Hypergeometric functions & Multiple Series:

reduction, ε-expansion, Feynman Diagrams

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Motivation

Any dimensionally-regularized multiloop Feynman diagram with propagators $1/(p^2 - m^2)$ can be written in the form of a finite sum of multiple Mellin-Barnes integrals obtained via a Feynman-parameter or "α" representation:

$$F\left(a_{js}, b_{km}, c_i, d_j, \vec{x}\right) = \int_{\gamma + i\mathbb{R}} d\vec{z} \frac{\Pi_{j=1}^{p} \Gamma\left(\sum_{s=1}^{r} a_{js} z_s + c_j\right)}{\Pi_{k=1}^{q} \Gamma\left(\sum_{m=1}^{r} b_{km} z_m + d_k\right)} x_1^{-z_1} \ldots x_r^{-z_r} \,,$$

Formally,this integral can be expressed in terms of a sum of residues of the integrated expression

$$F\left(a_{js}, b_{km}, \vec{c}, \vec{d}, \vec{\alpha}, \vec{x}\right) = \sum_{\vec{\alpha}} B_{\vec{\alpha}} \vec{x}^{\,\vec{\alpha}} \Phi(\vec{\gamma}; \vec{\sigma}; \vec{x}) \,,$$

with

$$\Phi(\vec{\gamma}; \vec{\sigma}; \vec{x}) = \sum_{m_1, m_2, \cdots, m_r = 0}^{\infty} \left(\frac{\Pi_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\Pi_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \,,$$

Algebraic Reduction of Gauss hypergeometric function:

M.K., JHEP, 2006

As is known, for any three contiguous Gauss hypergeometric functions there is a contiguous relation, which is a linear relation with coefficients being rational functions in the parameters A, B, C and argument z.

$$P_1(A,B,B,z)\;{}_2F_1\left(\left.\begin{matrix} A\pm 1, B \\ C \end{matrix}\right| z\right) + P_2(A,B,B,z)\;{}_2F_1\left(\left.\begin{matrix} A, B\pm 1 \\ C \end{matrix}\right| z\right)$$
$$+P_3(A,B,B,z)\;{}_2F_1\left(\left.\begin{matrix} A, B \\ C\pm 1 \end{matrix}\right| z\right) = 0$$

Any Gauss hypergeometric function with arbitrary parameters is reduced to the linear combination of two (our basis):

$$P(a,b,c,z)_2F_1(a+I_1,b+I_2;c+I_3;z)$$
$$= Q_1(a,b,c,z)_2F_1(a+1,b+1;c+1;z) + Q_2(a,b,c,z)_2F_1(a,b;c;z)\ ,$$

where a, b, c, are any fixed numbers, P, Q_1, Q_2 are polynomial in parameters a, b, c and argument z, and I_1, I_2, I_3 any integer numbers.

Γ-functions: ε-expansion

The starting point of ε-expansion is the Taylor expansion of the Γ function.

$$\ln \frac{\Gamma(k+1+\frac{p}{q}+j+z)}{\Gamma(k+1+\frac{p}{q}+j)} = \ln \frac{\Gamma(k+1+\frac{p}{q}+z)}{\Gamma(k+1+\frac{p}{q})} - \sum_{m=1}^{\infty} \frac{(-z)^m}{m} \sum_{r=1}^{j} \frac{1}{\left(r+k+\frac{p}{q}\right)^m}$$

$$= \ln \frac{\Gamma\left(1+\frac{p}{q}+z\right)}{\Gamma\left(1+\frac{p}{q}\right)} - \sum_{m=1}^{\infty} \frac{(-z)^m}{m} \sum_{r=1}^{j+k} \frac{1}{\left(r+\frac{p}{q}\right)^m} \,,$$

In particular, for $p = 0$, we have

$$\ln \frac{\Gamma(1+j+z)}{\Gamma(1+z)} = \ln \Gamma(1+j) - \sum_{m=1}^{\infty} \frac{(-z)^m}{m} S_m(j) \,,$$

where $S_a(j)$ is the harmonic sum defined as $S_a(j) = \sum_{k=1}^{j} \frac{1}{k^a}$.

