schechter formalism

- consider variance of the convolved density field

$$\sigma_R^2 = \frac{1}{2\pi^2}\int dk\, k^2 P(k) W(kR)^2 \tag{29}$$

with a top-hat filter function of size R

- convolved field $\bar{\delta}$ has a Gaussian statistic with the variance σ_R^2

$$p(\bar{\delta}, a)d\bar{\delta} = \frac{1}{\sqrt{2\pi\sigma_R^2}} \exp\left(-\frac{\bar{\delta}^2}{2\sigma_R^2(a)}\right) \tag{30}$$

with $\sigma_R^2(a) = \sigma_R^2 D_+(a)$

- condition for halo formation: $\bar{\delta} > \delta_c$
- fraction of cosmic volume filled with haloes of mass M

$$F(M, a) \int_{\delta_c}^{\infty} d\bar{\delta}\, p(\bar{\delta}, a) = \frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R(a)}\right) \tag{31}$$

Press-Schechter formalism

- distribution of haloes with mass M: $\partial F(M)/\partial M \rightarrow$ add relation between M and R

$$\frac{\partial F(M)}{\partial M} = \frac{1}{\sqrt{2\pi}} \frac{\delta_c}{\sigma_R D_+(a)} \frac{d\ln\sigma_R}{dM} \exp\left(-\frac{\delta_c}{2\sigma_R^2 D_+^2(a)}\right) \quad (32)$$

after using the derivative

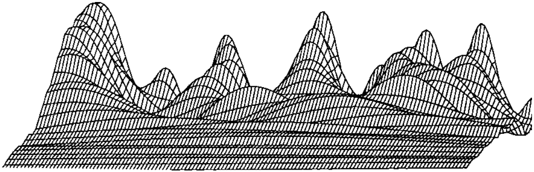
$$\frac{d}{dx}\mathrm{erfc}(x) = -\frac{2}{\sqrt{\pi}}\exp(-x^2) \quad (33)$$

- comoving number density: divide occupied cosmic volume fraction by halo volume M/ρ_0

$$n(M,a)dM = \frac{\rho_0}{\sqrt{2\pi}} \frac{\delta_c}{\sigma_R D_+(a)} \frac{d\ln\sigma_R}{d\ln M} \exp\left(-\frac{\delta_c^2}{2\sigma_R^2 D_+^2(a)}\right) \frac{dM}{M^2} \quad (34)$$

- normalisation is not right by a factor of 2, but there is an elaborate argument for fixing it

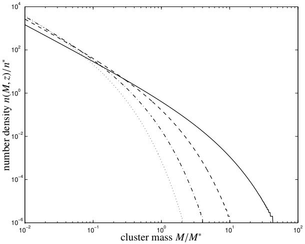
halo formation as a random walk



source: Bond et al. (1991)

- if the density is smoothed with $R = \infty$, the mean density of any perturbation is $\delta = 0$ and $\rho = \bar{\rho} = \Omega_m \rho_{crit}$
- reduce filter scale: density field will develop fluctuations
- if a density on scale R exceeds the threshold δ_c, it will collapse and form an object of mass $M = 4\pi\rho_0\delta R^3/3$
- at a single point in space: δ as a function of R performs a random walk (for k-space top-hat filter)
- probability of $\delta > \delta_c$ is given by $\mathrm{erfc}(\delta_c/(\sqrt{2}\sigma(M))$

CDM mass functions



CDM mass function: comoving number density of haloes (redshifts $z = 0, 1, 2, 3$)

