QCD and Monte Carlo simulation III

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http://www.desy.de/~jung/qcd_and_mc_2009

Outline of the lectures

- 10 July Intro to Monte Carlo techniques and structure of matter
- 17 July DGLAP and small x: solution with MCs
- 23 July Small x, CCFM and BFKL
- 30 July W/Z production in pp and soft gluon resummation NEW

- Lectures will be recorded and made available immediately .....

- Exercises in the afternoons: 14:00 – 16:30 in sem 1
  Assistant: A. Grebenyuk

- Discussion forum online:
  see link from web page or

http://www.terascale.de/research_topics/rt1_physics_analysis/monte_carlo_generators/discussion_forum/discussion_forum_lecture_2
_monte_carlos

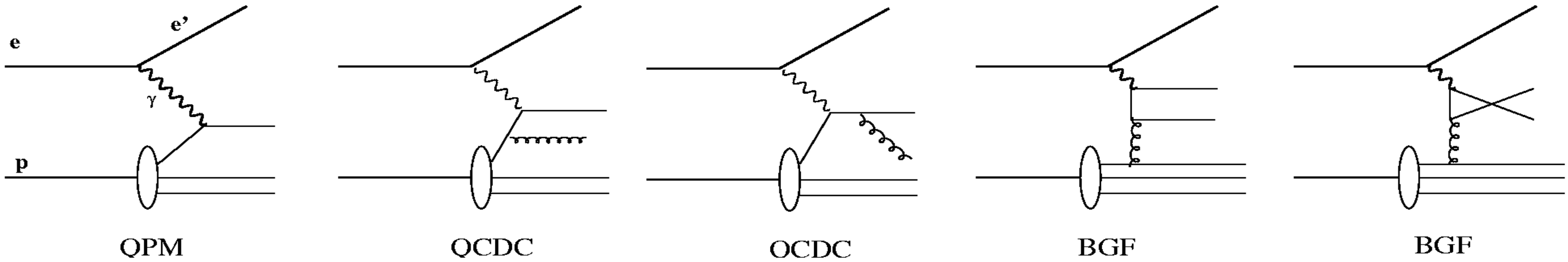
Requests to you ...

- If things go wrong .. lecture is too easy... too trivial ... too complicated, too chaotic or too boring ...
- **PLEASE complain immediately !**
- PLEASE ask questions any time !

Questions from
last lecture and
last exercise ?

Recap from last lecture …

recap: Higher order corrections to DIS



- lowest order: $e + q \to e' + q' \quad \mathcal{O}(\alpha_s^0)$
- higher order: $e + q \to e' + q' + g, \quad e + g \to e' + q + \bar{q} \quad \mathcal{O}(\alpha_s^1)$

  - What is the dominant part of the x-section ?
    - Investigate full x-section of QCDC and BGF
    - dominant part comes from small transverse momenta ...
    - rewrite x-section in terms of $k_\perp$
    - use small t limit:

$$\frac{d\sigma}{dk_\perp} = \frac{d\sigma}{dt}\frac{1}{(1-z)} = \frac{1}{(1-z)}\frac{1}{F}dLips|ME|^2$$

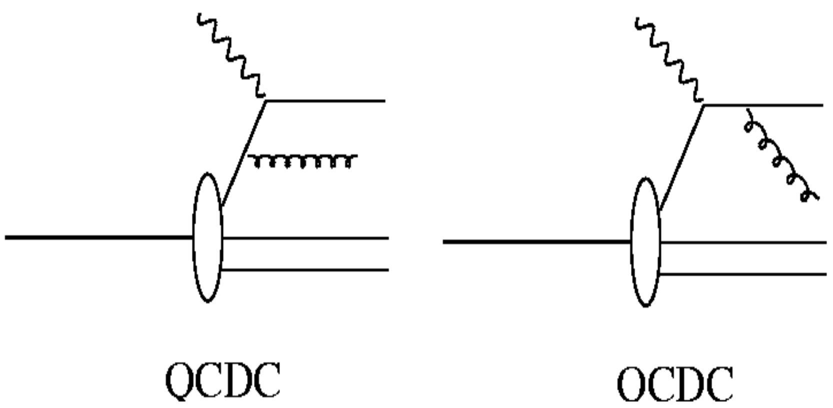
$$= \frac{1}{(1-z)}\frac{1}{16\pi}\frac{1}{\hat{s}+Q^2}\frac{1}{\hat{s}}|ME|^2$$

recap: QCDC - contribution

$$|ME|^2 = 32\pi^2 \left(e_q^2 \alpha\alpha_s\right) \frac{4}{3} \left[\frac{-\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}} \right]$$

$$= 32\pi^2 \left(e_q^2 \alpha\alpha_s\right) \frac{4}{3} \frac{-1}{t} \left[\frac{Q^2(1+z^2)}{z(1-z)} + \cdots \right]$$

- integrate over *kt* generates *log*, BUT what is the lower limit



$$\frac{d\sigma}{dk_\perp^2} = \sigma_0 e_q^2 \frac{\alpha_s}{2\pi} \frac{1}{k_\perp^2} \left[P_{qq}(z) + \cdots \right]$$

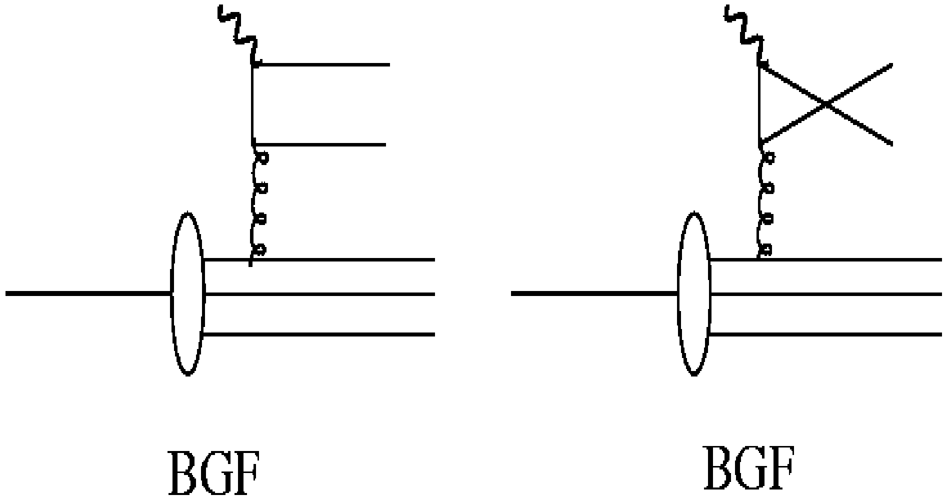
$$P_{qq}(z) = \frac{4}{3} \frac{1+z^2}{1-z} \qquad \sigma_0 = \frac{4\pi^2\alpha}{\hat{s}}$$

$$\sigma^{QCDC} = \sigma_0 e_q^2 \frac{\alpha_s}{2\pi} \left[P_{qq}(z) \log\left(\frac{Q^2(1-z)}{\chi^2 z} \right) + \cdots \right]$$

recap: boson gluon fusion

$$|M|^2 = 32\pi^2 \left(e_q^2 \alpha \alpha_s\right) \frac{1}{2} \left[\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} - \frac{2\hat{s}Q^2}{\hat{t}\hat{u}} \right]$$

$$= 32\pi^2 \left(e_q^2 \alpha \alpha_s\right) \frac{-1}{t} \frac{Q^2}{z} \frac{1}{2} \left(z^2 + (1-z)^2 + \cdots\right)$$



$$\frac{d\sigma}{dk_\perp^2} = \sigma_0 e_q^2 \frac{\alpha_s}{2\pi} \frac{1}{k_\perp^2} \left[P_{qg}(z) + \cdots\right]$$

$$P_{qg}(z) = \frac{1}{2}\left(z^2 + (1-z)^2\right)$$

- integrate over kt generates log, BUT what is the lower limit

$$\sigma^{BGF} = \sigma_0 e_q^2 \frac{\alpha_s}{2\pi} \left[P_{qg}(z) \log\left(\frac{Q^2(1-z)}{\chi^2 z} \right) + \cdots \right]$$

Collinear factorization: DGLAP

- introduce new scale $\mu^2 \gg \chi^2$ and include soft, non-perturbative physics into renormalized parton density:

$$q_i(x,\mu^2) = q_i^0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[q_i^0(\xi)P_{qq}\left(\frac{x}{\xi}\right) + g^0(\xi)P_{qg}\left(\frac{x}{\xi}\right)\right]\log\left(\frac{\mu^2}{\chi^2}\right)$$

- **D**okshitzer **G**ribov **L**ipatov **A**ltarelli **P**arisi equation:

V.V. Gribov and L.N. Lipatov Sov. J. Nucl. Phys. 438 and 675 (1972) 15, L.N. Lipatov Sov. J. Nucl. Phys 94 (1975) 20, G. Altarelli and G. Parisi Nucl.Phys.B 298 (1977) 126, Y.L. Dokshitser Sov. Phys. JETP 641 (1977) 46

$$\frac{dq_i(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[q_i(\xi,\mu^2)P_{qq}\left(\frac{x}{\xi}\right) + g(\xi,\mu^2)P_{qg}\left(\frac{x}{\xi}\right)\right]$$

- BUT there are also gluons....

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[\sum_i q_i(\xi,\mu^2)P_{gq}\left(\frac{x}{\xi}\right) + g(\xi,\mu^2)P_{gg}\left(\frac{x}{\xi}\right)\right]$$

- DGLAP is the analogue to the beta function for running of the coupling

recap: Solving DGLAP equations ...

- Different methods to solve integro-differential equations
  - brute-force (BF) method (M. Miyama, S. Kumano CPC 94 (1996) 185)

$$\frac{df(x)}{dx} = \frac{f(x)_{m+1} - f(x)_m}{\Delta x_m} \qquad \int f(x)dx = \sum f(x)_m \Delta x_m$$

  - Laguerre method (S. Kumano J.T. Londergan CPC 69 (1992) 373, and L. Schoeffel Nucl.Instrum.Meth.A423:439-445,1999)
  - Mellin transforms (M. Glueck, E. Reya, A. Vogt Z. Phys. C48 (1990) 471)
  - QCDNUM: calculation in a grid in x,Q2 space (M. Botje Eur.Phys.J. C14 (2000) 285-297)
  - CTEQ evolution program in x,Q2 space: http://www.phys.psu.edu/~cteq/
  - QCDFIT program in x,Q2 space (C. Pascaud, F. Zomer, LAL preprint LAL/94-02, H1-09/94-404,H1-09/94-376)
  - MC method using Markov chains (S. Jadach, M. Skrzypek hep-ph/0504205)
  - Monte Carlo method from iterative procedure
- brute-force method and MC method are best suited for detailed studies of branching processes !!!

recap: Divergencies again...

- collinear divergencies factored into renormalized parton distributions $z \to 1$
- what about soft divergencies ?

  treated with "plus" prescription

$$\frac{1}{1-z} \to \frac{1}{1-z}_{+} \quad \text{with} \quad \int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{(1-z)}$$

- soft divergency treated with Sudakov form factor:

$$\Delta(t) = \exp\left[- \int_{t_0}^{t} \frac{dt'}{t'} \int^{z_{max}} dz \frac{\alpha_s}{2\pi} \tilde{P}(z) \right]$$

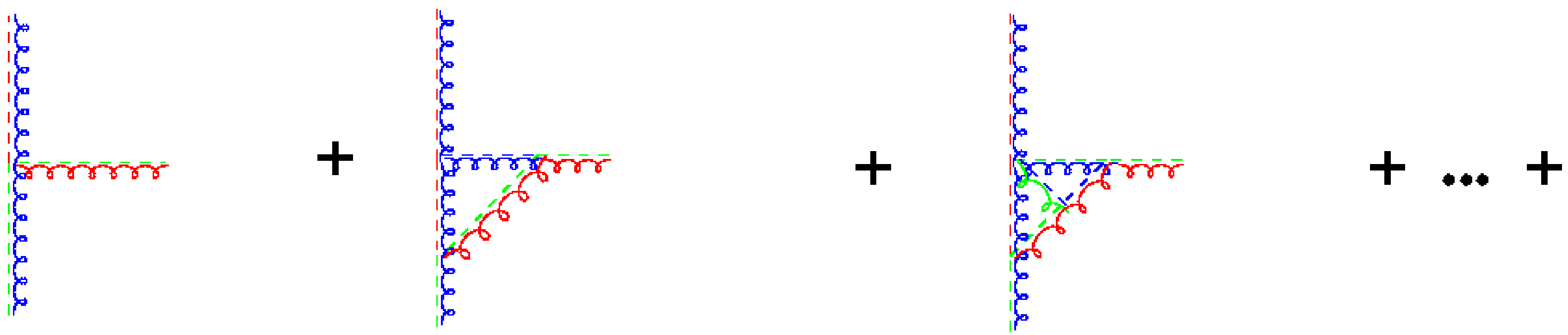
recap: Sudakov form factor: all loop resum

$g \to gg$ Splitting Fct $\tilde{P}(z) = \frac{\bar{\alpha}_s}{1-z} + \frac{\bar{\alpha}_s}{z} + \ldots$

- Sudakov form factor .... all loop resummation

$$\Delta_S = \exp\left(- \int dz \int \frac{dq'}{q'} \frac{\alpha_s}{2\pi} \tilde{P}(z) \right)$$

$$\Delta_S = 1 + \left(- \int dz \int \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z) \right)^1 + \frac{1}{2!} \left(- \int dz \int \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z) \right)^2 \ldots$$



$$\tilde{P}(z) \left[1 - \int \int dz \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z) + \frac{1}{2!} \left(- \int \int dz \frac{dq}{q} \frac{\alpha_s}{2\pi} \tilde{P}(z) \right)^2 - \ldots - \right]$$

recap: DGLAP evolution again....

- differential form: $$t\frac{\partial}{\partial t} f(x,t) = \int \frac{dz}{z}\frac{\alpha_s}{2\pi} P_+(z)\, f\left(\frac{x}{z},t\right)$$
- differential form using f/Δ_s with

$$\Delta_s(t) = \exp\left(- \int_x^{z_{max}} dz \int_{t_0}^{t} \frac{\alpha_s}{2\pi}\frac{dt'}{t'}\tilde{P}(z)\right)$$

$$t\frac{\partial}{\partial t}\frac{f(x,t)}{\Delta_s(t)} = \int \frac{dz}{z}\frac{\alpha_s}{2\pi}\frac{\tilde{P}(z)}{\Delta_s(t)} f\left(\frac{x}{z},t\right)$$

- integral form

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z}\int \frac{dt'}{t'}\cdot\frac{\Delta_s(t)}{\Delta_s(t')}\tilde{P}(z) f\left(\frac{x}{z},t'\right)$$

no - branching probability from t_0 to t

recap: Solving integral equations

- Integral equation of Fredholm type: $\phi(x) = f(x) + \lambda \int_a^b K(x,y)\phi(y)dy$
- solve it by iteration (Neumann series):

$$\begin{aligned}
\phi_0(x) &= f(x) \\
\phi_1(x) &= f(x) + \lambda \int_a^b K(x,y) f(y) dy \\
\phi_2(x) &= f(x) + \lambda \int_a^b K(x,y_1) f(y_1) dy_1 + \lambda^2 \int_a^b \int_a^b K(x,y_1) K(y_1,y_2) f(y_2) dy_2 dy_1 \\
\phi_n(x) &= \sum_{i=0}^{n} \lambda^i u_i(x) \\
u_0(x) &= f(x) \\
u_1(x) &= \int_a^b K(x,y) f(y) dy \\
u_n(x) &= \int_a^b \int_a^b K(x,y_1) K(y_1,y_2) \cdots K(y_{n-1},y_n) f(y_n) dy_2 \cdots dy_n
\end{aligned}$$

with the solution: $$\phi(x) = \lim_{n\to\infty} q_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \lambda^i u_i(x)$$

recap: DGLAP re-sums leading logs...

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

- solve integral equation via iteration:

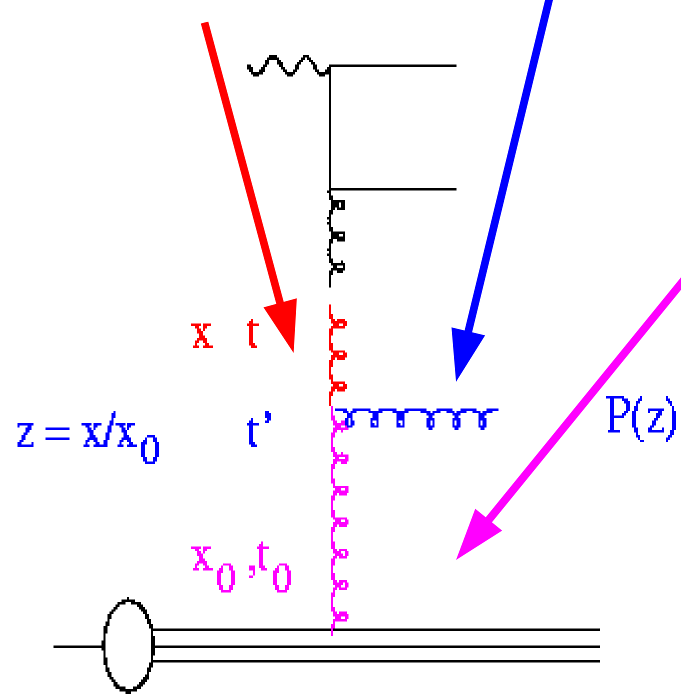
from t' to t w/o branching

branching at t'

from t_0 to t' w/o branching

$$f_0(x,t) = f(x,t_0)\Delta(t)$$

$$f_1(x,t) = f(x,t_0)\Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \tilde{P}(z) f(x/z, t_0)\Delta(t')$$



recap: DGLAP re-sums leading logs...

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

- solve integral equation via iteration:

from t' to t w/o branching

branching at t'

from t_0 to t' w/o branching

$$f_0(x,t) = f(x,t_0)\Delta(t)$$

$$f_1(x,t) = f(x,t_0)\Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \tilde{P}(z) f(x/z, t_0)\Delta(t')$$

$$= f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A \otimes \Delta(t) f(x/z, t_0)$$

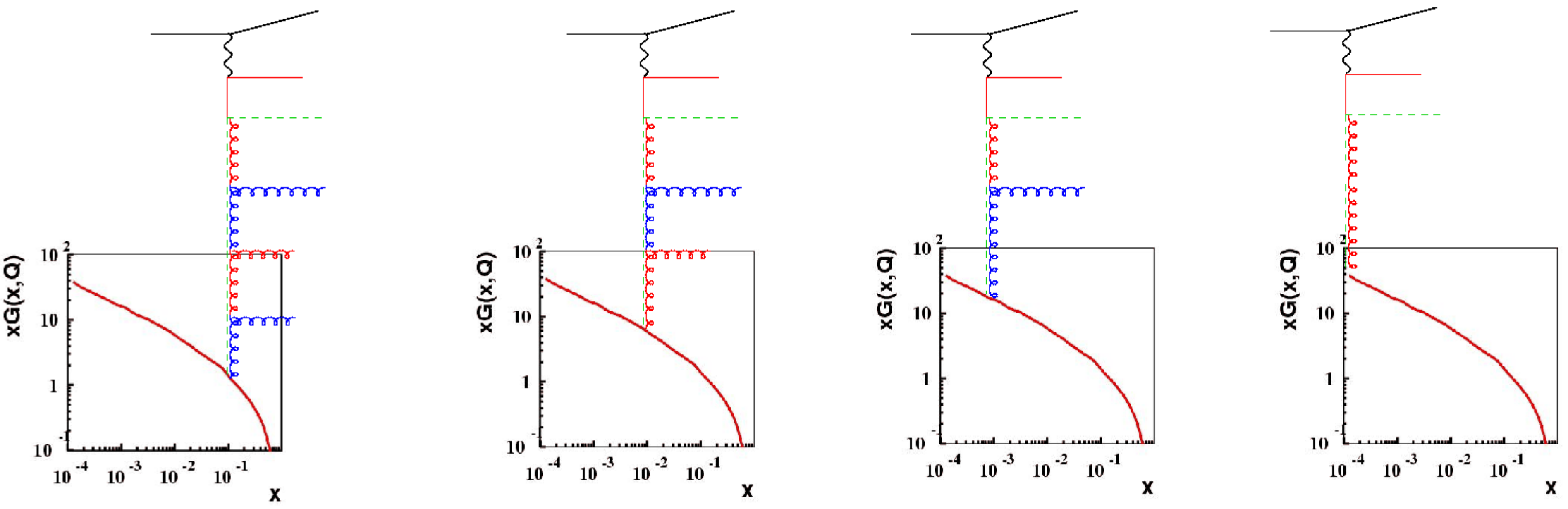
$$f_2(x,t) = f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A \otimes \Delta(t) f(x/z, t_0) + \frac{1}{2}\log^2\frac{t}{t_0} A \otimes A \otimes \Delta(t) f(x/z, t_0)$$

$$f(x,t) = \lim_{n\to\infty} f_n(x,t) = \lim_{n\to\infty} \sum_n \frac{1}{n!} \log^n\left(\frac{t}{t_0}\right) A^n \otimes \Delta(t) f(x/z, t_0)$$

DGLAP re-sums $\log t$ to all orders !!!!!!!!!!!!!!!

recap: DGLAP evolution equation, again

- for fixed x and Q^2 chains with different branchings contribute
- iterative procedure, spacelike parton showering



$$f(x,t) = f_0(x,t_0)\Delta_s(t) + \sum_{k=1}^{\infty} f_k(x_k,t_k)$$

recap: What is happening at small x ?