Gauss hypergeometric functions: ε-expansion

Algebraic Reduction

$$ {}_2F_1(I_1+a_1\varepsilon, I_2+a_2\varepsilon; I_3-p/q+c\varepsilon; z) \to {}_2F_1(1+a_1\varepsilon, 2+a_2\varepsilon; 2-p/q+c\varepsilon; z) \ , $$

$$ {}_2F_1\left(\begin{array}{c} 1+a_1\varepsilon, 1+a_2\varepsilon \\ 2-\frac{p}{q}+c\varepsilon \end{array} \middle| z \right) = \frac{1}{z}\left(1-\frac{p}{q}+c\varepsilon\right) \sum_{j=1}^{\infty} z^j \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1-\frac{p}{q}+j\right)} \Delta \ , $$

where

$$ \Delta = \exp\left[\sum_{k=1}^{\infty} \frac{(-\varepsilon)^k}{k} \left(-A_k S_k(j-1) + c^k \mathrm{S}_k^{[q-p,q]}(j-1) \right) \right] \ , $$

$$ A_k = a_1^k + a_2^k \ , $$

$$ \mathrm{S}_k^{[p,q]}(j) = \sum_{r=1}^{j} \frac{1}{\left(r+\frac{p}{q}\right)^k} . \quad S_k(j) \equiv \mathrm{S}_k^{[0,q]}(j) \ . $$

Gauss hupergeometric function: ε-expansion

In particular, the first few coefficients of the ε expansion read:

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1+j-\frac{p}{q}\right)} \exp\left[\sum_{k=1}^{\infty} \frac{(-\varepsilon)^k}{k} \left(-A_k S_k(j-1) + c^k \mathrm{S}_k^{[q-p,q]}(j-1)\right)\right]$$

$$= \sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1+j-\frac{p}{q}\right)} \Bigg\{ 1 + \varepsilon \left[A_1 S_1(j-1) - c\mathrm{S}_1^{[q-p,q]}(j-1)\right]$$

$$+\frac{1}{2}\varepsilon^2 \Bigg[A_1^2 S_1^2(j-1) - A_2 S_2(j-1) - 2cA_1 S_1(j-1)\mathrm{S}_1^{[q-p,q]}(j-1)$$

$$+c^2 \left(\left[\mathrm{S}_1^{[q-p,q]}(j-1)\right]^2 + \mathrm{S}_2^{[q-p,q]}(j-1) \right) \Bigg] + O(\varepsilon^3) \Bigg\} .$$

Subclass of multiple sums

There is an important subclass of *multiple inverse rational sums*, which are defined as

$$\Sigma^{[p,q]}_{a_1,\cdots,a_k;\,-;c;-}(z) \equiv \sum_{j=1}^{\infty} \frac{z^j}{j^c} \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1-\frac{p}{q}+j\right)} S_{a_1}(j-1) S_{a_2}(j-1) \cdots S_{a_k}(j-1) \;,$$

where $a_1, \cdots, a_k, c$ are arbitrary positive integers. The number $w = c+1+a_1+\cdots+a_k$ is called the *weight* $d = k$ the *depth* of the sums.

Comments

The series representation is an intensively studied approach. Particularly impressive results were derived in the framework of the nested-sum approach for hypergeometric functions with a balanced set of parameters by
Moch,Uwer,Weinzierl,2002; Weinzierl,2004;
Computer realizations of nested sums approach to expansion of hypergeometric functions are given in
Weinzierl, 2002; Moch & Uwer, 2006;
Huber & Maître, 2006, 2008

Generating-function approach have been applied to construction of ε-expansion for hypergeometric functions with one unbalanced set of parameters
M.K.,Davydychev,2004; M.K.,Ward,Yost,2007; M.K.,Kniehl,2008

Generating Function Approach

Let us rewrite the multiple sum in the form

$$\Sigma^{[p,q]}_{\vec{a};-;c;-}(z) = \sum_{j=1}^{\infty} z^j \eta_{\vec{a};-;c;-}(j) \; .$$