- shape of mass function: power law with exponential cut-off
- CDM:
  - hierarchical structure formation: more massive objects form later
  - cut-off scale $M_* \propto D_+(z)^3$ (dark energy influence!)
- normalisation: $\simeq 100$ clusters and $\simeq 10^4$ galaxies in a cube with side length 100 Mpc/h today ($a = 1$, $z = 0$)

cosmological parameter from cluster surveys

- mass function (comoving number density of haloes of mass M)

$$n(M,z)\mathrm{d}M = \sqrt{\frac{2}{\pi}}\rho_0\Delta(M,z)\frac{\mathrm{d}\ln\sigma(M)}{\mathrm{d}\ln M}\exp\left(-\frac{\Delta^2(M,z)}{2}\right)\frac{\mathrm{d}M}{M^2} \qquad (35)$$

  with $\rho_0 = \Omega_m\rho_{crit}$

- Δ describes the ratio between collapse overdensity and variance of the fluctuation strength on the mass scale M:

$$\Delta(M,z) = \frac{\delta_c(z)}{D_+(z)\sigma(M)} \qquad (36)$$

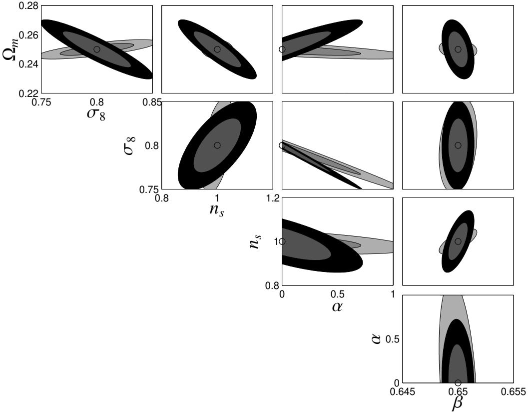
- comoving space is a theoretical construct, we observe redshifts!

$$N(z) = \frac{\Delta\Omega}{4\pi}\frac{\mathrm{d}V}{\mathrm{d}z}\int_{M_{min}(z)}^{\infty}\mathrm{d}M\, n(M,z) \qquad (37)$$

- comoving volume element, with the angular diameter distance d_A:

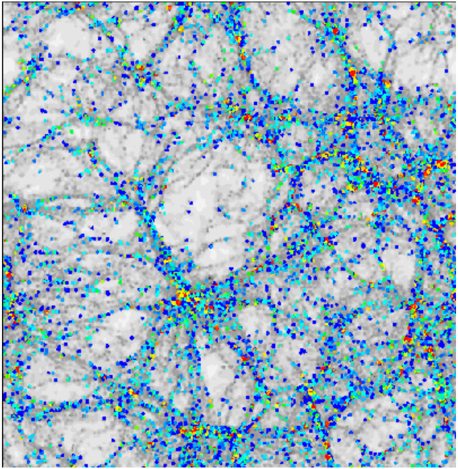
$$\mathrm{d}V \quad d_A^2(a)$$

cosmological parameter from cluster surveys



cosmological parameters from cluster surveys

galaxy biasing



GIF-simulation, Kaufmann et al.

galaxy bias models

- galaxies trace the distribution of dark matter
- simplest (local, linear, static, morphology and scale-indep.) relation:

$$\frac{\delta n}{\langle n \rangle} = b \frac{\rho}{\langle \rho \rangle} \tag{39}$$

  with **bias parameter** b
- bias models:
  - massive objects are more clustered (larger b) than low-mass objects
  - red galaxies are stronger clustered than blue galaxies
  - bias is slowly time evolving and **decreases**
- physical explanation: galaxies form at local peaks in the dark matter field, and reflect the local matter density directly
- naturally: $\xi_{galaxy}(r) = b^2 \xi_{CDM}(r)$ for the above model

question

are there more galaxies if b is larger?

galaxy formation: Jeans instability

- galaxies form by condensation of baryons inside potential wells formed by dark matter
- cooling process: needs to be fast, for overcoming the negative specific heat of a self-gravitating system
- hydrostatic equilibrium: balance **pressure** and **gravity**

$$\frac{dp}{dr} = -\frac{GM}{r^2}\rho \tag{40}$$

- collapse: internal pressure smaller than gravity, which happens if M is large, or the temperature small (small pressure)