- For $x \to 0$ only gluon splitting function matters:

$$P_{gg} = 6\left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right) = 6\left(\frac{1}{z} - 2 + z(1-z) + \frac{1}{1-z}\right)$$

$$P_{gg} \sim 6\frac{1}{z} \text{ for } z \to 0$$

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} g(\xi,\mu^2) P_{gg}\left(\frac{x}{\xi}\right)$$

- evolution equation is then:

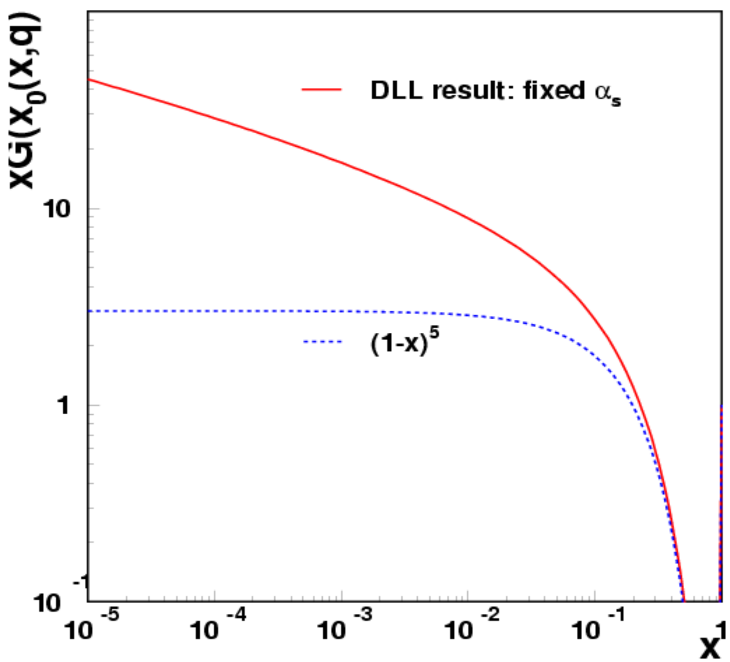
$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} g(\xi,\mu^2) P_{gg}\left(\frac{x}{\xi}\right)$$

$$xg(x,t) = \frac{3\alpha_s}{\pi}\int_{t_0}^{t} d\log t' \int_x^1 \frac{d\xi}{\xi}\xi g(\xi,t') \quad \text{with} \quad t = \mu^2$$

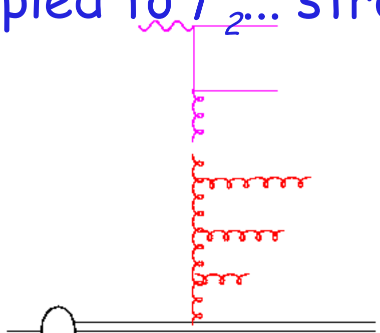
recap: Results from DLL approximation

- DLL arise from taking small x limit of splitting fct:
  - $\log 1/x$ from small x limit of splitting fct
  - $\log t/t_0$ from t integration
  - strong ordering in x from small x limit
  - strong ordering in t from small t limit of ME...

$$xg(x,t) \sim C \exp\left(2\sqrt{\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}}\right)$$



- DLL gives rapid increase of gluon density from a flat starting distribution
- gluons are coupled to F_2... strong rise of F_2 at small x:



- consequences:
- rise continues forever ???
- what happens when too high gluon density ?
-

There are different divergencies ...

Divergencies, everywhere ...

- $\frac{1}{t}$ singularities in $\mathcal{O}(\alpha_s)$ matrix elements

  ➔ leads to redefinition of PDFs and collinear factorization

- $\frac{1}{1-z}$ singularities in splitting functions

  ➔ treated by virtual corrections via “+” prescription or Sudakov

- $\frac{1}{z}$ singularities in splitting functions

  ➔ treated by dedicated small x evolution equations: BFKL/CCFM

Divergencies, everywhere ...

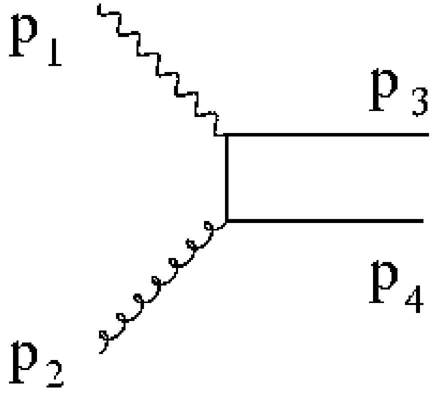
- $\frac{1}{t}$ singularities in $\mathcal{O}(\alpha_s)$ matrix elements

  ➔ leads to redefinition of PDFs and collinear factorization

Calculating x-sections

- partonic x-section:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi}\frac{1}{\hat{s}+Q^2}\frac{1}{\hat{s}}|M|^2$$



  - with

$$|M|^2(\gamma^* g \to q\bar{q}) \quad = \quad 32\pi^2\left(e_q^2\alpha\alpha_s\right)\frac{1}{2}\left[\frac{\hat{u}}{\hat{t}}+\frac{\hat{t}}{\hat{u}}-\frac{2\hat{s}Q^2}{\hat{t}\hat{u}}\right]$$

- photon-proton x-section:

$$\frac{d\sigma(\gamma^* p \to X)}{dx_g d\cos\theta} = g(x_g,\mu^2)\frac{d\hat{\sigma}(\mu^2,\hat{s},....)}{d\cos\theta}$$

  - with $t = -2E_2E_4(1-\cos\theta)$

Jet Cross Sections in $O(\alpha_s)$

S. Schilling desy-thesis-00-040

- From differential x-section

$$\frac{d\sigma(\gamma g \to q\bar{q})}{dx\, dz\, d\hat{t}} = K \sum_{\text{quarks } a} e_a^2 \frac{\alpha_s(Q^2)}{4\pi^2} \frac{2z}{Q^2} z f_g(\frac{x}{z}, Q^2) \times$$

$$\times \left\{ \frac{1}{4} \left(\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} - 2\frac{\hat{s}Q^2}{\hat{u}\hat{t}} + 4\frac{\hat{s}Q^2}{(\hat{s}+Q^2)^2} \right) \right\}$$

➔ obtain:

J. Collins JHEP 0005:004,2000

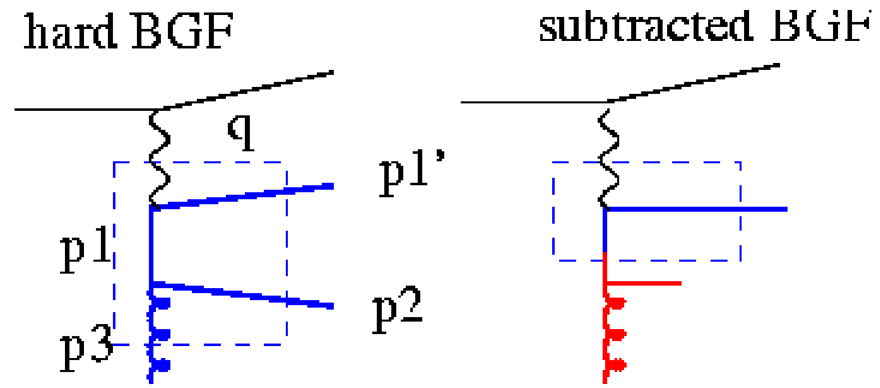
$$\frac{d\sigma(\gamma g \to q\bar{q})}{dx\, dz\, d\cos\theta} = K \sum_{\text{quarks } a} e_a^2 \frac{\alpha_s(Q^2)}{4\pi^2} z f_g(\frac{x}{z}, Q^2) \times$$

$$\times \left\{ P(z) \left[\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \right] - \frac{1}{2} + 3z(1-z) \right\},$$

BGF: LO and subtraction

$$z = \frac{p_3 q}{pq}$$

$$u = \frac{-Q^2(1+cos\theta)z}{2x}$$



➔ and obtain singular piece separately:

$$\frac{d\sigma^{\text{subtract}}(\gamma g \to q\bar{q})}{dx\, dy\, dz\, \mathrm{d}\cos\theta} = K \sum_{\text{quarks } a} e_a^2 \frac{\alpha_s(Q^2)}{4\pi^2} z f_g(\frac{x}{z}, Q^2) P(z) \left[\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \right]$$

➔ problem: kinematics ... avoid to subtract too much

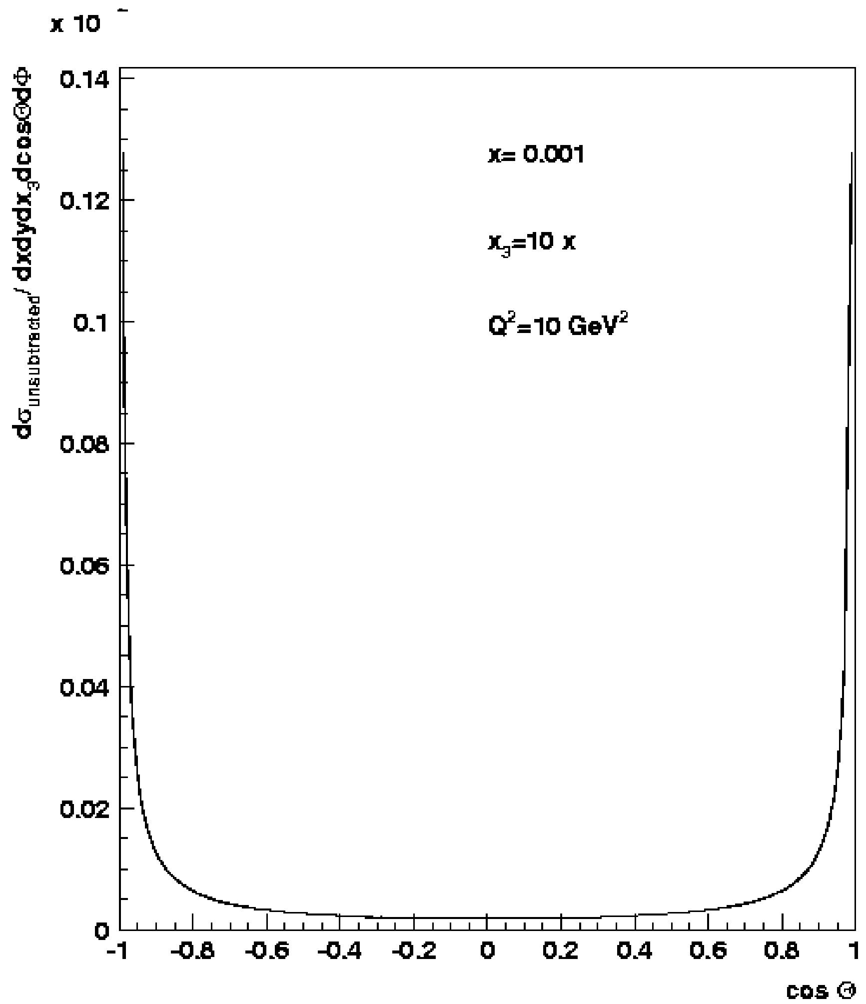
$$\frac{d\sigma_{\text{hard}}}{dx\, dy\, dz\, \mathrm{d}\cos\theta} = K \sum_{\text{quarks } a} e_a^2 \frac{\alpha_s(Q^2)}{4\pi^2} z f_g(\frac{x}{z}, Q^2) \times$$

$$\times \left\{ \frac{P(z)(1-C(-t))}{1-\cos\theta} + \frac{P(z)(1-C(-u))}{1+\cos\theta} - \frac{1}{2} + 3z(1-z) \right\}$$

- with $C(a) = \Theta(Q^2 - a)$

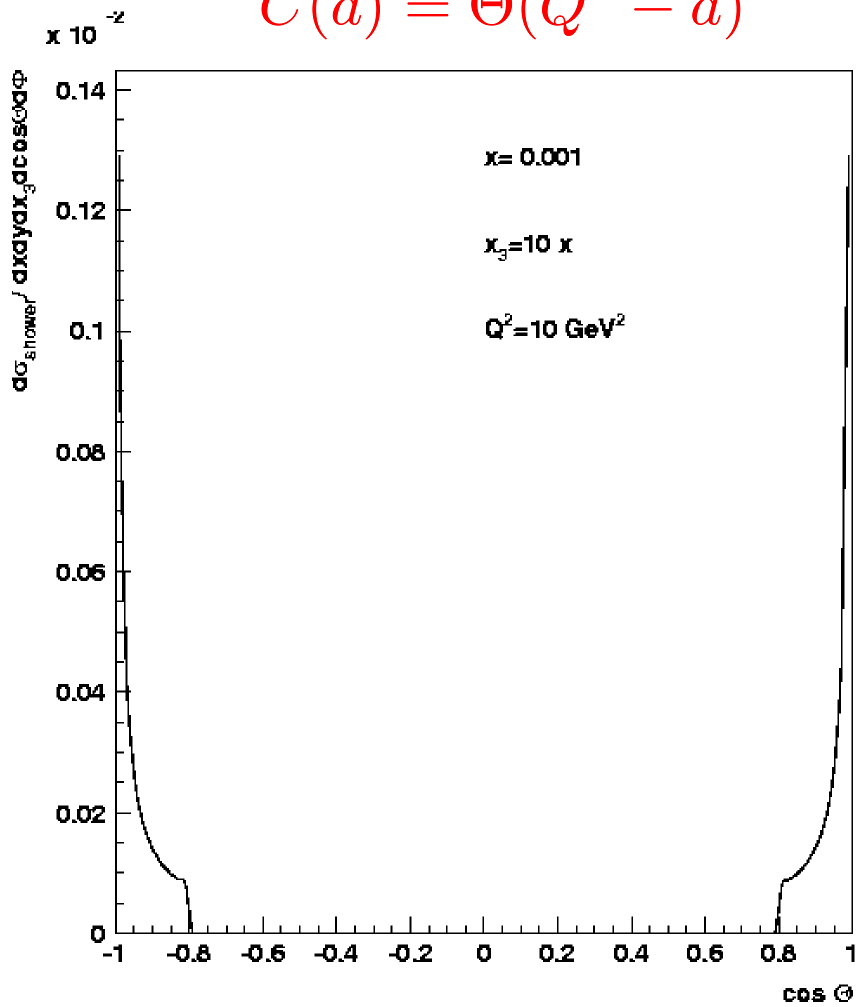
BGF x-section

full BGF contribution



collinear contribution

$$C(a) = \Theta(Q^2 - a)$$

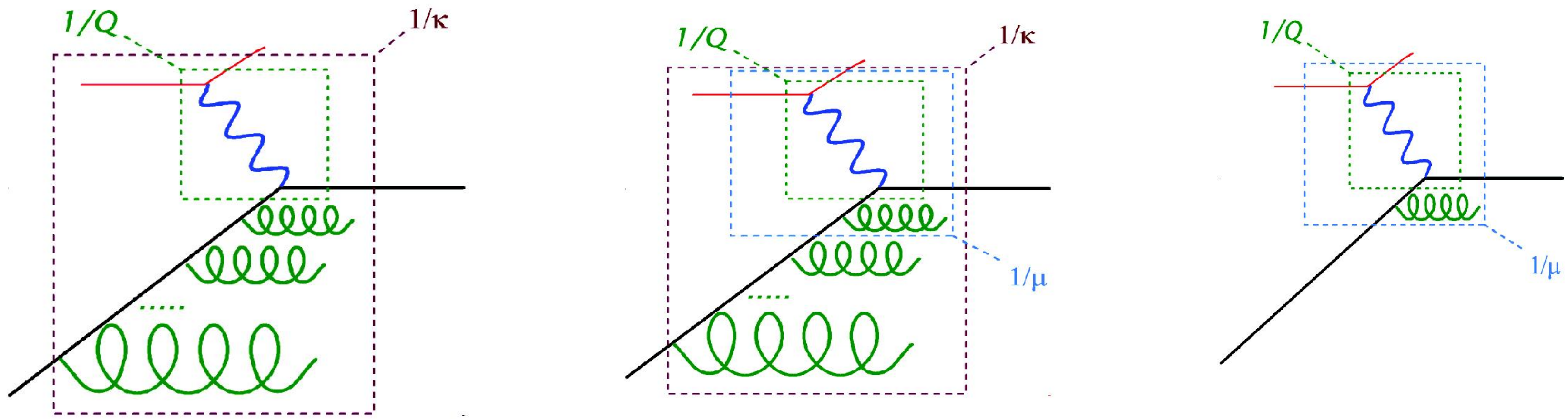


Calculating x-sections

- what is μ^2 ?
- remember how we obtained scale dependent PDFs:

$$q_i(x,\mu^2) = q_i^0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[q_i^0(\xi)P_{qq}\left(\frac{x}{\xi}\right)\log\left(\frac{\mu^2}{\chi^2}\right) + C_q\left(\frac{x}{\xi}\right)\right] + \ldots$$

From S. Ellis, Lecture 2, 2003



- use pt2 as scale ?

Divergencies, everywhere ...

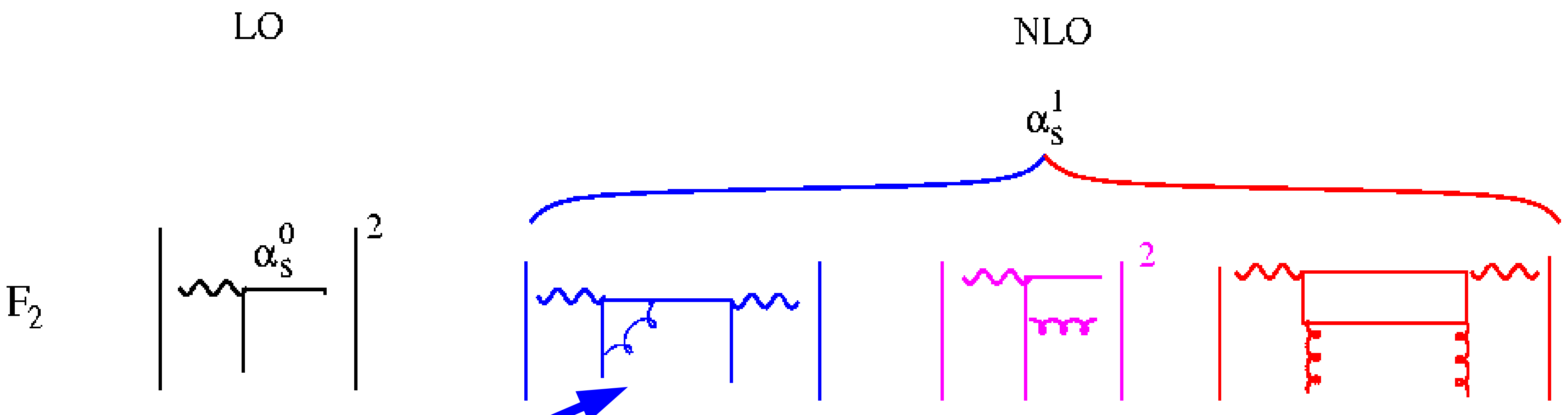
- $\frac{1}{t}$ singularities in $\mathcal{O}(\alpha_s)$ matrix elements

  ➔ leads to redefinition of PDFs and collinear factorization

- $\frac{1}{1-z}$ singularities in splitting functions

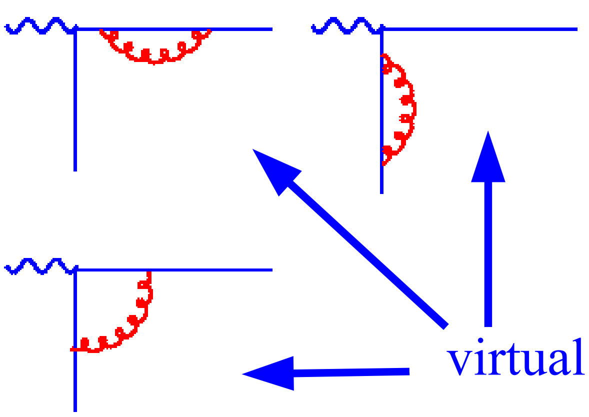
  ➔ treated by virtual corrections via "+" prescription or Sudakov

NLO contributions to $F_2(x,Q^2)$



virtual corrections cancel collinear singularity

$$P_{qq} = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$$



virtual corrections to QPM

$$P_{qg} = \frac{1}{2}\left(z^2 + (1+z)^2\right)$$

no collinear singularity

Virtual corrections ...

- collinear divergencies factored into renormalized parton distributions
- what about soft divergencies ? $z \to 1$

  treated with "plus" prescription

$$\frac{1}{1-z} \to \frac{1}{1-z}_+$$

with

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z) - f(1)}{(1-z)}$$

- soft divergency treated with Sudakov form factor:

$$\Delta(t) = \exp\left[- \int_{t_0}^{t} \frac{dt'}{t'} \int^{z_{max}} dz \frac{\alpha_s}{2\pi} \tilde{P}(z) \right]$$

resulting in

$$t \frac{\partial}{\partial t} \left(\frac{f}{\Delta} \right) = \frac{1}{\Delta} \int^{z_{max}} \frac{dz}{z} \frac{\alpha_s}{2\pi} \tilde{P}(z) f(x/z, t)$$

and

$$f(x,t) = \Delta(t) f(x, t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int^{z_{max}} \frac{dz}{z} \frac{\alpha_s}{2\pi} \tilde{P}(z) f(x/z, t')$$

Before discussing other divergencies....
calculate x-sections ...

Calculating x-sections (QCDC)

- partonic x-section:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi}\frac{1}{\hat{s}+Q^2}\frac{1}{\hat{s}}|M|^2$$

  - with

$$|M|^2(\gamma^* q \to qg) = 32\pi^2\left(e_q^2\alpha\alpha_s\right)\frac{4}{3}\left[\frac{-\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}}\right]$$

- photon-proton x-section:

$$\frac{d\sigma(\gamma^* p \to X)}{dt} = \int dx q(x,\mu^2)\frac{d\hat{\sigma}(\mu^2,\hat{s},\ldots.)}{dt}$$

  - with $q^+ = xP^+$

What happens at small pt ?

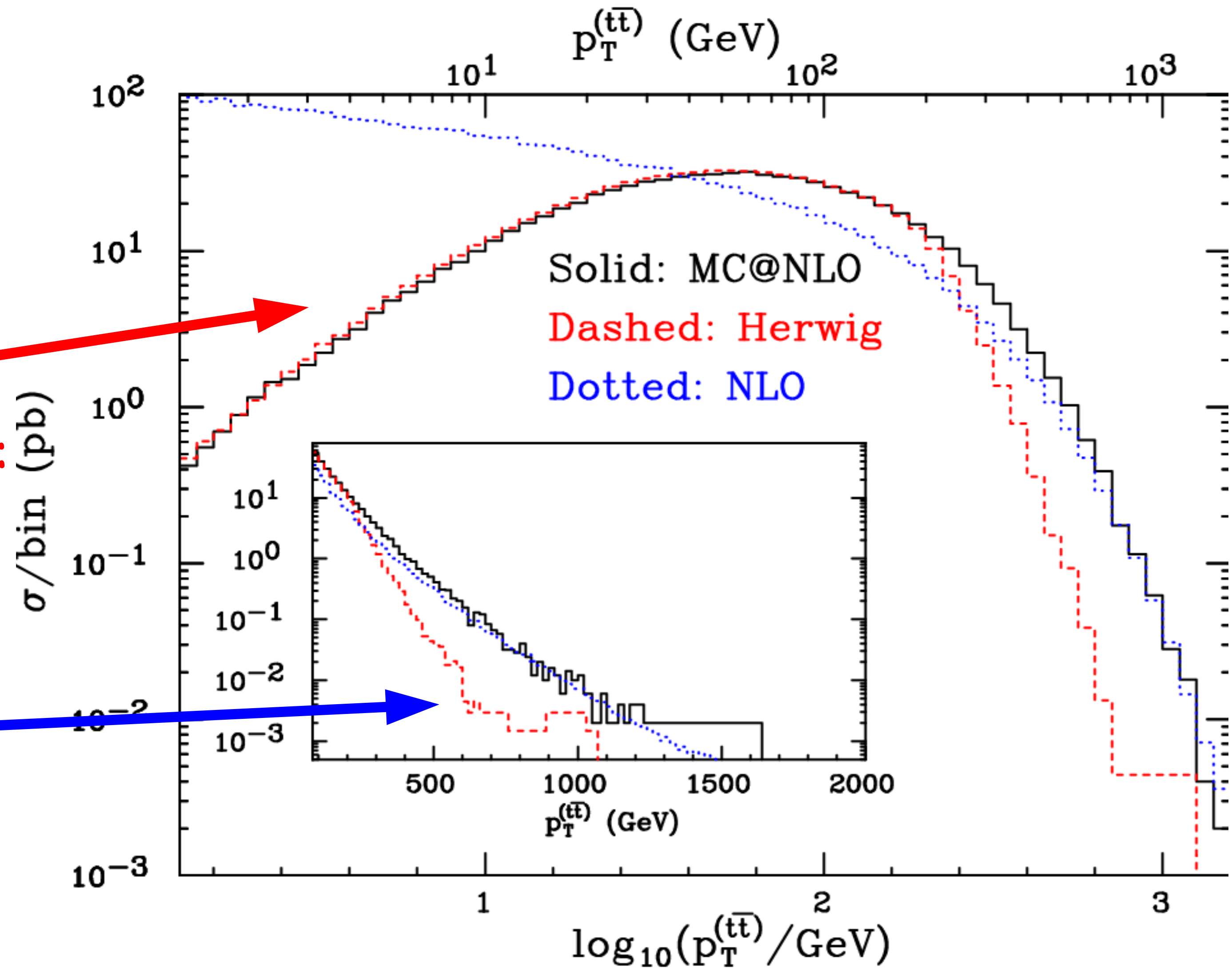
Blackboard

- soft gluon resummation
- sudakov effects:
  - probability for no radiation of any soft gluon at fixed scales is very small

Small pt in heavy quark prod.

- Compare fixed NLO calculation of top production with resummed calculation from Monte Carlo
- Similar effects at small pt are observed: Suppression of xsection at small pt
- At large pt, resummation is too small, NLO is better

Frixione et al, hep-ph/035252

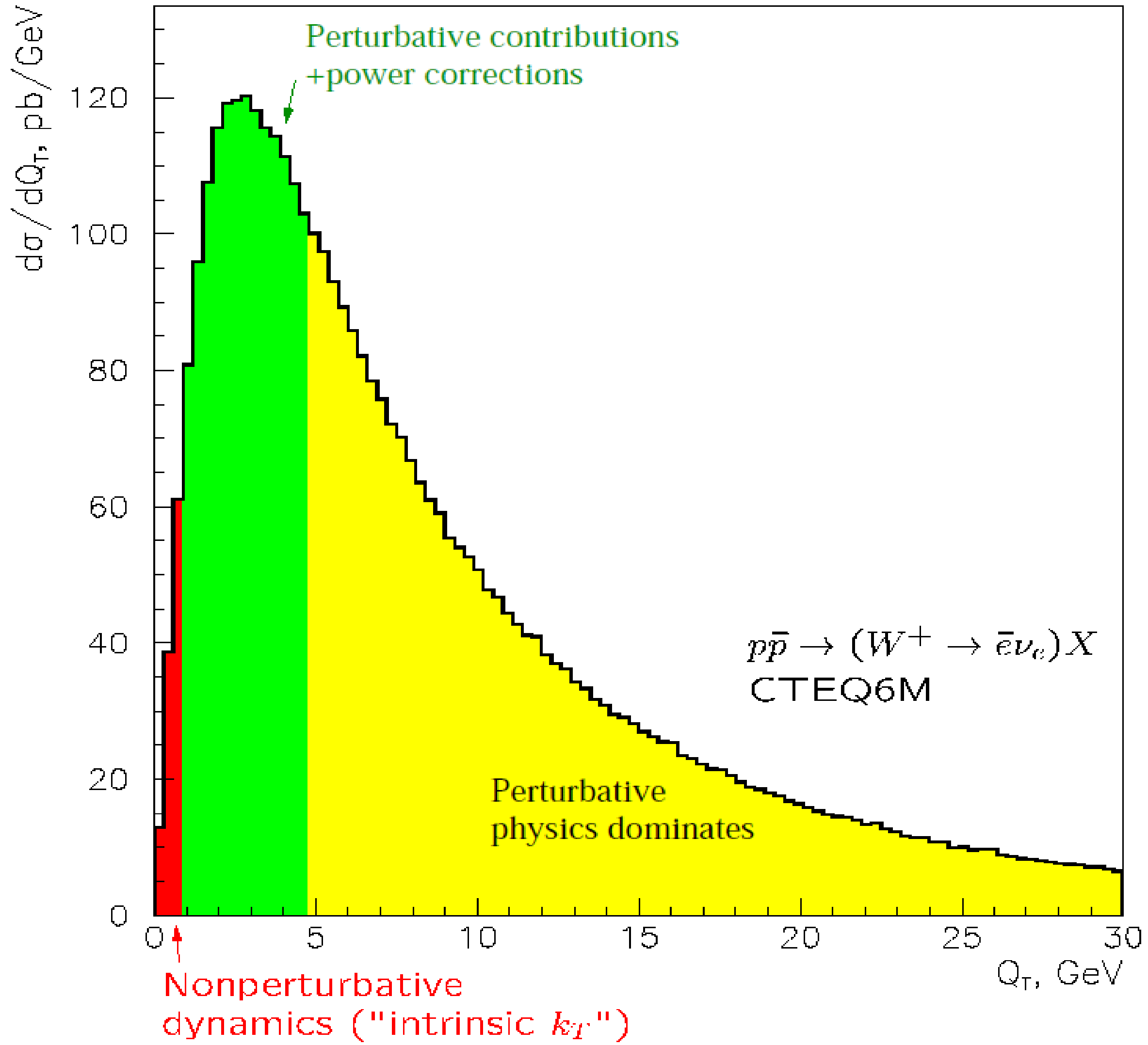


Transverse Momentum of W/Z

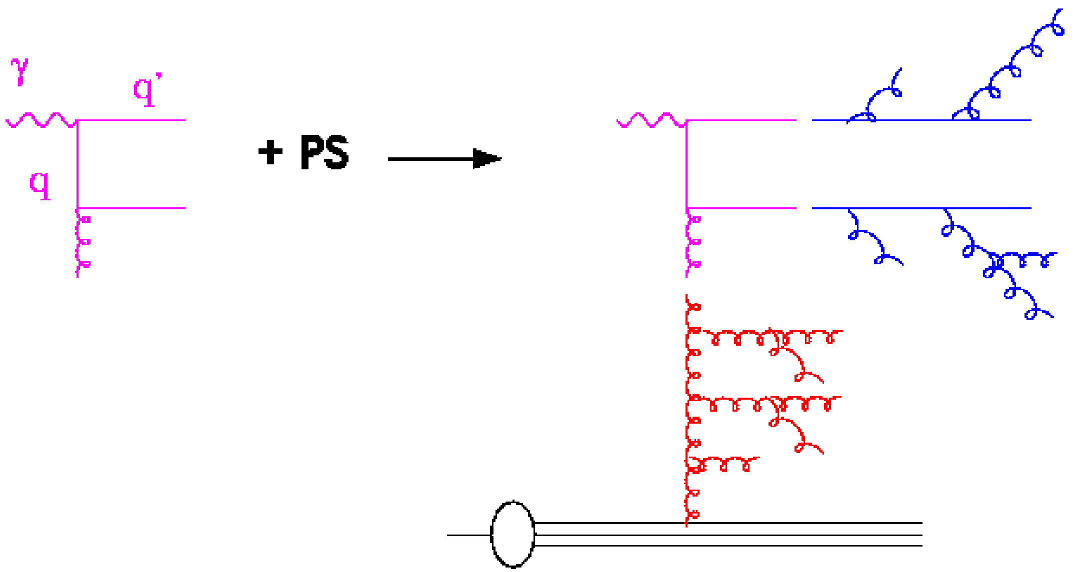
The complete P_T spectrum for the W boson

Fred Olness, CTEQ
summerschool 2003

The full P_T spectrum for the W-boson showing the different theoretical regions



Inconsistency: example from HERA



- **Collinear approach:** incoming/outgoing partons are on mass shell

  $(\gamma+q)^2 = q'^2$, $-Q^2 + x\,y\,s = 0 \rightarrow x = Q^2/(ys)$
- **BUT** final state radiation:

  $(\gamma+q)^2 = q'^2$, $-Q^2 + x\,y\,s = m^2 \rightarrow x = (Q^2+m^2)/(ys)$
- **AND** initial state radiation:

  $(\gamma+q)^2 = q'^2$, $-Q^2 + x\,y\,s + q^2 = 0 \rightarrow x = (Q^2-q^2)/(ys)$
- **Collinear approach: $q'^2 = q^2 = 0$, order by order .....**
- NLO corrections... better treatment of kinematics... but still not all....

Blackboard

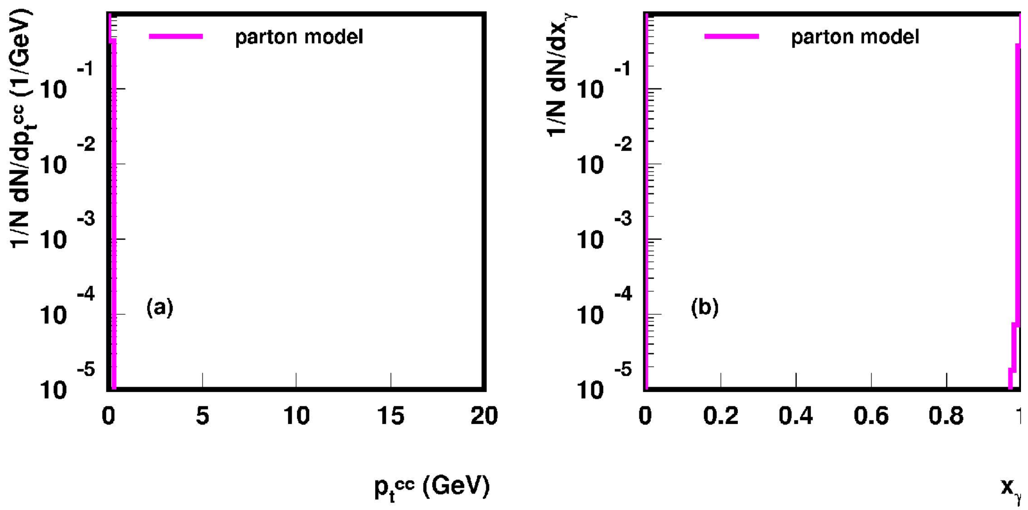
Can one do better ?

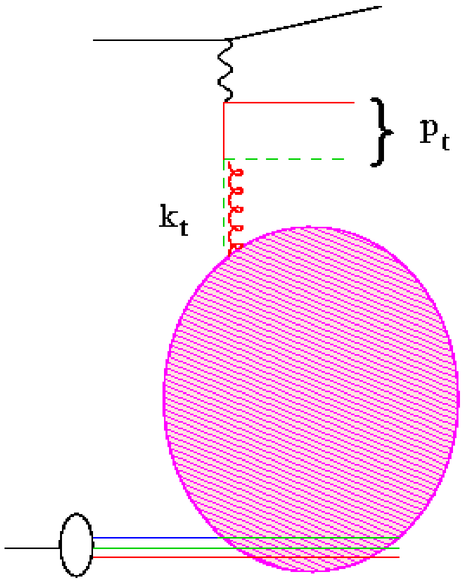
Need for uPDFs

J. Collins, H. Jung, hep-ph/0508280

Define: p_{qq}

- $$x_\gamma = \frac{\sum_{i=q,\bar{q}}(E_i - p_{z\ i})}{2yE_e} = \frac{p^-_{q\bar{q}}}{q^-}$$
- parton kinematics



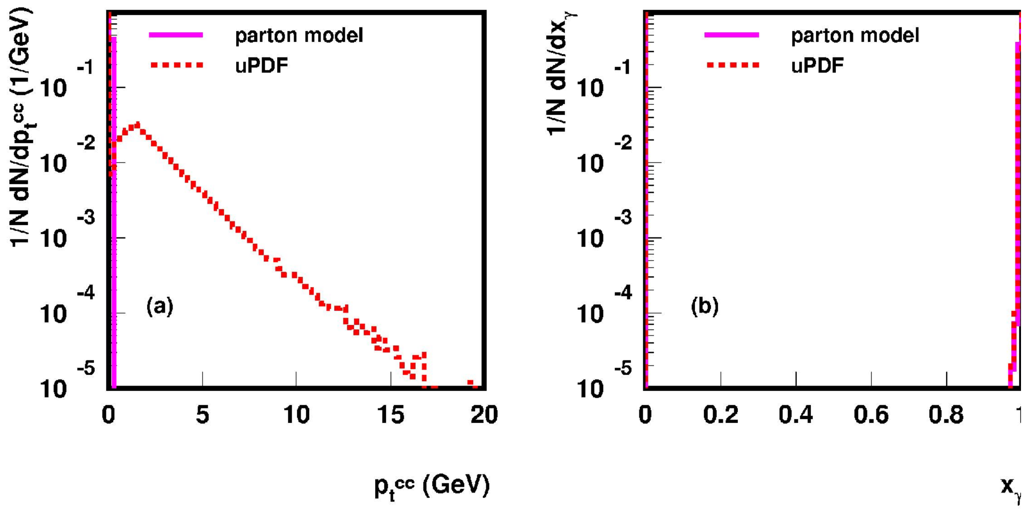


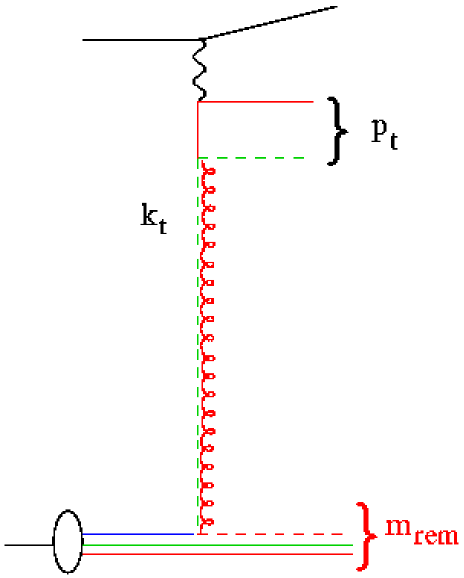
Need for uPDFs

J. Collins, H. Jung, hep-ph/0508280

Define: $p_{T\,q\bar{q}}$

- $$x_\gamma = \frac{\sum_{i=q,\bar{q}}(E_i - p_{z\,i})}{2yE_e} = \frac{p^-_{q\bar{q}}}{q^-}$$
- parton kinematics
- uPDFs



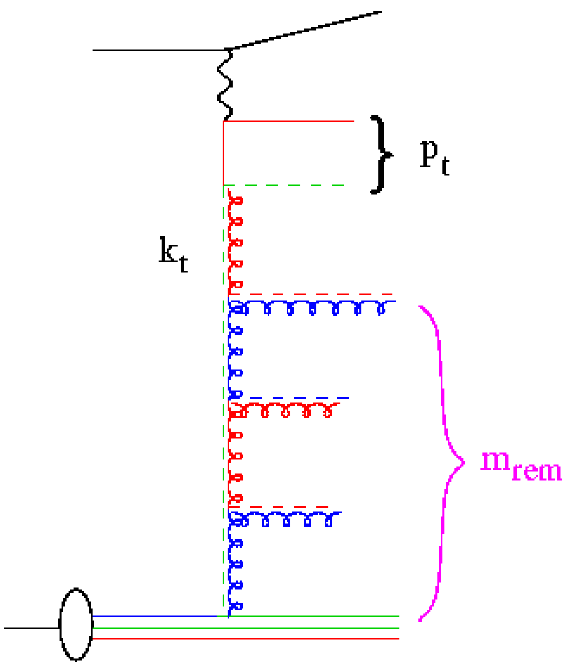


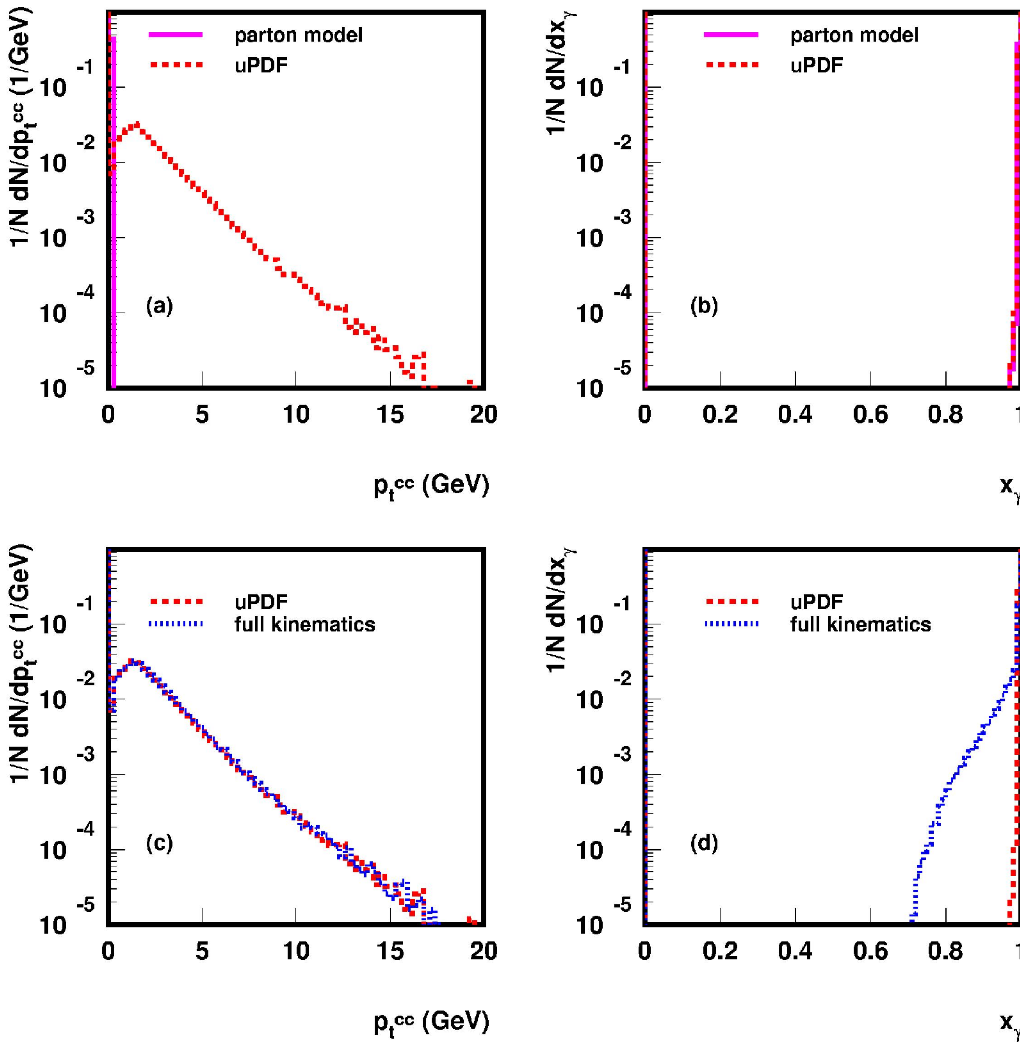
Need for uPDFs

J. Collins, H. Jung, hep-ph/0508280

Define: $p_{T\,q\bar{q}}$

- $$x_\gamma = \frac{\sum_{i=q,\bar{q}}(E_i - p_{z\,i})}{2yE_e} = \frac{p^-_{q\bar{q}}}{q^-}$$
- parton kinematics
- uPDFs
- full kinematics