Jeans mass

Jeans mass is the **minimum mass** for galaxy formation

Jeans-scale: derivation

- initially: spherical gas cloud of radius R and mass M
- compress cloud slightly: pressure wave will propagate through it, and establish new equilibrium
  - pressure equilibration = sound crossing time $t_{sound} = \frac{R}{c_s}$
  - gravitational collapse = free-fall time scale $t_{grav} = \frac{1}{\sqrt{G\rho}}$
- compare time scales
  - $t_{grav} > t_{sound}$ pressure wins, system settles in new equilibrium
  - $t_{grav} < t_{sound}$ gravity wins, system undergoes spherical collapse
- Jeans length $R_J = c_s t_{grav}$ allows to determine Jeans mass M_J:

$$M_J = \frac{4\pi}{3}\rho\left(\frac{R_J}{2}\right)^3 = \frac{\pi}{6}\frac{c_s^3}{G^{1.5}\rho^{0.5}} \tag{41}$$

stability of elliptical galaxies

- stabilisation of elliptical galaxies → **velocity dispersion**
- Jeans equations are 2 coupled nonlinear PDEs for the evolution of collisionless systems
  - first moment: continuity

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{\upsilon}) = 0 \tag{42}$$

  - second moment: momentum equation

$$\frac{\partial\vec{\upsilon}}{\partial t} + \vec{\upsilon}\nabla\vec{\upsilon} = -\nabla\Phi - \mathrm{div}(\rho\sigma^2) \tag{43}$$

- no viscosity, and velocity dispersion tensor $\sigma_{ij}^2 = \langle\upsilon_i\upsilon_j\rangle - \langle\upsilon_i\rangle\langle\upsilon_j\rangle$ **emulates** (possibly anisotropic) pressure
- gravitational potential: self-consistently derived from Poisson's equation $\Delta\Phi = 4\pi G\rho$, closed system!
- in a virialised elliptical galaxy, σ_{ij} corresponds to $\langle V\rangle$ → **stability**

stability of spiral galaxies

- collisionless fluids can not build up pressure against gravity
- a rotating system can provide force balancing → **centrifugal force**
- spin-up: explained by tidal torquing
- spin-parameter λ

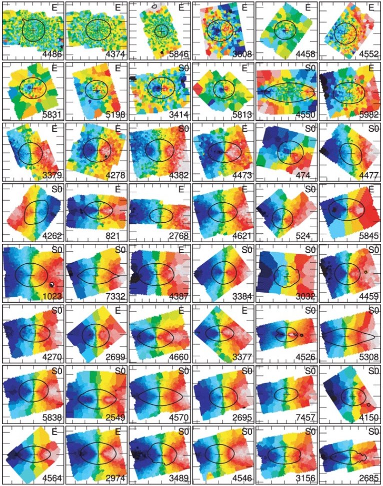
$$\lambda \equiv \frac{\omega}{\omega_0} = \frac{L/(MR^2)}{\sqrt{GM/R^3}} = \frac{L\sqrt{E}}{GM^{5/2}} \quad (44)$$

- specific angular momentum necessary for rotational support
- $\lambda \simeq 1/2$ in spirals in ΛCDM cosmologies, rotation is the dominant supporting mechanism

question

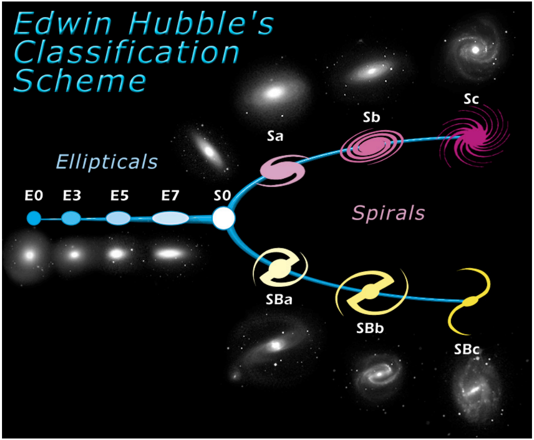
why is the definition of λ sensible?