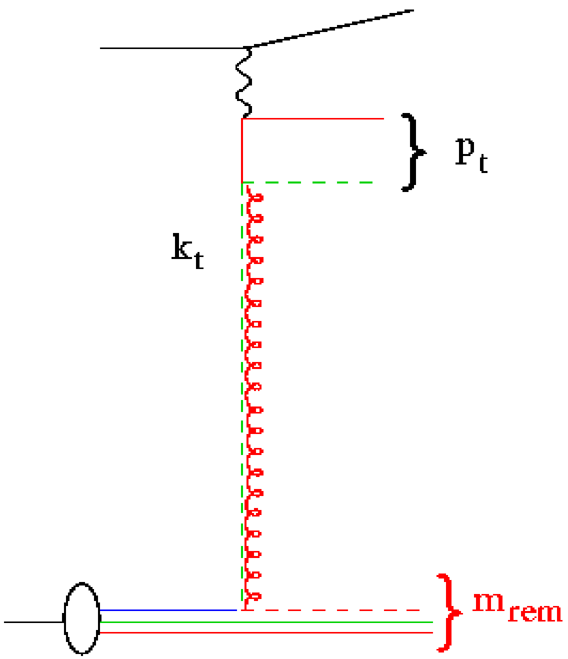


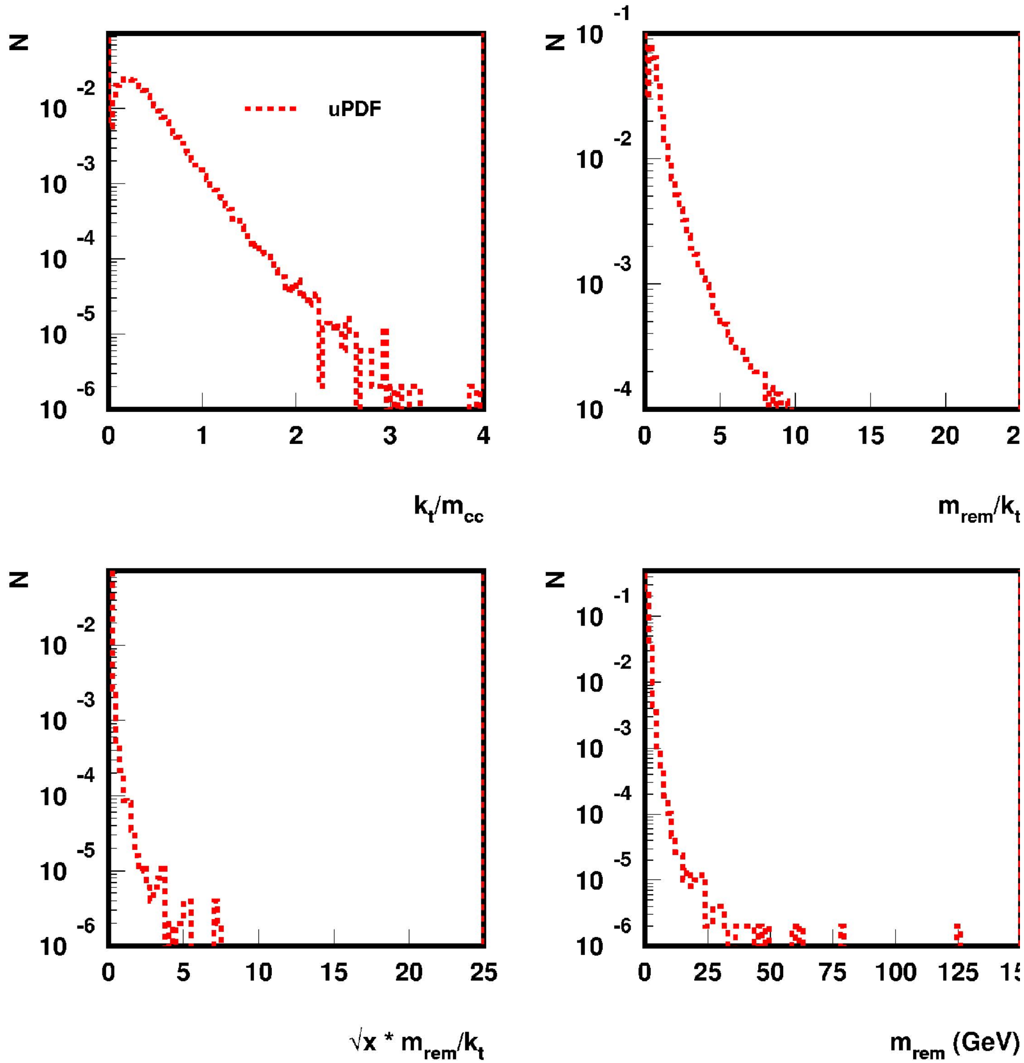


Need for uPDFs

J. Collins, H. Jung, hep-ph/0508280

$$k^2 = -\frac{k_t^2}{1-x}$$



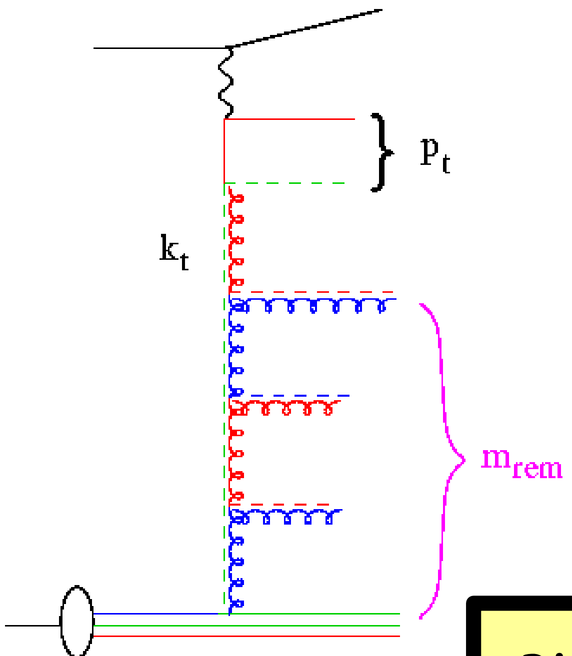


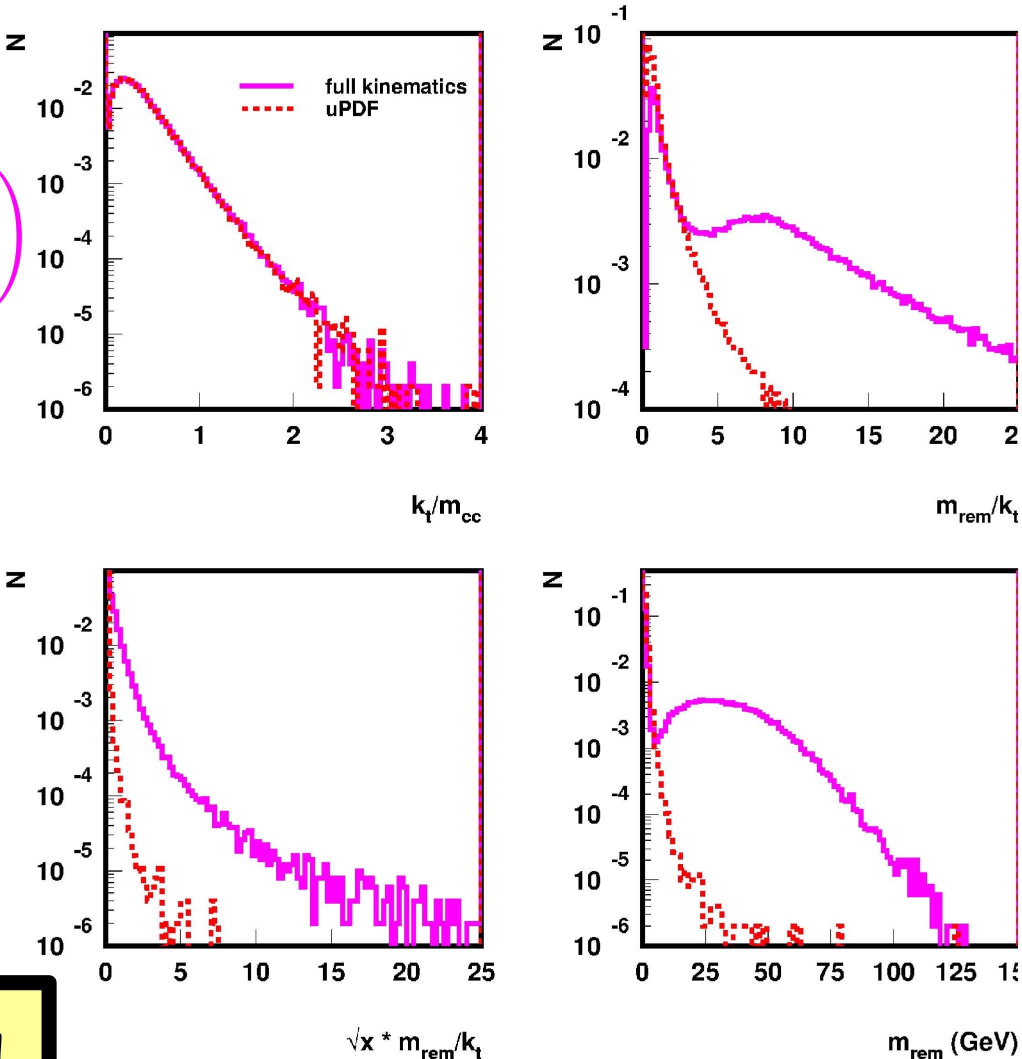
Need for uPDFs

J. Collins, H. Jung, hep-ph/0508280

$$k^2 = -\frac{k_t^2}{1-x}$$

$$k^2 = -\frac{k_\perp^2}{1-x}\left(1 + x\frac{m_{rem}^2}{k_\perp^2}\right)$$





Blackboard

Solving evolution equation with Monte Carlo

Evolution equation and Monte Carlo

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z}\int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')}\tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

- solve integral equation via iteration:

$$f_0(x,t) = f(x,t_0)\Delta(t)$$

- generate t according to Sudakov $\Delta(t_0,t) = \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int^{z_{max}} dz \frac{\alpha_s}{2\pi}\tilde{P}(z)\right]$

$$\log \Delta_s(t_0,t) = \log R$$

➔ solve it for t

$$\frac{d\Delta_x(t_0,t)}{dt} = \Delta(t_0,t)\frac{1}{t}\exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int^{z_{max}} dz \frac{\alpha_s}{2\pi}\tilde{P}(z)\right]$$

- generate z according to

$$\int_{\epsilon}^{z} dz \frac{\alpha_s}{2\pi} P(z) = R\int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_s}{2\pi}P(z)$$

Evolution equation and Monte Carlo

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

- solve integral equation via iteration:

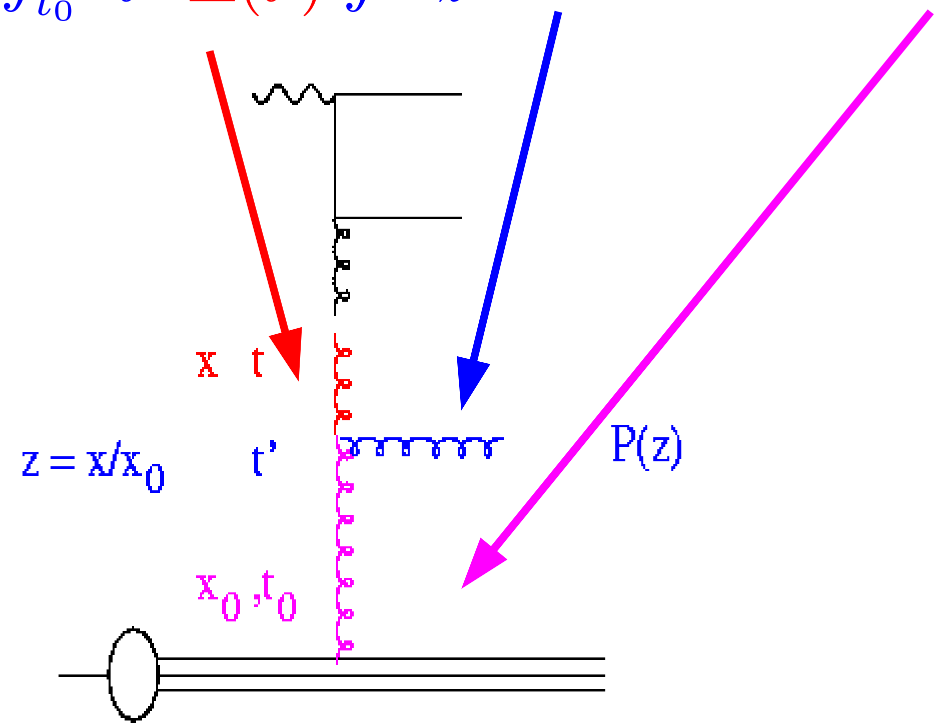
from t' to t w/o branching

branching at t'

from t_0 to t' w/o branching

$$f_0(x,t) = f(x,t_0)\Delta(t)$$

$$f_1(x,t) = f(x,t_0)\Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \tilde{P}(z) f(x/z, t_0)\Delta(t')$$



Evolution equation and Monte Carlo

$$f(x,t) = f(x,t_0)\Delta_s(t) + \int \frac{dz}{z} \int \frac{dt'}{t'} \cdot \frac{\Delta_s(t)}{\Delta_s(t')} \tilde{P}(z) f\left(\frac{x}{z}, t'\right)$$

- solve integral equation via iteration:

from t' to t w/o branching

branching at t'

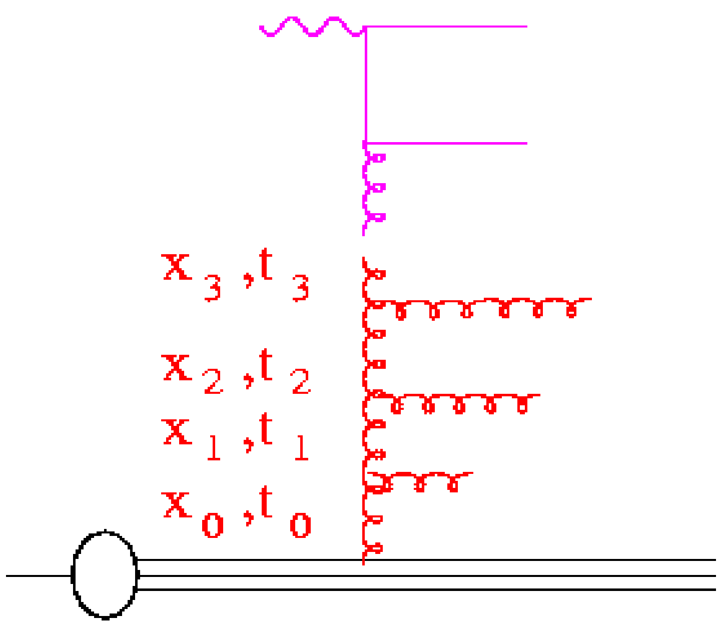
from t_0 to t' w/o branching

$$\begin{aligned}
f_0(x,t) &= f(x,t_0)\Delta(t) \\
f_1(x,t) &= f(x,t_0)\Delta(t) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \tilde{P}(z) f(x/z,t_0)\Delta(t') \\
&= f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A \otimes \Delta(t) f(x/z,t_0) \\
f_2(x,t) &= f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A \otimes \Delta(t) f(x/z,t_0) + \\
&\quad \frac{1}{2}\log^2\frac{t}{t_0} A \otimes A \otimes \Delta(t) f(x/z,t_0) \\
f(x,t) &= \lim_{n\to\infty} f_n(x,t) = \lim_{n\to\infty} \sum_n \frac{1}{n!} \log^n\left(\frac{t}{t_0}\right) A^n \otimes \Delta(t) f(x/z,t_0)
\end{aligned}$$

summing up all contribution up to t … advantage of importance sampling....

Including kinematic effects into evolution ?

Approximations so far ....

- Only inclusive quantities were considered:
  - nothing was said about "real" emissions or gluons or quarks although implicitly assumed....
  - in deriving DGLAP splitting functions we assumed: $\hat{t} \ll \hat{s}$
  - 

$$\hat{t} \sim \frac{-k_\perp^2}{1-z}$$

- neglect *t* in previous branchings
  $t_0 \ll t_1 \ll t_2 \ll t_3 \cdots \ll \mu^2$
  - strong ordering condition
  - strong ordering: neglect all kinematics of previous branchings...
- ordering in *x*
  $x_0 > x_1 > x_2 > x_3$

Better treatment including kt ...

- start from integral equation:

$$f(x,q) = f(x,Q_0)\Delta_s(q) + \int \frac{dz}{z} \int \frac{d^2q'}{\pi q'^2} \cdot \frac{\Delta_s(q)}{\Delta_s(q')} \tilde{P}(z) f\left(\frac{x}{z}, q'\right)$$

- use un-integrated pdfs: $\mathcal{A}(x,k,q)$

$$x\mathcal{A}(x,k_\perp,q) = x\mathcal{A}_0(x,k_\perp)\Delta_s(q) + \int dz \int \frac{d^2q'}{\pi q'^2} \cdot \Delta_s(q, f(q')) \tilde{P}(z,q',k_\perp) \Theta(\mathcal{O}) \frac{x}{z} \mathcal{A}\left(\frac{x}{z}, k'_\perp, q'\right)$$

Blackboard

because of phi integration:

$$\frac{dt}{t} \rightarrow \frac{dq^2}{q^2} \rightarrow \frac{d^2q}{\pi q^2}$$

define updf:

$$xg(x,Q) = \int \frac{d^2k_\perp}{\pi} x\mathcal{A}(x,k_\perp,Q)\Theta(Q-k_\perp)$$

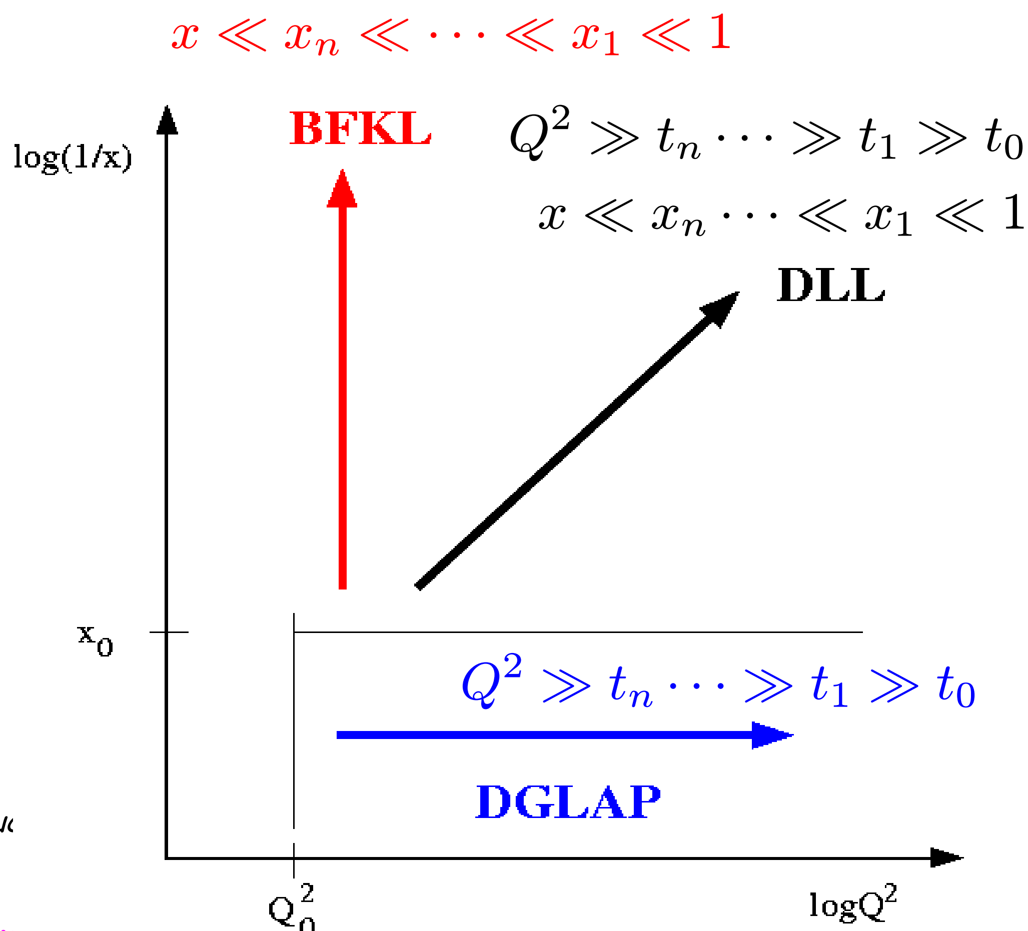
- same as before.... but included explicitly dependence on transverse momentum k_t in addition to evolution scale q
- what are the ordering constraints $f(q')$ and $\Theta(\mathcal{O})$?
- what is the splitting function?

Divergencies, everywhere ...

- $\frac{1}{t}$ singularities in $\mathcal{O}(\alpha_s)$ matrix elements
  - ➔ leads to redefinition of PDFs and collinear factorization
- $\frac{1}{1-z}$ singularities in splitting functions
  - ➔ treated by virtual corrections via "+" prescription or Sudakov
- $\frac{1}{z}$ singularities in splitting functions
  - ➔ treated by dedicated small x evolution equations: BFKL/CCFM

Kinematic regions: new evolution ..

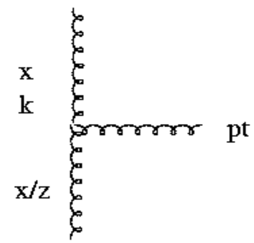
- DGLAP:
  - strong ordering in t
- DLL:
  - strong ordering in t
  - strong ordering in x
- what happens if strong t ordering relaxed ?
  - **B**alitskii**F**adin**K**uraev**L**ipatov evolution
    E. Kuraev, L. Lipatov, V.Fadin, *Sov. Phys. JETP* 44 (1976),443., E. Kuraev, L. Lipatov, V. Fadin, *Sov. Phys. JETP* 45,(1977),199., Y. Balitskii, L. Lipatov, *Sov. J. Nuc Phys.* 28,(1978), 822.
  - **C**atani**C**iafaloni**F**iorani**M**archesini evolution
    M. Ciafaloni, Nucl. Phys. B 296,(1988),49. S. Catani, F. Fiorani, G. Marchesini, Phys. Lett. B 234, (1990), 339, S. Catani, F. Fiorani, G. Marchesini, Nucl. Phys. B 336, (1990),18, G. Marchesini, Nucl. Phys. B 445, (1995), 49.

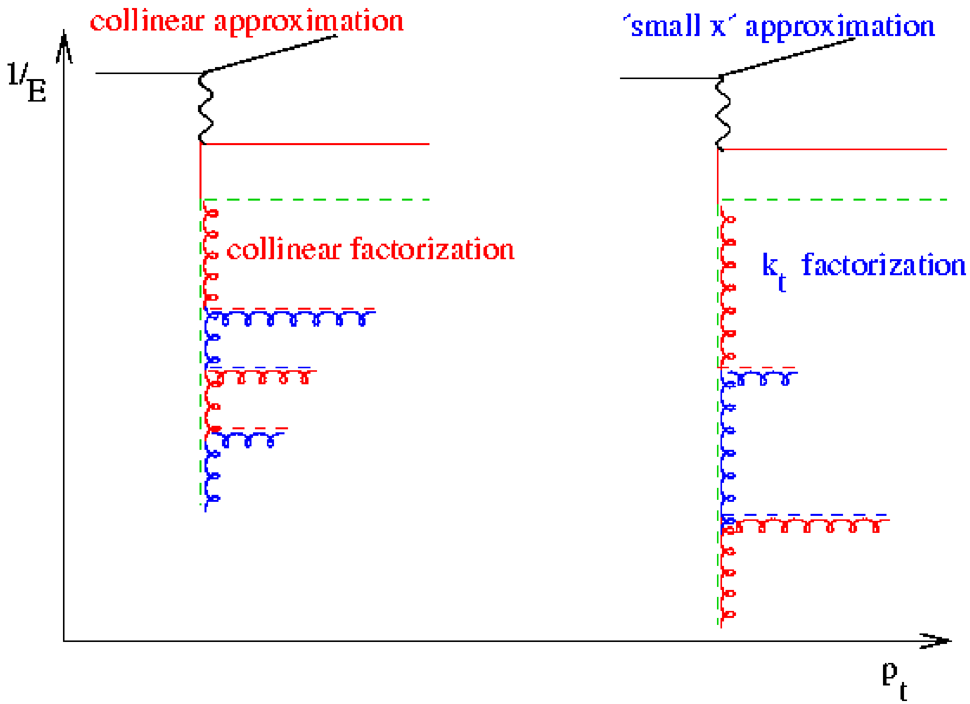


Approximations to higher orders ...

gluon bremsstrahlung

$$\sim \frac{1}{k^2}\left(\frac{1}{z} + \cdots\right)$$





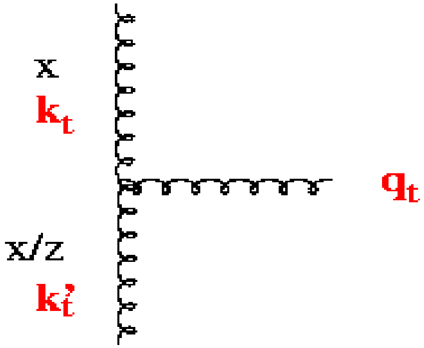
DGLAP

- collinear singularities factorized in pdf
- evolution in $Q^2 \sim k^2, \text{ or } k_t^2 \text{ or ?}$
- $$\sigma = \sigma_0 \int \frac{dz}{z} C^a\left(\frac{x}{z}\right) f_a(z, Q^2)$$

BFKL

- k_t dependent pdf → unintegrated pdf
- evolution in x
- $$\sigma = \int \frac{dz}{z} d^2k_t \hat{\sigma}\left(\frac{x}{z}, k_t\right) \mathcal{F}(z, k_t)$$

Approximations: Double Leading Log



$$\vec{k_t}' = \vec{k_t} + \vec{q}$$

- Recover DLL:
- use $P_{gg} \to \frac{3\alpha_s}{\pi}\frac{1}{z}$

$$\Delta_s \to 1$$

$$\Theta(\mathcal{O}) \to \Theta(k_t - k_t')$$

- Obtain from:

$$x\mathcal{A}(x, k_\perp, q) = x\mathcal{A}_0(x, k_\perp)\Delta_s(q) + \int dz \int \frac{d^2q'}{\pi q'^2} \cdot \Delta_s(q, f(q'))\tilde{P}(z, q', k_\perp)\Theta(\mathcal{O})\frac{x}{z}\mathcal{A}\left(\frac{x}{z}, k'_\perp, q'\right)$$

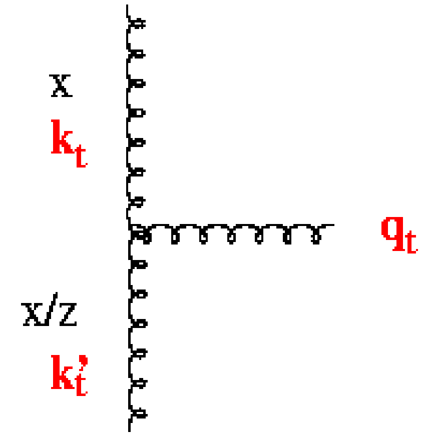
previous result in Double Leading Log approximation (upon integration over kt)

$$xg(x,t) = \frac{3\alpha_s}{\pi}\int_{t_0}^{t} d\log t' \int_x^1 \frac{d\xi}{\xi}\xi g(\xi, t')$$

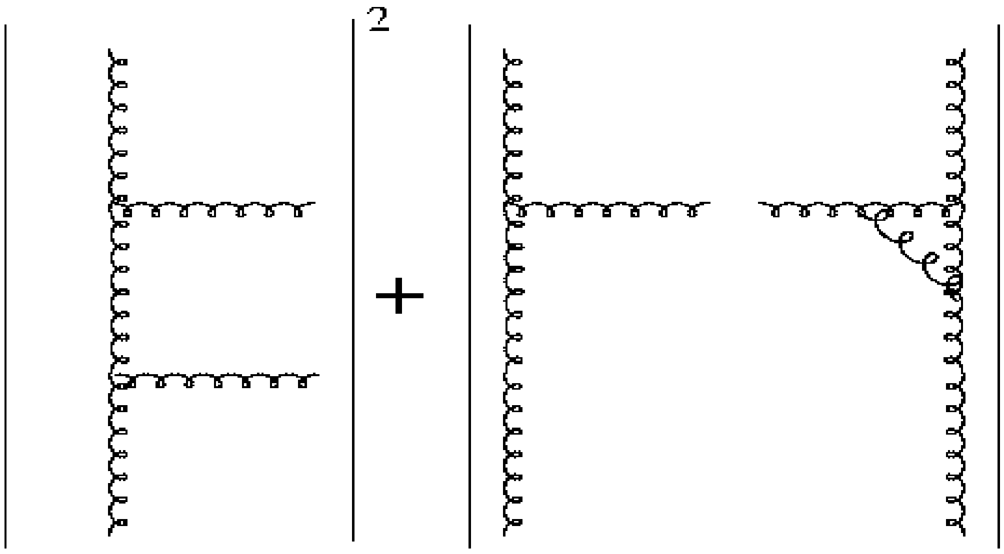
Approximations: BFKL

- At small z, divergency of gluon splitting function:

$$P_{gg} \sim \frac{1}{z}$$



- analogy with large z divergency:
  - cancelled by virtual corrections



for $k_t \sim k_t'$

$q_t \to 0$

but still z small
parallel to k, k'

  - similar to Sudakov, but NOW at small x .... “non” Sudakov (or Regge form factor)

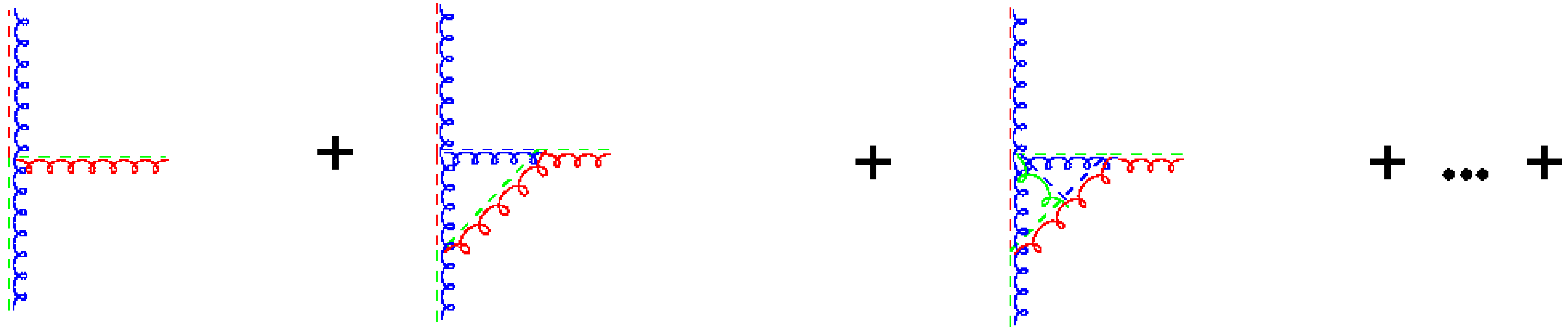
Non-Sudakov form factor: all loop re-sum...

$g \to gg$ Splitting Fct $\tilde{P}(z) = \frac{\bar{\alpha}_s}{1-z} + \frac{\bar{\alpha}_s}{z} + \ldots$

- Non - Sudakov form factor .... all loop resummation

$$\Delta_{ns} = \exp\left[-\bar{\alpha}_s(k_t^2)\int_0^1 \frac{dz'}{z'}\int \frac{dq^2}{q^2}\Theta(k_t - q)\Theta(q-\mu_0)\right]$$

$$\Delta_{ns} = 1 + \left(-\bar{\alpha}_s(k_t^2)\int \frac{dz'}{z'}\int \frac{dq^2}{q^2}\right)^1 + \frac{1}{2!}\left(-\bar{\alpha}_s(k_t^2)\int \frac{dz'}{z'}\int \frac{dq^2}{q^2}\right)^2 \ldots$$



$$\bar{\alpha}_s(k_t)\frac{1}{z}\left[+\ \bar{\alpha}_s \log(z)\log\left(\frac{k_t^2}{\mu_0^2}\right) + \frac{1}{2!}\left(\bar{\alpha}_s \log(z)\log\left(\frac{k_t^2}{\mu_0^2}\right)\right)^2 + \cdots\right]$$

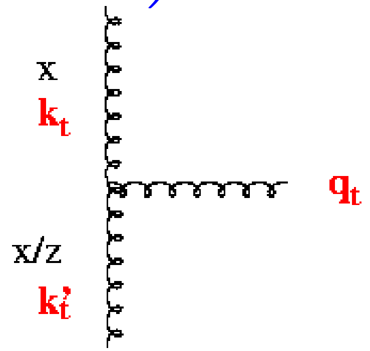
BFKL equation

J. Kwiecinski, A. Martin, P. Sutton PRD 52 (1995) 1445

- Non-Sudakov form factor screens 1/z singularity,

  ..... as the Sudakov does for 1/(1-z)

$$\Delta_{ns} = \exp\left(-\bar{\alpha}_s \int \frac{dq^2}{q^2} \int_z^1 \frac{dz'}{z'} \Theta(k_\perp^2 - q^2)\Theta(q^2 - \mu_0^2)\right)$$
$$= \exp\left(\bar{\alpha}_s \log z \log \frac{k_\perp^2}{\mu_0^2}\right)$$
$$= z^\omega \text{ with } \omega = \bar{\alpha}_s \log \frac{k_\perp^2}{\mu_0^2}$$



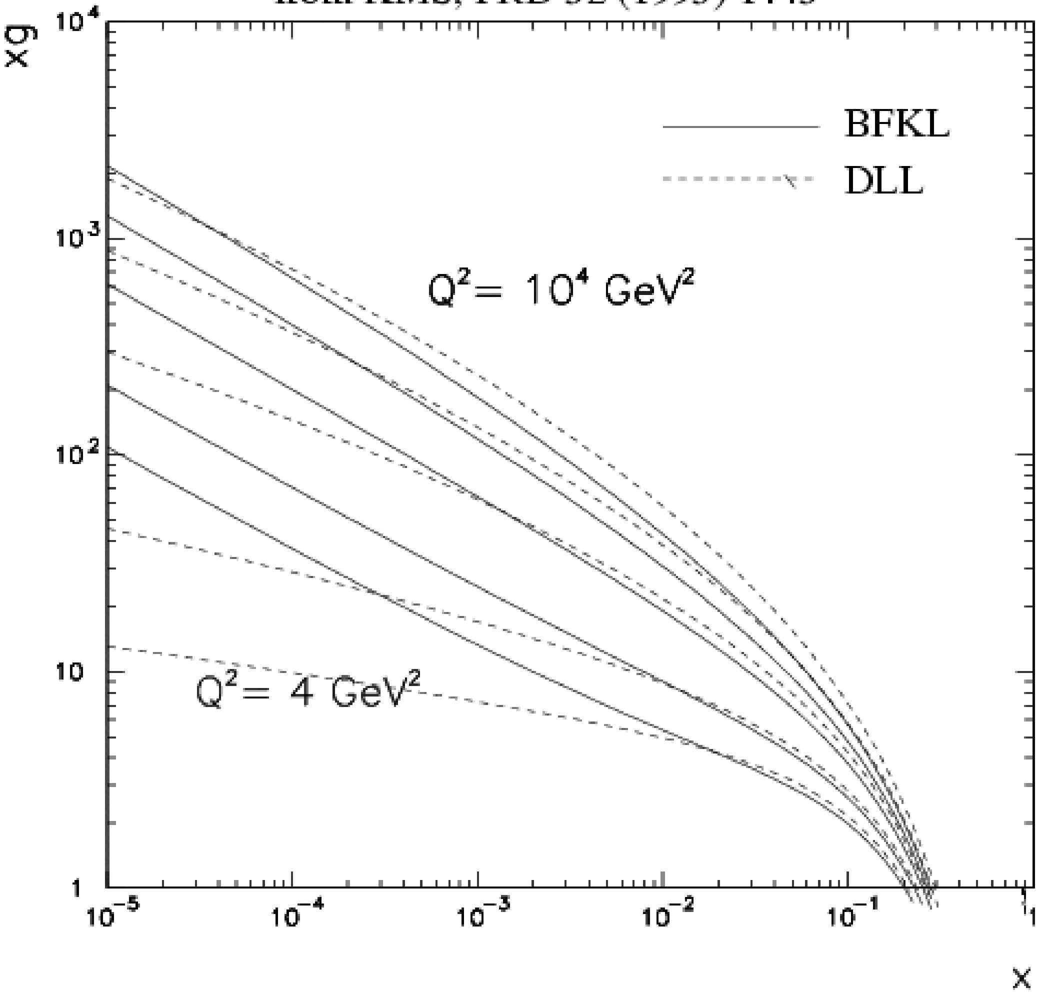
$$x\mathcal{A}(x, k_\perp, q) = x\mathcal{A}_0(x, k_\perp) + \int \bar{\alpha}_s dz z^\omega \int \frac{d^2q'}{\pi q'^2} \frac{x}{z} \mathcal{A}\left(\frac{x}{z}, k'_\perp, q'\right)$$

- here use: $\vec{k_\perp}' = \vec{k_\perp} + \vec{q}$
- recursive equation for BFKL, solve it numerically with iteration...

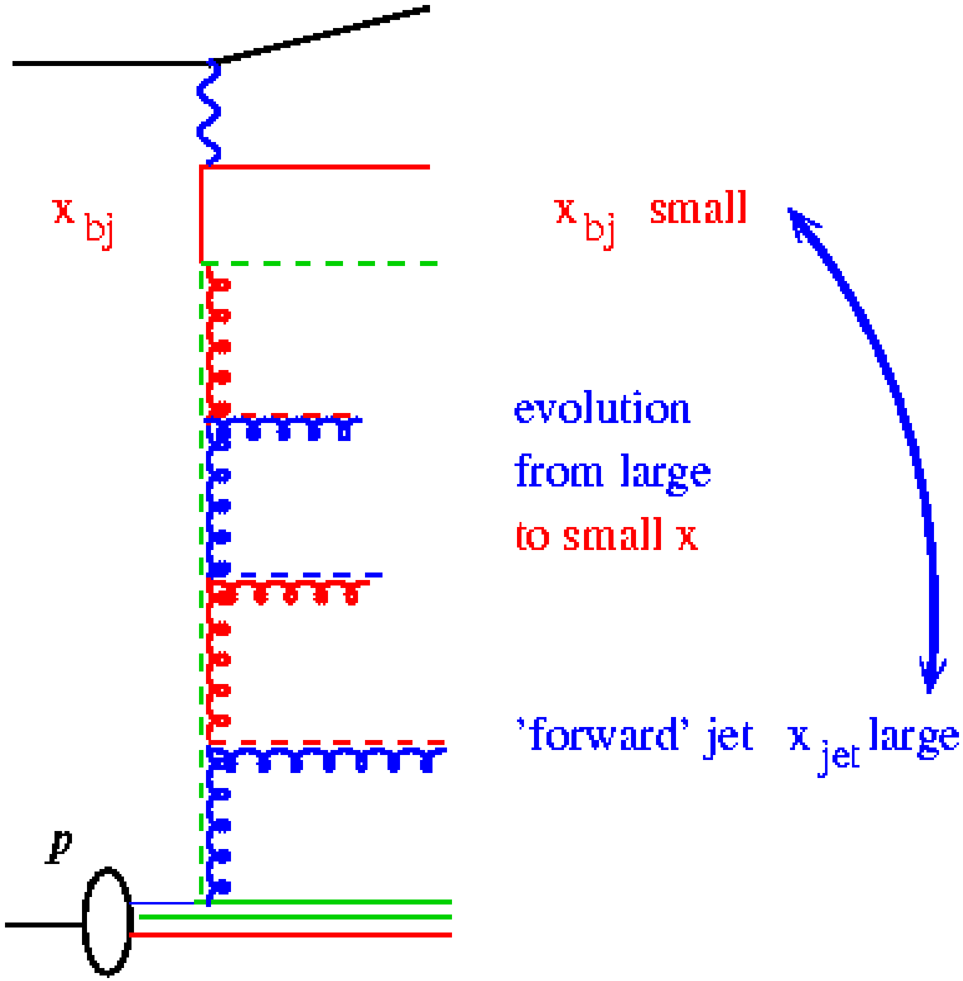
small x: BFKL/DLL comparison

- relaxing strong ordering of virtualities gives fast increase of gluon at small x
- BFKL gluon increases even faster than with DLL
- instead of increasing virtualities.... perform a random walk ... increasing or decreasing transverse momentum...
- can even reach non-perturbative region ... need cutoff normally set to 1 GeV ...
- but result depends on non-perturbative input....

from KMS, PRD 52 (1995) 1445



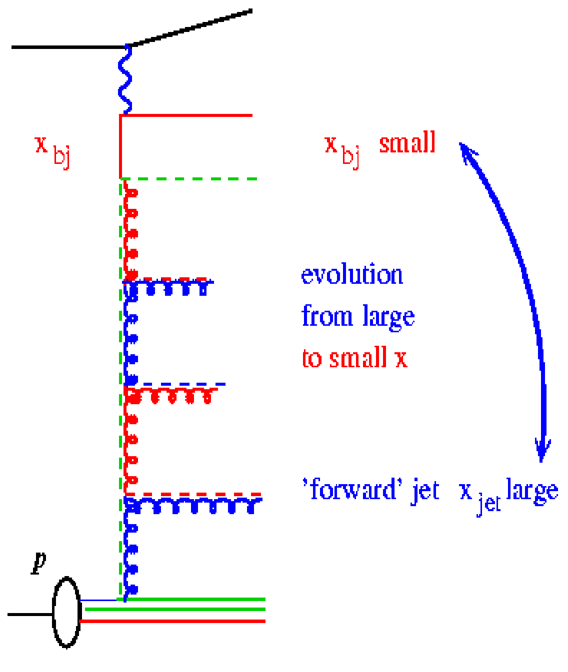
forward jet production and BFKL

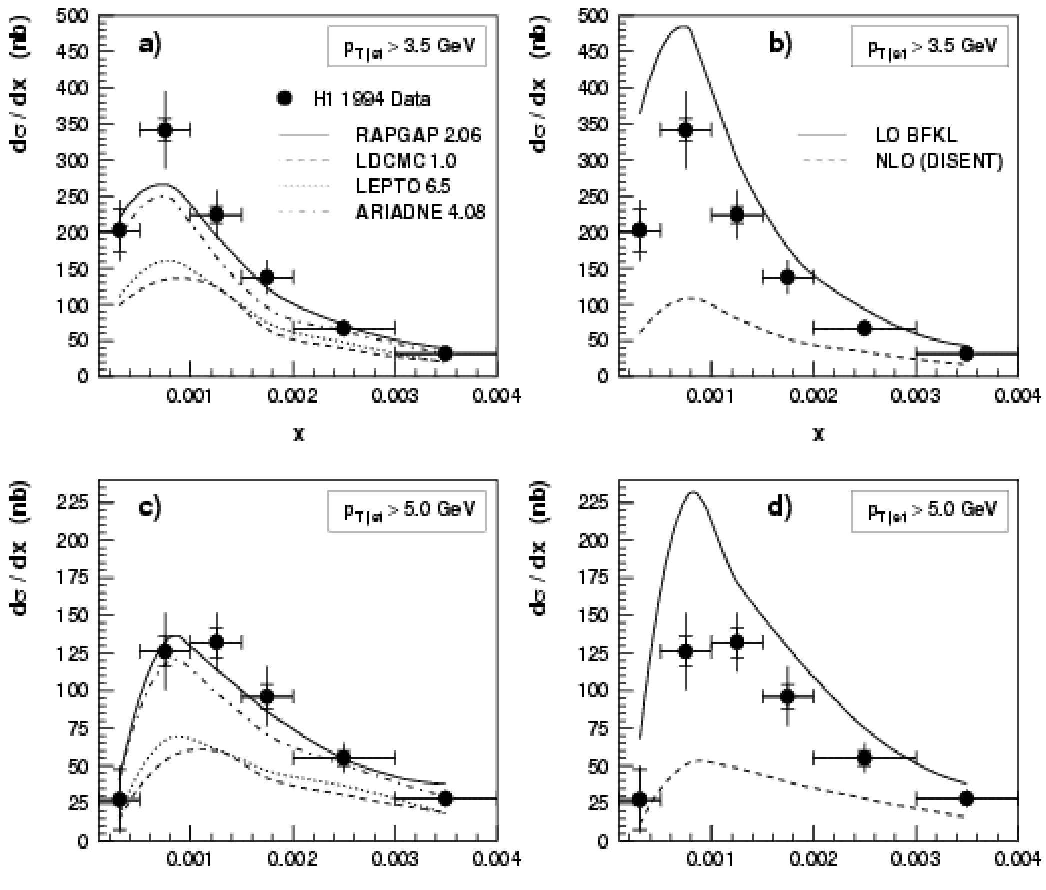


- DIS and forward jet:
  $1.7 < \eta_{jet} < 2.8$
  $x_{jet} > 0.035$
  $0.5 < \frac{p_{t\,jet}^2}{Q^2} < 5$
  $\sigma(\text{fwd jet})/\sigma(\text{DIS}) \sim 1\%$