SAURON observations of galaxies



source: SAURON experiment

galaxy morphologies: 'tuning fork' diagramme



source: wikipedia

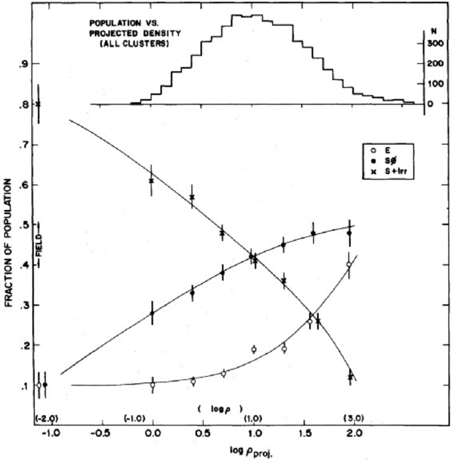
merging of haloes

- contary to Hubble's hypothesis: merging activity and tidal interaction influence galaxy morphologies and convert spirals into ellipticals → **density-morphology relation**
- confusing nomenclature remains:

elliptical	early-type	old stars
spiral	late-type	young stars

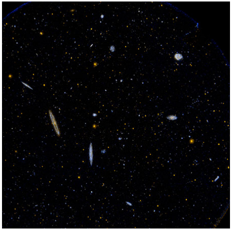
- merging generates heavy haloes from low-mass systems and wipes out the kinematical structure by violent relaxation → **bottom-up structure formation**
- merging activity depends on the cosmology, and causes the mass function to evolve

density-morphology relation



density-morphology relation, source: Dressler et al. (1980s)

galaxy clusters



Perseus cluster (source: NASA/JPL) Virgo cluster (source: USM)

- largest, gravitationally bound objects, with $M > M_*$
- quasar host structures at high redshift
- historically
  - visual identification (Abell catalogue)
  - need for dark matter: dynamical mass $\gg$ sum of galaxies (Zwicky)
- large clusters have masses of $10^{15} M_\odot/h$ and contain $\sim 10^3$ galaxies

X-ray emission of clusters

- the intra-cluster medium of clusters of galaxies is so hot ($T \simeq 10^7$K) that is produces thermal X-ray radiation
- the plasma is in hydrostatic equilibrium with gravity, therefore the density profile can be computed

$$\frac{dp}{dr} = -\frac{GM(r)}{r^2}\rho \rightarrow \frac{k_B T}{m}\frac{d\rho}{dr} + \frac{\rho k_B}{m}\frac{dT}{dr} = -\frac{GM}{r^2}\rho \qquad (45)$$

for ideal gas with $p = \rho k_B T/m$

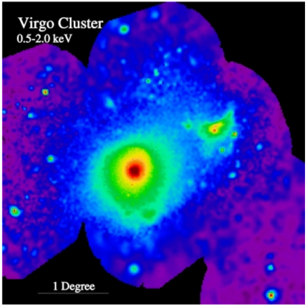
- determination of mass: from measurement of the density and temperature profile:

$$M(r) = -\frac{r k_B T}{Gm}\left(\frac{d\ln\rho}{d\ln r} + \frac{d\ln T}{d\ln r}\right) \qquad (46)$$

question

what can one do if the cluster is not spherically symmetric?

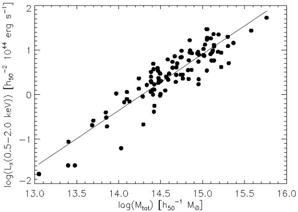
X-ray emission of clusters: ROSAT data



VIRGO cluster as seen by ROSAT

- cluster is in hydrostatic equilibrium
- X-ray emissivity is $\propto \sqrt{T}\rho^2 \rightarrow$ fuzzy blobs

scaling relations



scaling relation between L_X and M from the ROSAT survey

- virial relation allow the prediction of simple **scaling relations**
- valid for fully virialised systems, where the temperature reflects the release in gravitational binding energy
  - potential energy $\langle V\rangle \propto -GM^2/R$
  - size $M \propto R^3 \rightarrow \langle V\rangle \propto -M^{5/3}$
  - kinetic energy $\langle T\rangle \propto TM$
  - virial relation $2\langle T\rangle = -\langle V\rangle \rightarrow T \propto M^{2/3}$
  - X-ray luminosity $L_X \propto M^2\sqrt{T}/R^3 \propto M^{4/3} \propto T^2$