Aim is to investigate and test small x evolution ...
Supress contribution from known DGLAP region of phase space

forward jet production





- DIS and forward jet:

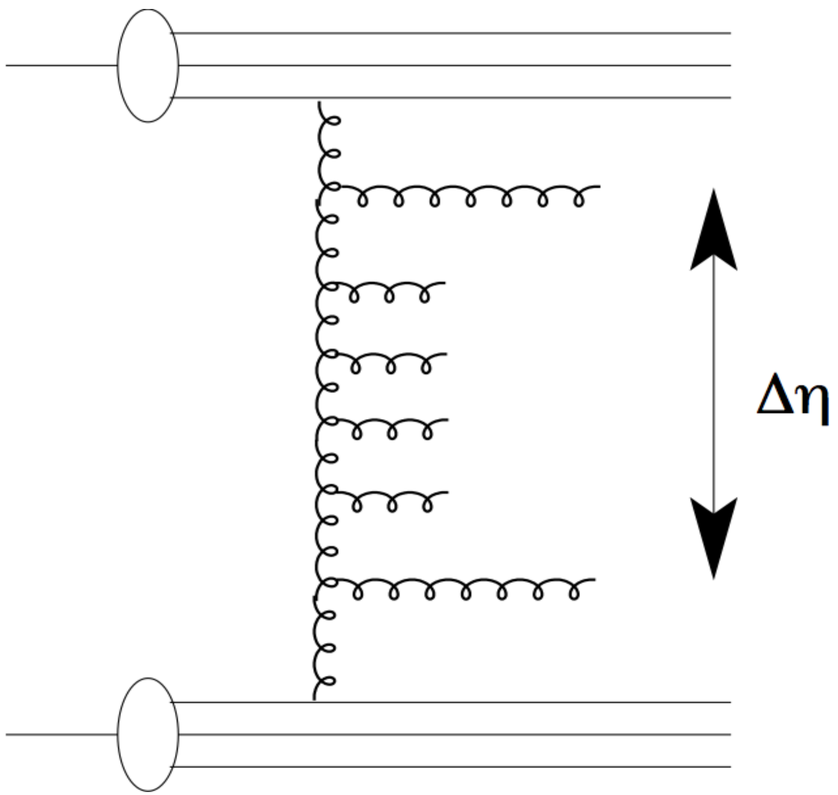
$$1.7 < \eta_{jet} < 2.8$$

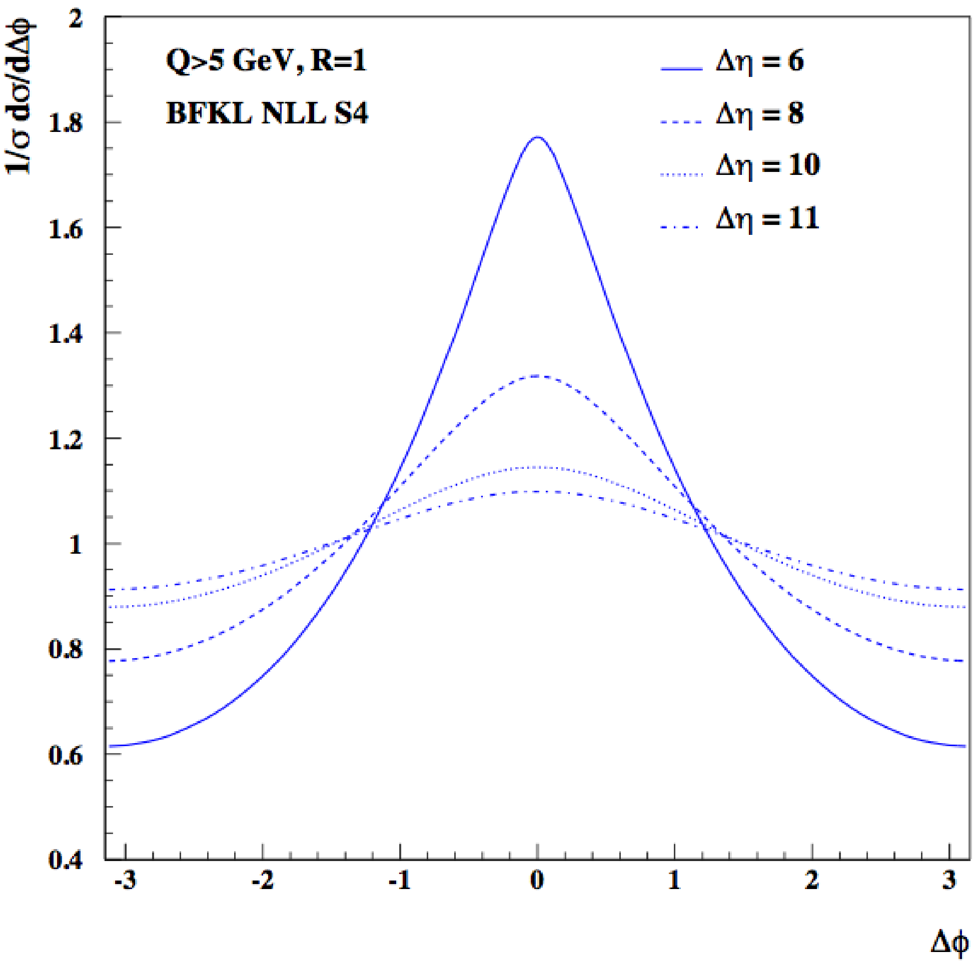
$$x_{jet} > 0.035$$

$$0.5 < \frac{p_{t\ jet}^2}{Q^2} < 5$$

- **BFKL evolution closer to data**
- **DGLAP/NLO too small**

Mueller-Navelet jets in pp





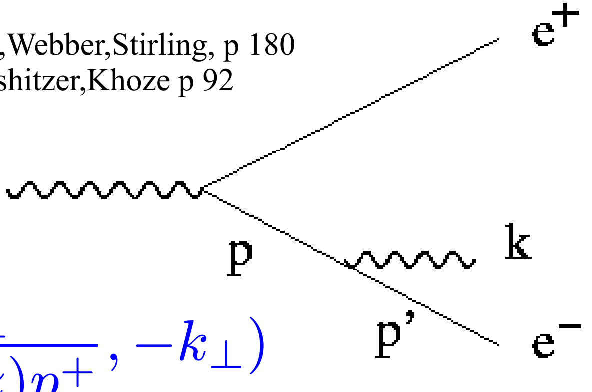
- Require large η separation
- similar p_t
- look at ϕ decorrelation

Can one do even better...
matching small and large x ?

Reconsider ordering
conditions

Angular ordering in QED

- assume QED
- use light-cone vectors:



$$
\begin{aligned}
p &= (p^+, p^-, 0) = (p^+, 0, 0) \\
p' &= ((1-z)p^+, p'^-, -k_\perp) = ((1-z)p^+, \frac{k_\perp^2}{(1-z)p^+}, -k_\perp) \\
k &= (zp^+, k^-, k_\perp) = (zp^+, \frac{k_\perp^2}{zp^+}, k_\perp)
\end{aligned}
$$

- use energy imbalance:

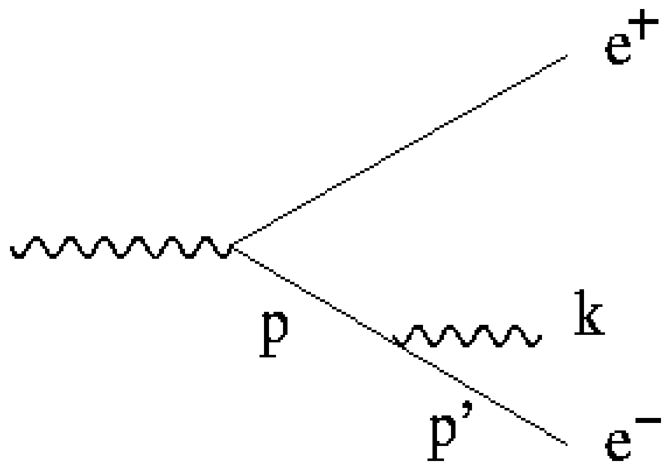
$$\Delta E \sim \frac{k_t^2}{zp^+} = zp^+ \Theta_{e\gamma}^2$$

- define transverse wavelength:

$$k_\perp = k\Theta_{e\gamma} = \lambda_\perp^{-1}$$

- from uncertainty principle: $\Delta t = \dfrac{\lambda_\perp}{\Theta_{e\gamma}}$
- during Δt e^+e^- pair has travelled a distance: $\rho_t^{e^+e^-} = \Delta x \Delta t \sim \Theta_{e^+e^-} \dfrac{\lambda_\perp}{\Theta_{e\gamma}}$

Angular ordering in QED



Ellis,Webber,Stirling, p 180
Dokshitzer,Khoze p 92

- photon emissions allowed for:

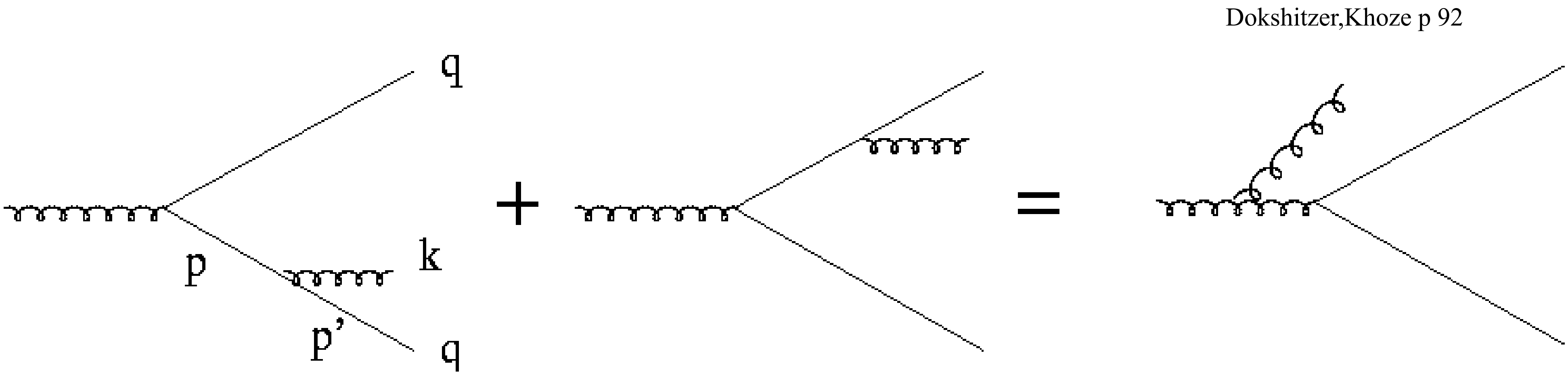
  for $\Theta_{\gamma,e} < \Theta_{e^+,e^-}$

- radiation strongly suppressed for:

  for $\Theta_{\gamma,e} > \Theta_{e^+,e^-}$

➔ since photon cannot resolve any structure of e^+e^- pair

Angular ordering and color coherence



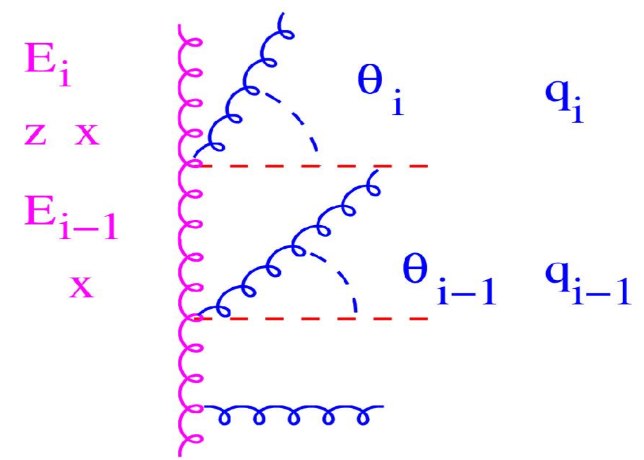
- gluon emissions are allowed

$$\begin{array}{ll} \text{off } q_1 & \text{for } \Theta_{kq_1} < \Theta_{q_1,q_2} \\ \text{off } q_2 & \text{for } \Theta_{kq_2} < \Theta_{q_1,q_2} \\ \text{off parent } g & \text{for } \Theta_{kg} > \Theta_{q_1,q_2} \end{array}$$

- calculations done explicitly in Ellis,Stirling & Webber

CataniCiafaloniFioraniMarchesini evolution

$$p_{ti} = |q_i^0| \sin\Theta_i$$
$$z = \frac{E_i}{E_{i-1}}$$
$$E_{i-1} = E_i + q_i^0 = zE_{i-1} + q_i^0,$$
$$\rightarrow q_i^0 = (1-z)E_{i-1}$$
$$p_{ti} = q_i^0 \sin\Theta_i \simeq (1-z)E_{i-1}\Theta_i$$
$$\frac{p_{ti}}{1-z} \simeq E_{i-1}\Theta_i$$
$$\text{with: } q_i = \frac{p_{ti}}{1-z_i}$$
$$\rightarrow \Theta_i = \frac{q_i}{E_{i-1}}$$
$$\Theta_{i+1} = \frac{q_{i+1}}{E_i}$$



- Apply color coherence in form of angular ordering
- $\bar{q} > z_n q_n, q_n > z_{n-1} q_{n-1}, \ldots, q_1 > Q_0$
- true angular ordering (in terms of rescaled momentum):

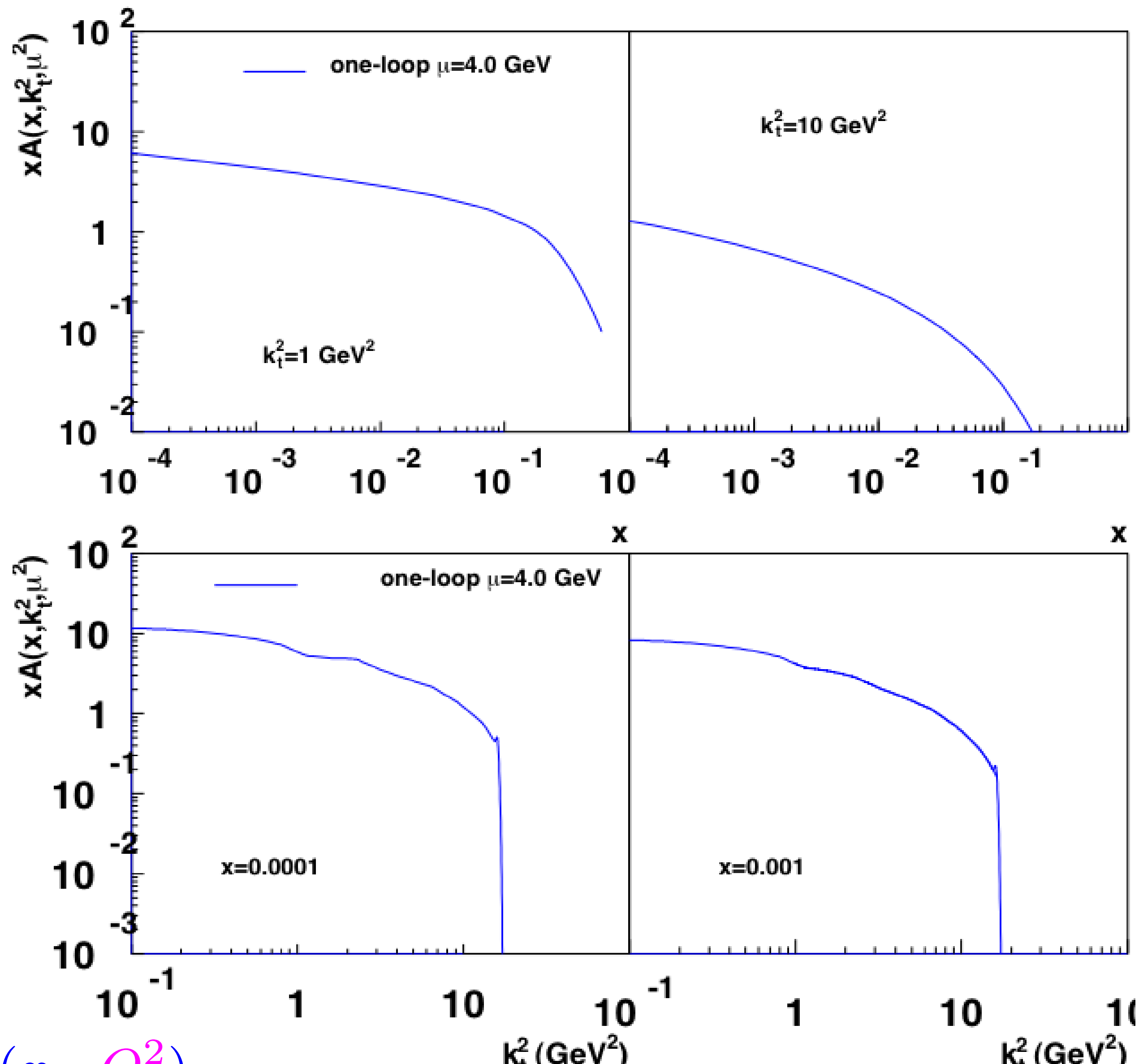
$$q_i > z_{i-1} q_{i-1}$$

“semi-angular” ordering and uPDF

- standard DGLAP evolution from small to large scale :

$$t_0 \ll t_1 \ll t_2 \ll t_3 \cdots \ll \mu^2$$

- strong ordering: neglect all kinematics of previous branchings...
- at hard scattering, parton have a transverse momentum: uPDF

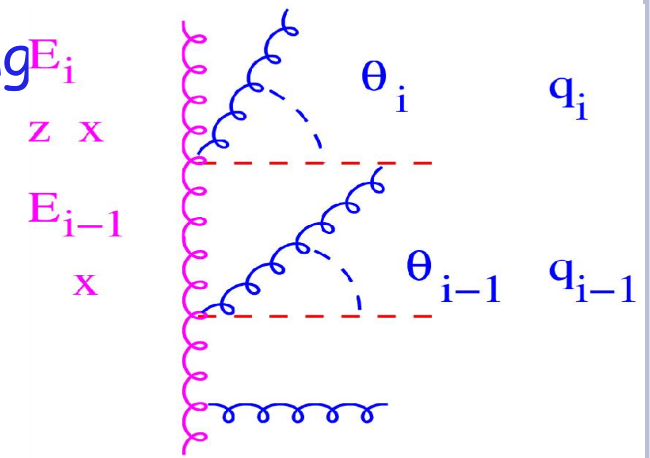


$$\int d^2k_t x_g \mathcal{A}(x_g, k_t, \bar{q}) = x_g G(x_g, Q^2)$$

CataniCiafaloniFioraniMarchesini evolution

- Apply color coherence in form of angular ordering

  $\bar{q} > z_n q_n, q_n > z_{n-1} q_{n_1}, \ldots, q_1 > Q_0$



- with:

$$\tilde{P}(z, q, k_\perp) = \frac{\bar{\alpha}_s(q(1-z))}{1-z} + \frac{\bar{\alpha}_s(k_t)}{z} \Delta_{ns}(z, q, k_\perp)$$

- gives:

$$x\mathcal{A}(x, k_\perp, q) = x\mathcal{A}_0(x, k_\perp)\Delta_s(q) + \int dz \int \frac{d^2 q'}{\pi q'^2} \Theta(\bar{q} - zq) \cdot \Delta_s(q, zq') \tilde{P}(z, q', k_\perp) \frac{x}{z} \mathcal{A}\left(\frac{x}{z}, k'_\perp, q'\right)$$

- integration much more complicated due to angular constraints

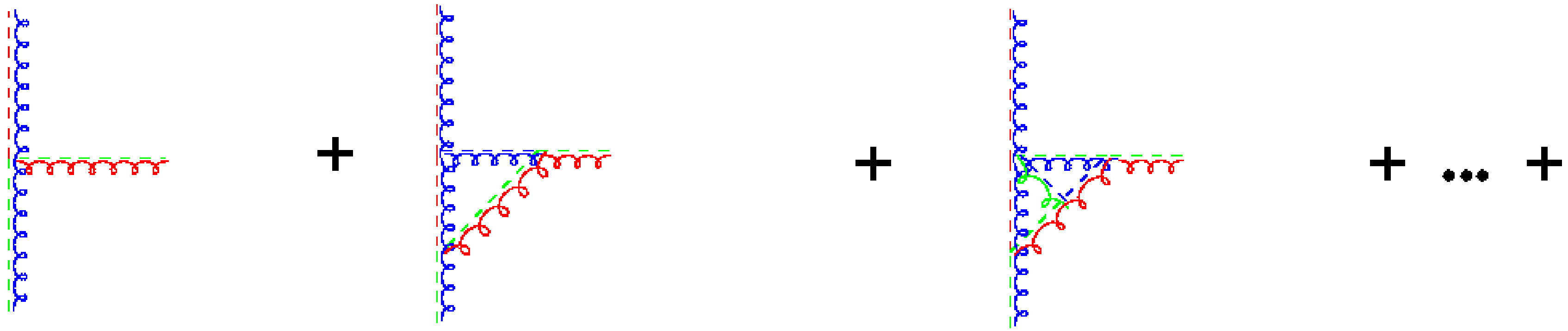
Non-Sudakov form factor: all loop re-sum...

$g \to gg$ Splitting Fct $\tilde{P}(z) = \frac{\bar{\alpha}_s}{1-z} + \frac{\bar{\alpha}_s}{z} + \ldots$

- Non - Sudakov form factor .... all loop resummation

$$\Delta_{\rm ns} = \exp\left[-\bar{\alpha}_s(k_t^2) \int_0^1 \frac{dz'}{z'} \int \frac{dq^2}{q^2} \Theta(k_t - q)\Theta(q - z' q_t) \right]$$

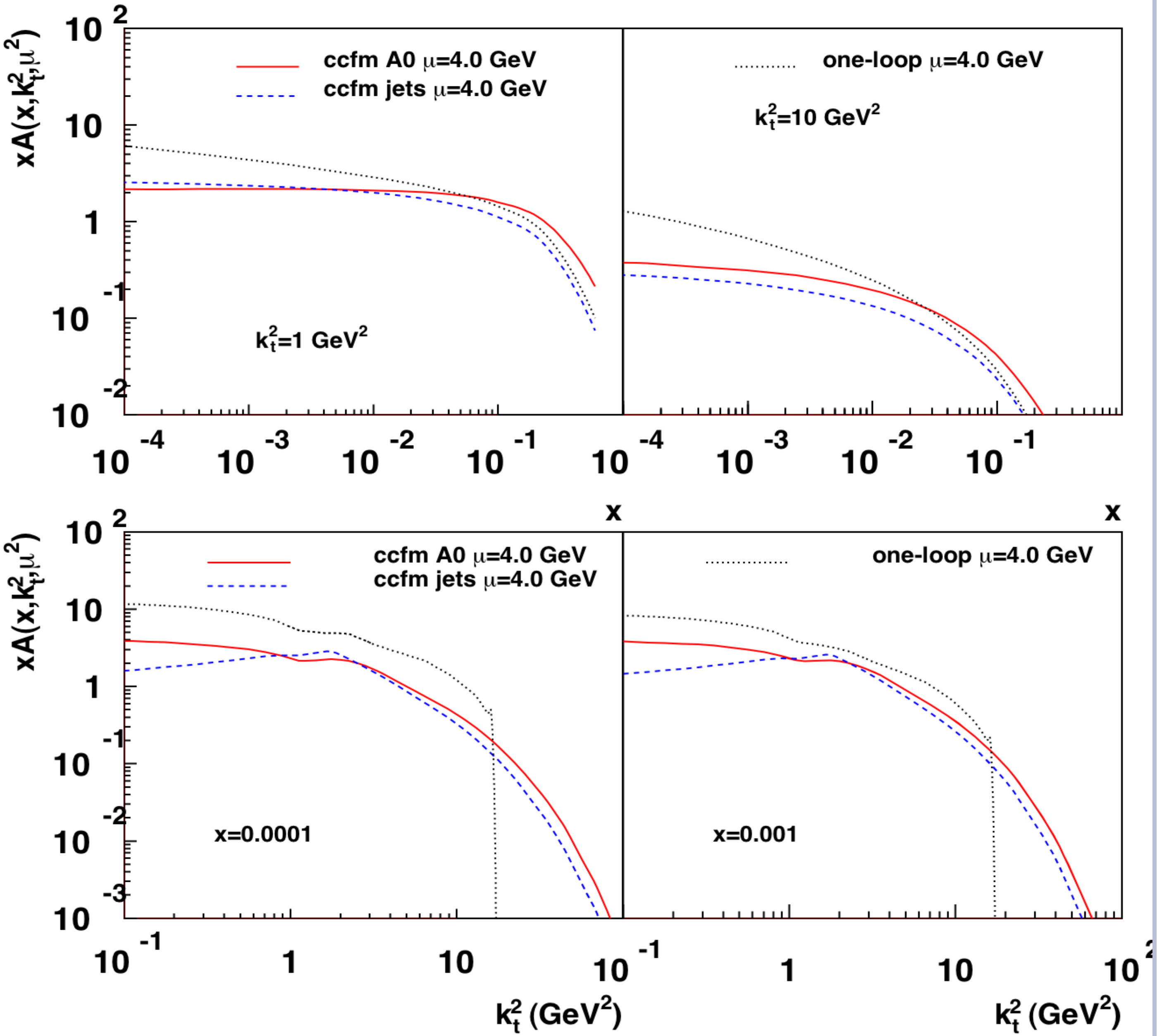
$$\Delta_{\rm ns} = 1 + \left(-\bar{\alpha}_s(k_t^2) \int \frac{dz'}{z'} \int \frac{dq^2}{q^2} \right)^1 + \frac{1}{2!} \left(-\bar{\alpha}_s(k_t^2) \int \frac{dz'}{z'} \int \frac{dq^2}{q^2} \right)^2 \ldots$$



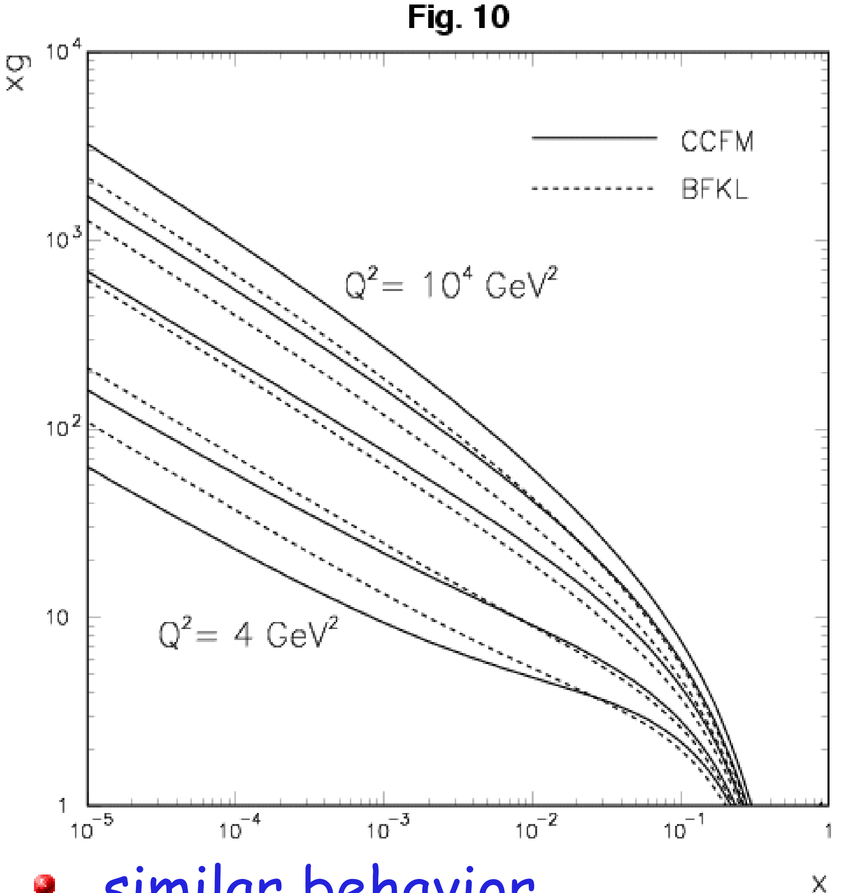
$$\bar{\alpha}_s(k_t)\frac{1}{z}\left[+ \quad \bar{\alpha}_s \log\left(\frac{z}{z_0}\right) \log\left(\frac{k_t^2}{z_0 z q^2}\right) + \frac{1}{2!}\left(\bar{\alpha}_s \log\left(\frac{z}{z_0}\right) \log\left(\frac{k_t^2}{z_0 z q^2}\right)\right)^2 + \cdots \right]$$

updf: CCFM - 1loop

- all uPDFs describe HERA measurements
- different intrinsic kt distributions
- different splitting functions

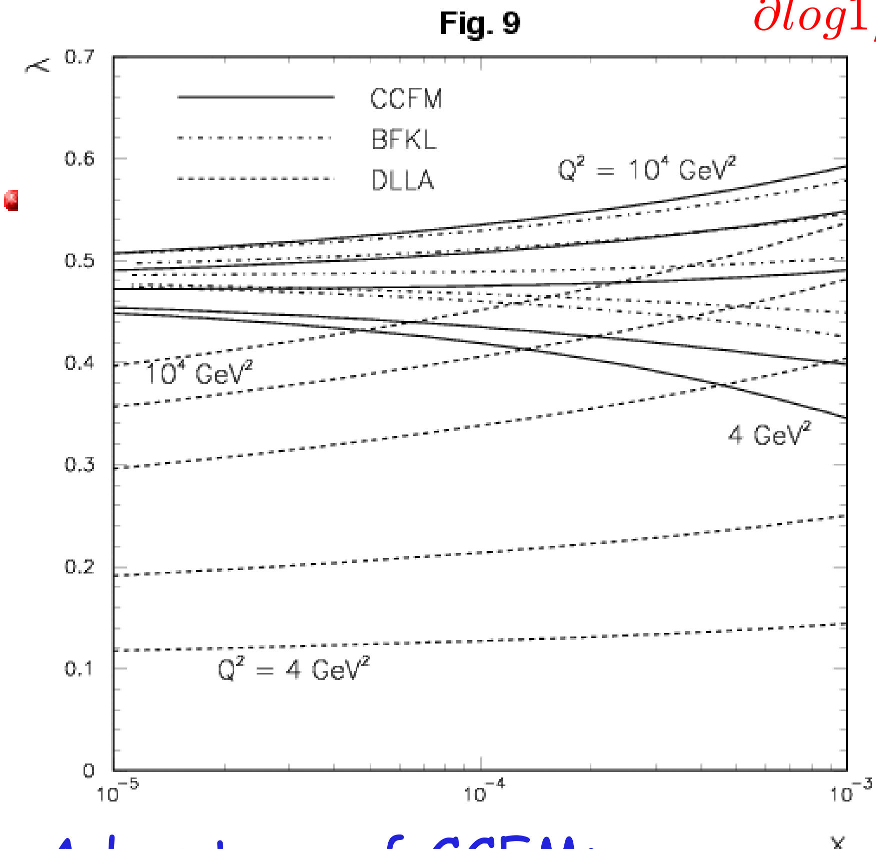


Comparison: CCFM and BFKL



J. Kwiecinski, A. Martin, P. Sutton PRD 52 (1005) 1445

$$\lambda = \frac{\partial F}{\partial log 1/x}$$



- similar behavior
  - BFKL indep. of Q2, effect comes from integration over uPDF
- details are different

- Advantage of CCFM:
  - attempt to describe emissions
  - unified for small and large x

How to calculate x-sections then ?

k_t-factorization

- use high energy (kt -) factorization:

  (Catani,Ciafaloni, Hautmann NPB 366 (1991) 135,
  Gribov, Levin, Ryskin, Phys. Rep.100 ,(1983),1,
  Collins, Ellis, NPB 360 ,(1991) ,3)

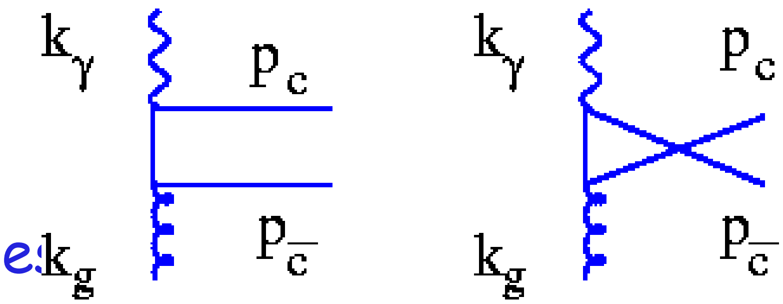
$$\sigma(\mathrm{ep} \to \mathrm{e}'\mathrm{q}\bar{\mathrm{q}}) = \int \frac{dy}{y} d^2Q \frac{dx_g}{x_g} \int d^2k_t \hat{\sigma}(\hat{s}, k_t, \bar{q}) x_g \mathcal{A}(x_g, k_t, \bar{q})$$

- with

$$\int^{Q^2} d^2k_t x_g \mathcal{A}(x_g, k_t, \bar{q}) \simeq x_g G(x_g, Q^2)$$

- t-channel gluon with virtuality $k^2 = -k_\perp^2$ dominates the process in the high energy limit $s \gg \hat{s}$
- collinear limit obtained by: $\hat{\sigma}(\hat{s}, 0, Q) \cdot \Theta(Q - k_\perp)$
- BUT k_t-factorization is proven only for small x ....

"off-shell" matrix elements



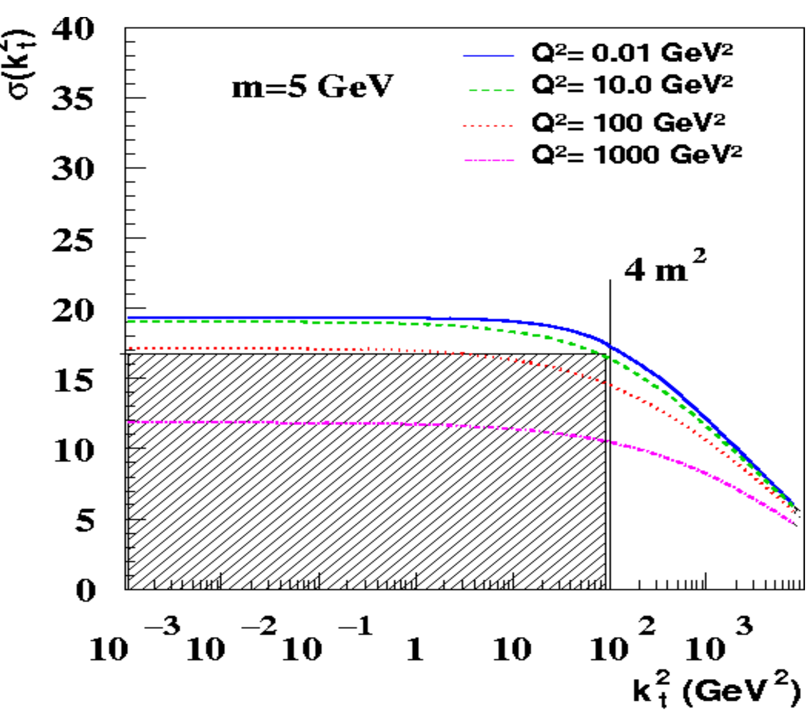
- calculation using standard Feynman rules

$$\mathcal{M}(\gamma g \to c\bar{c}) = \bar{u}(p_c)\left(\frac{\not\epsilon_\gamma\,(\not p_c - \not k_\gamma + m)\,\not\epsilon_g}{k_\gamma^2 - 2k_\gamma p_c} + \frac{\not\epsilon_g\,(\not p_c - \not k_g + m)\,\not\epsilon_\gamma}{k_g^2 - 2k_g p_c}\right)u(p_{\bar{c}})$$

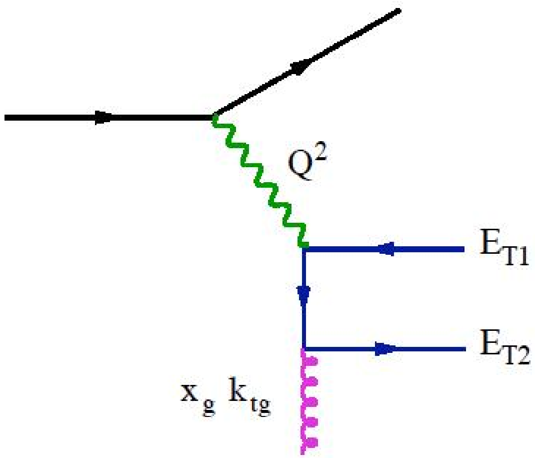
- use high-energy polarization projectio

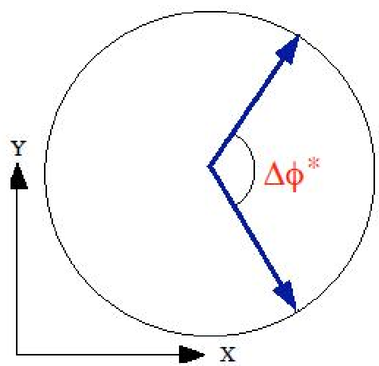
$$G^{\mu\nu} = \overline{\epsilon_g^\mu \epsilon_g^{*\nu}} = \frac{k_{t\,g}^\mu k_{t\,g}^\nu}{|k_{t\,g}|^2}$$

- ME is finite for $k_\perp \to 0$
- ME has tail to large k_t

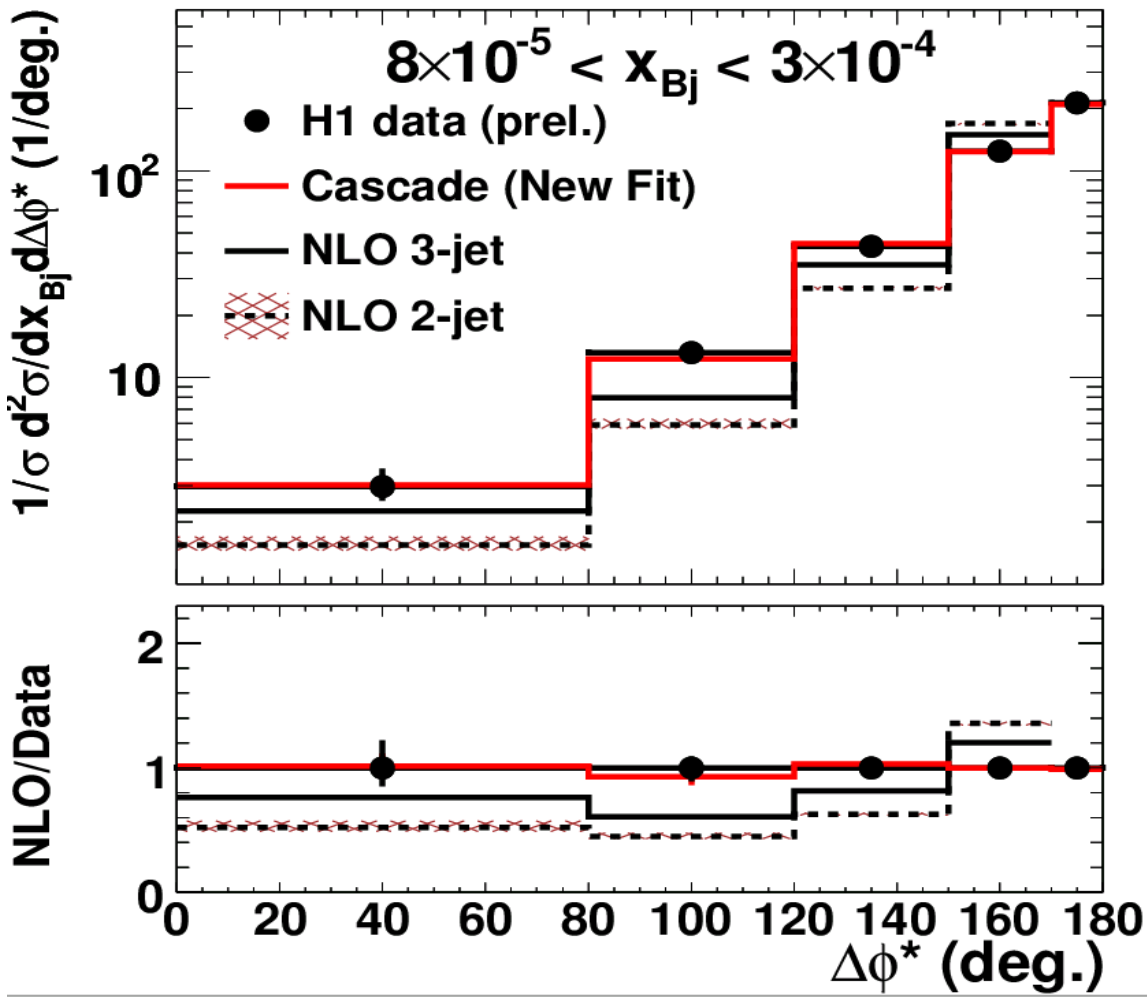


Dijets and uPDFs: azimuthal correlations





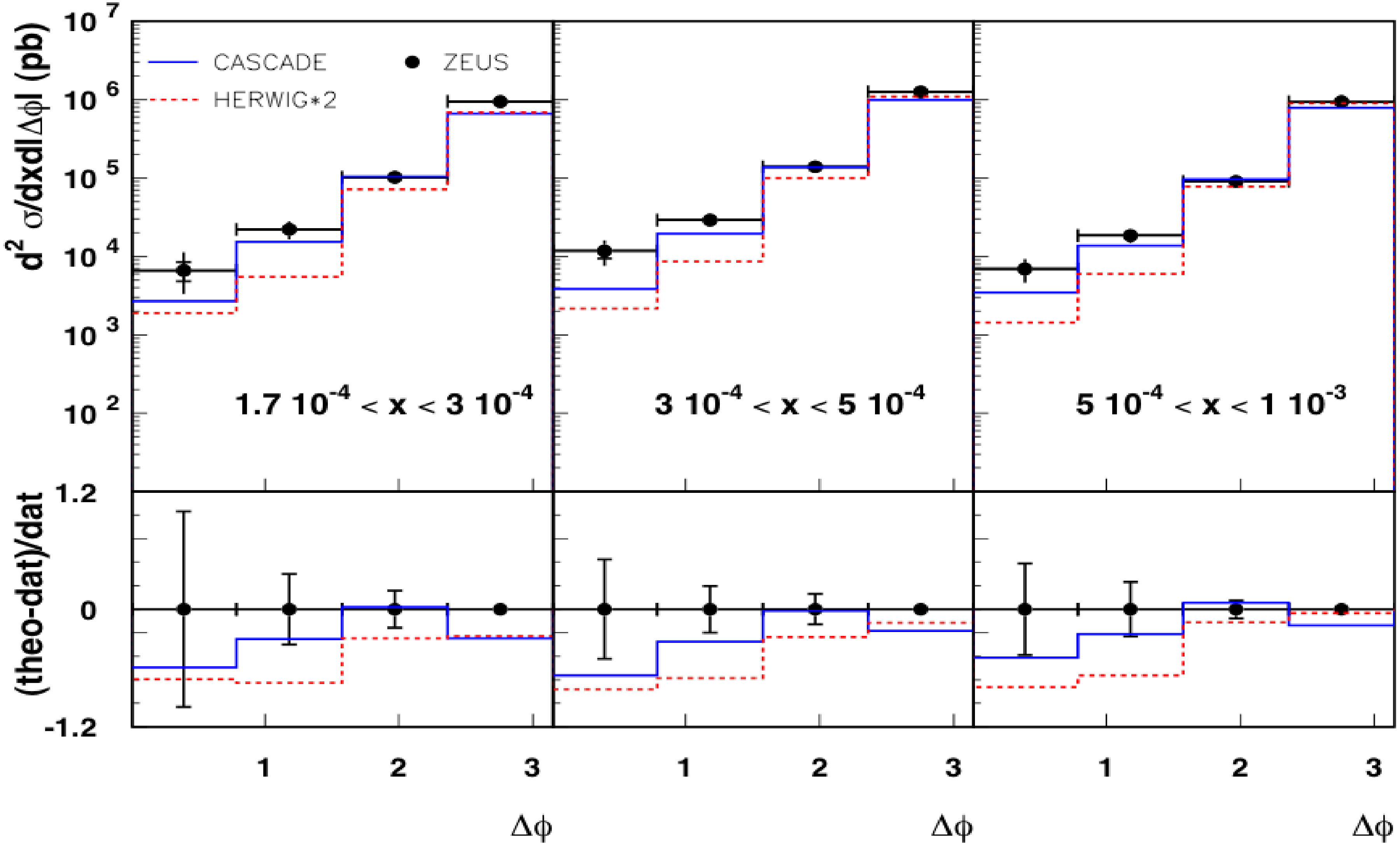
M. Hansson (Lund)



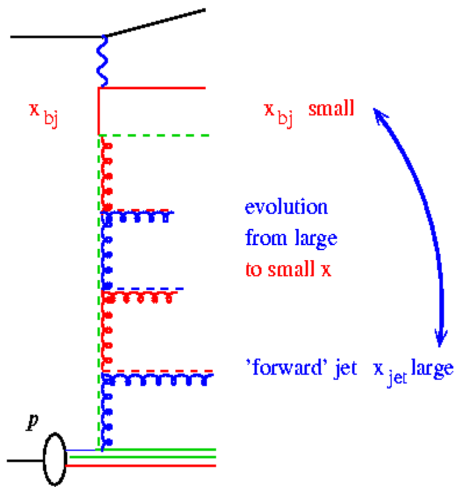
- **CCFM can better than NLO describe data !!!!**

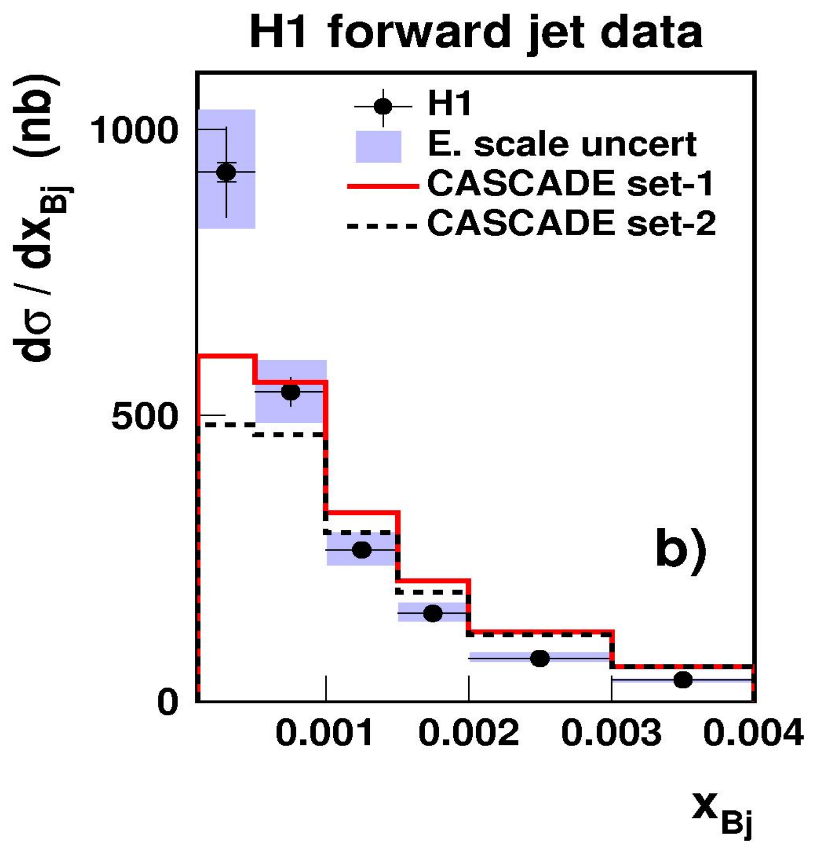
Comparison of angular orderings ...

- compare angular ordering versus semi angular-ordering (qt ordering at small x)



forward jet production





- DIS and forward jet:

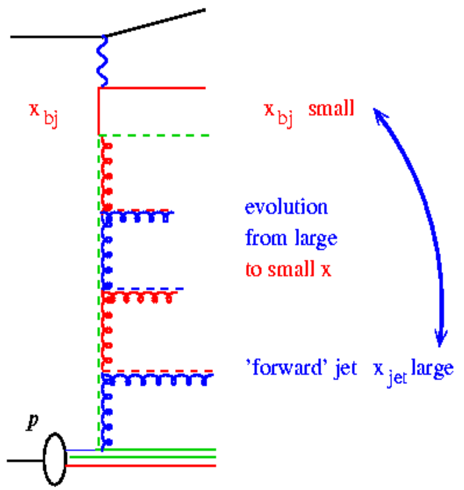
  $1.7 < \eta_{jet} < 2.8$

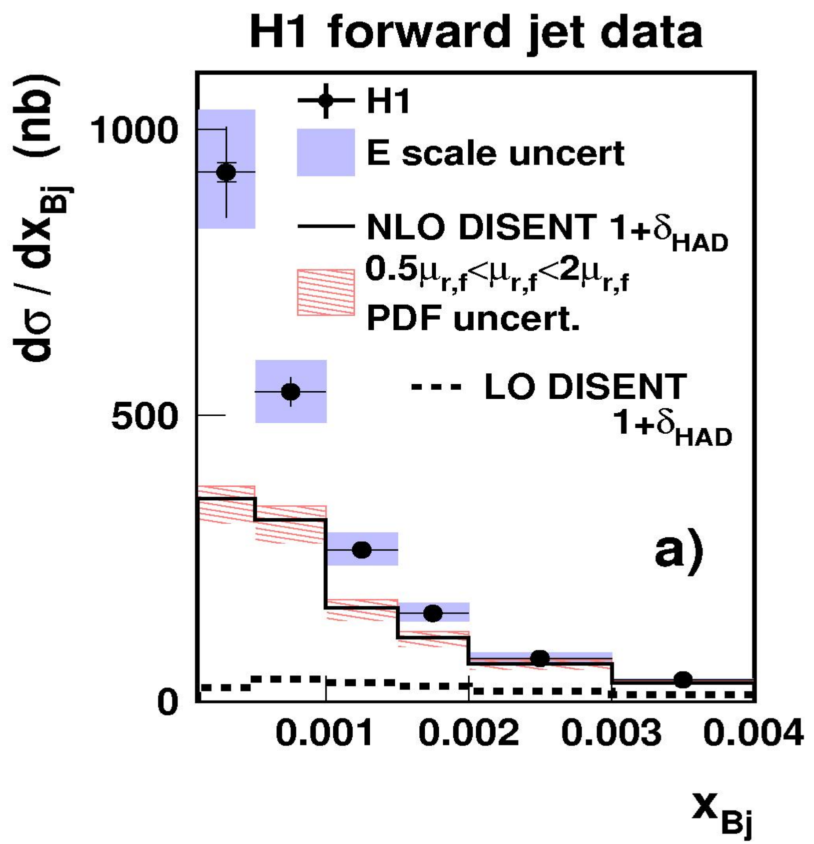
  $x_{jet} > 0.035$

  $0.5 < \frac{p_{t\,jet}^2}{Q^2} < 5$

- **CASCADE (CCFM) evolution closer to data**

forward jet production





- DIS and forward jet:

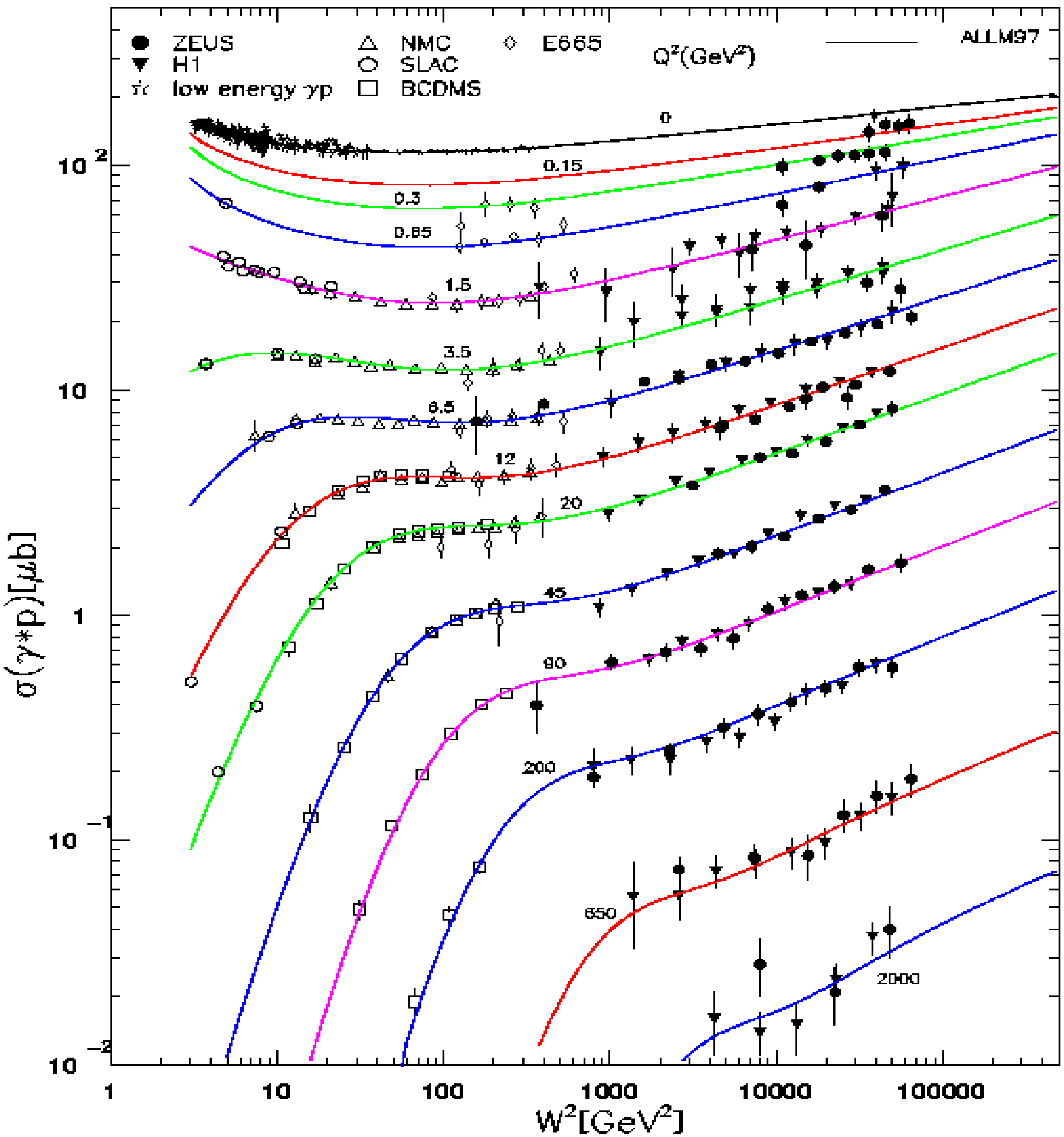
  $1.7 < \eta_{jet} < 2.8$

  $x_{jet} > 0.035$

  $0.5 < \frac{p_{t\,jet}^2}{Q^2} < 5$

- "NLO" too low

High energy behavior of x section



From H. Abramowicz A. Levy hep-ph/9712415

$$\sigma(\gamma^* p) = \frac{4\pi^2\alpha}{Q^2} F_2(x,Q^2)$$

$$= \frac{4\pi^2\alpha}{Q^2} \sum e_q^2 x q(x,Q^2)$$

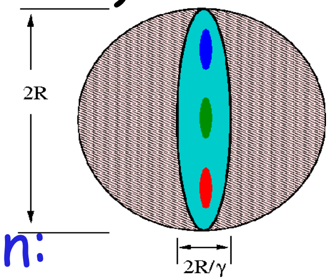
$$x = \frac{Q^2}{W^2+Q^2}$$

- rising x-section with W^2
- at large energies can become larger than σ_{tot}
- mechanism needed which tames rise at large energies

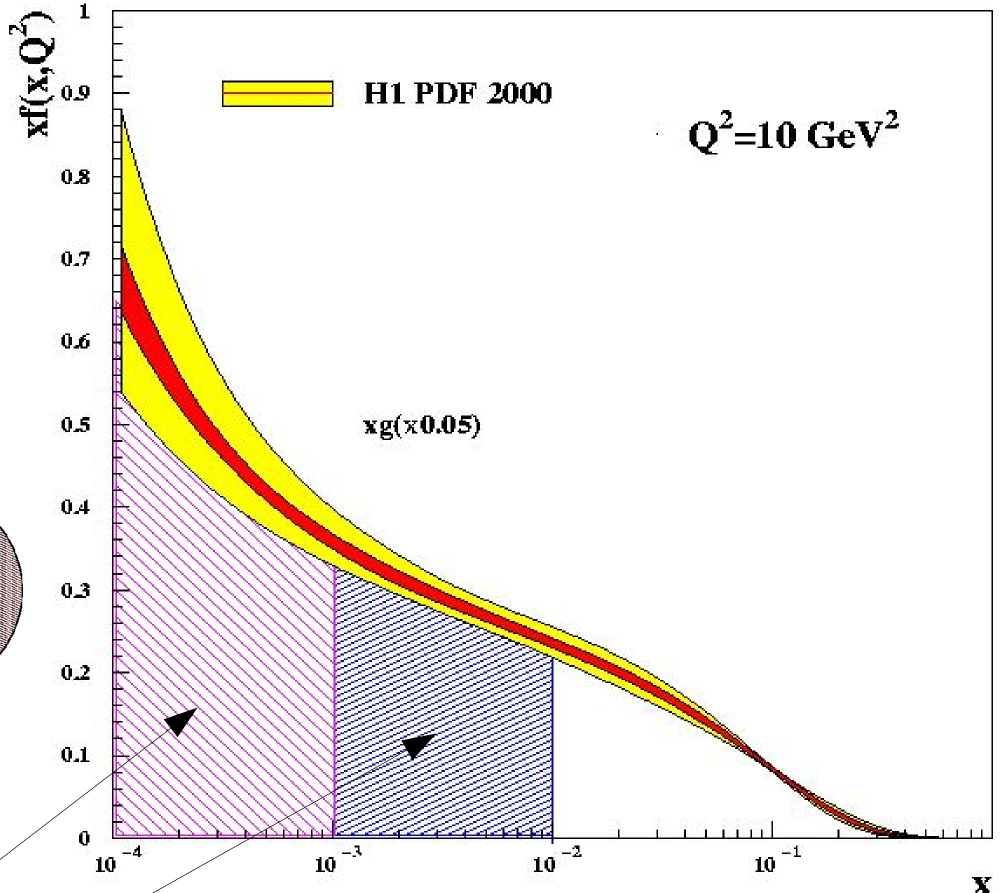
➔ saturation !!!

Parton Distribution Functions

- number of gluons in long. phase space dx/x : $xg(x,\mu^2)dx/x$
- occupation area:
  nr of gluons x (trans size)^2
  $g(x,\mu^2)\frac{1}{\mu^2}$



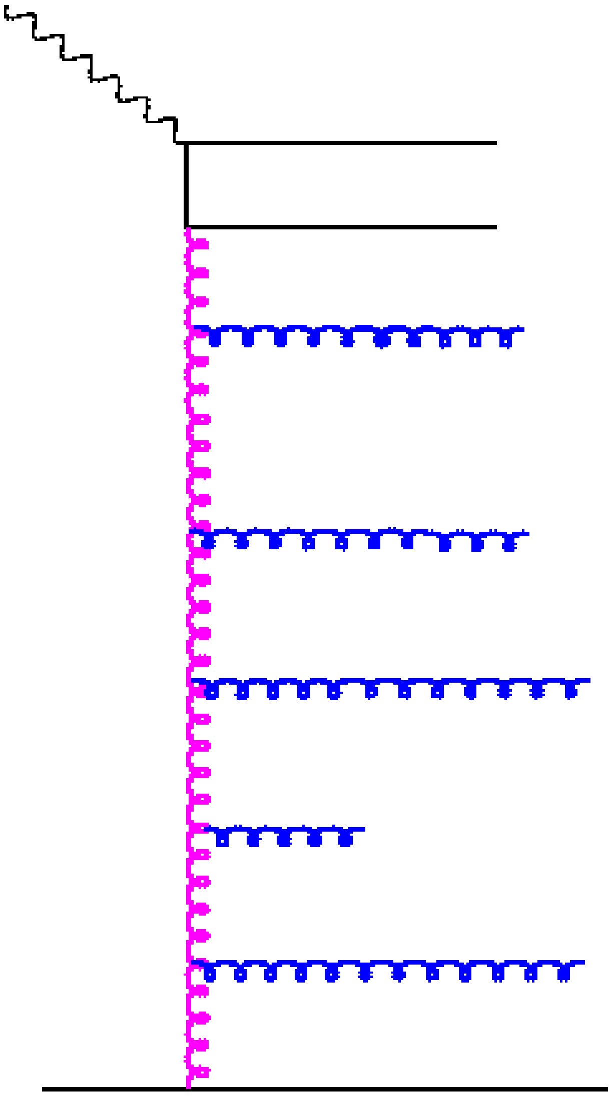
- saturation starts when:
  $$\frac{\alpha_s(\mu^2)}{\mu^2}xg(x,\mu^2)\frac{dx}{x} \geq \pi R^2$$



- gluon density is very large:~ 90 or 45 Gluons !!!!!
- with R ~ 1 GeV^{-1} we obtain: $$\frac{0.2}{10GeV^{-1}}100 \sim \pi$$! ! ! ! !

Parton evolution: gluon density

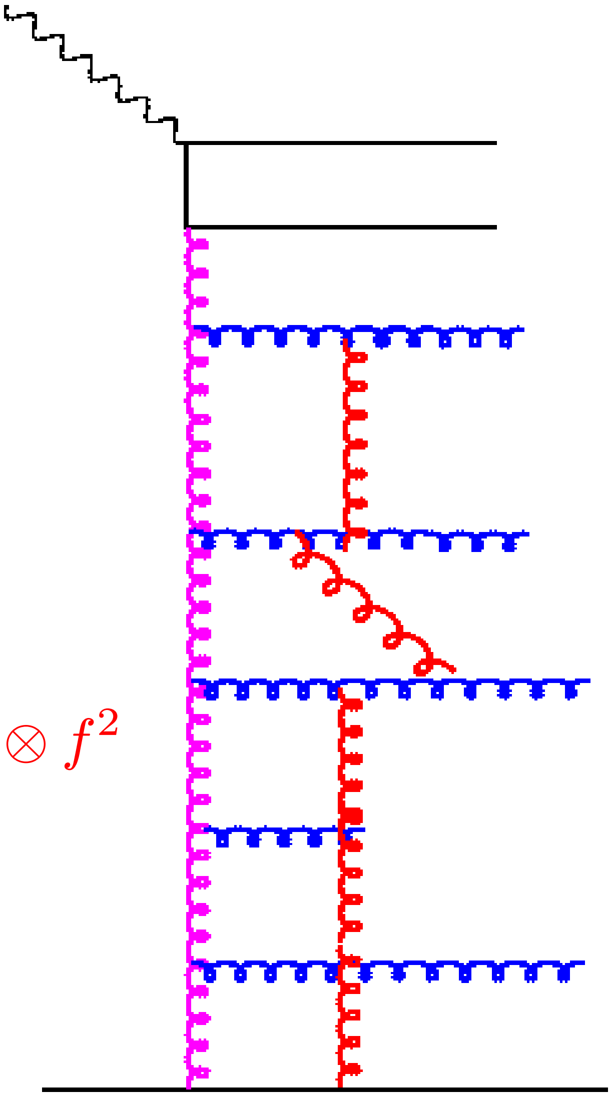
- Gluon splitting and evolution



Parton evolution: gluon density

- Gluon splitting and evolution
- High density of gluons
  - ➔ overlapping gluons
  - ➔ recombination
  - ➔ multiple scatterings
  - ➔ diffraction !!!!
- evolution equation including recombination effects:

$$f(x,k^2) = f^0(x,k^2) + K^1 \otimes f - \frac{1}{R^2} K^2 \otimes f^2$$



- GribovLevinRyskin equation (Phys.Rep. 100 1(1983))
- BalitskyKovchegov equation (NPB 463, 99 (1996), PRD 60 (1999) 034008, D62 (2000) 074018)