Difference Equation:

$$\left[j+1-\frac{p}{q} \right] (j+1)^c \eta^{[p,q]}_{\vec{a};-;c;-}(j+1) = j^{c+1} \eta^{[p,q]}_{\vec{a};-;c;-}(j) + r^{[p,q]}_{\vec{a};-}(j) \; ,$$

where

$$\frac{\Gamma\left(1+j-\frac{p}{q}\right)}{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)} r^{[p,q]}_{\vec{a};-}(j) = j \times \left\{ \prod_{r=1}^{k} \left[S_{a_r}(j-1) + \frac{1}{j^{a_r}} \right] - \prod_{r=1}^{k} S_{a_r}(j-1) \right\} \; .$$

$$\left[\left(\frac{1}{z} - 1 \right) z \frac{d}{dz} - \frac{1}{z}\frac{p}{q} \right] \left(z \frac{d}{dz} \right)^c \Sigma^{[p,q]}_{\vec{a};-;c;-}(z) = \delta_{\vec{a},0} + R^{[p,q]}_{\vec{a};-}(z) \; ,$$

System of Differential Equations

Starting Equation:

$$\left[\left(\frac{1}{z}-1\right) z\frac{d}{dz}-\frac{1}{z}\frac{p}{q}\right]\left(z\frac{d}{dz}\right)^{c}\Sigma^{[p,q]}_{\vec{a};-;c;-}(z) = \delta_{\vec{a},0} + R^{[p,q]}_{\vec{a};-}(z) \ ,$$

New variable

$$\xi = \left(\frac{z}{z-1}\right)^{\frac{1}{q}}$$

New system:

$$\left(\frac{1}{q}(1-\xi^q)\xi\frac{d}{d\xi}\right)^{c}\Sigma^{[p,q]}_{\vec{a};-;c;-}(\xi) = \xi^p \sigma^{[p,q]}_{\vec{a};-}(\xi) \ ,$$

$$-\frac{1}{q}\frac{1-\xi^q}{\xi^{q-p-1}}\frac{d}{d\xi}\sigma^{[p,q]}_{\vec{a};-}(\xi) = \delta_{\vec{a},0}+R^{[p,q]}_{\vec{a};-}(\xi) \ .$$

Iterated integral

The iterated integral is defined as

$$
\begin{aligned}
I(z; a_k, a_{k-1} \cdots, a_1) &= \int_0^z \frac{dt}{t - a_k} I(t; a_{k-1}, \cdots, a_1) \\
&= \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - a_1}
\end{aligned}
$$

where we put that all $a_k \neq 0$. In early consideration by Kummer, Poincare, Lappo-Danilevky this integral was called as hyperlogarithms One of the property of hyperlogarithms is the scaling invariance:

$$
I(z; a_1, \cdots, a_k) = I\left(1; \frac{a_1}{z}, \cdots, \frac{a_k}{z}\right) .
$$

A special case of this integral,

$$
\begin{aligned}
& G_{m_n, m_{n-1}, \cdots, m_1}(z; x_n, \cdots, x_1) \\
& \equiv I(z; \underbrace{0, \cdots, 0}_{m_n - 1 \text{ times}}, x_n, \underbrace{0, \cdots, 0}_{m_{n-1} - 1 \text{ times}}, x_{n-1}, \cdots, \underbrace{0, \cdots, 0}_{m_1 - 1 \text{ times}}, x_1)
\end{aligned}
$$

Multiple polylogarithms (MPL)

By definition, the multiple polylogarithm is defined by power series

$$\mathrm{Li}_{k_1,k_2,\cdots,k_n}(x_1,x_2,\cdots,x_n) = \sum_{m_n>\cdots>m_1>0}^{\infty} \frac{x_1^{m_1}}{m_1^{k_1}} \frac{x_2^{m_2}}{m_2^{k_2}} \cdots \frac{x_n^{m_n}}{m_n^{k_n}} ,$$

where **weight** $k = k_1 + k_2 + \cdots + k_n$ and **depth** is equal to n.

It is defined for $|x_n| < 1$ and admit an analytical continuation.

The MZV corresponds to $x_1 = \cdots = x_n = 1$.

The multiple polylogarithm is a special case of iterated integral:

$$\mathrm{Li}_{k_1,k_2,\cdots,k_n}(y_1,y_2,\cdots,y_n)$$
$$= (-1)^n G_{k_n,k_{n-1},\cdots,k_2,k_1}\left(1; \frac{1}{y_n}, \frac{1}{y_n y_{n-1}}, \cdots, \frac{1}{y_1 \cdots y_n}\right) ,$$

$$G_{m_n,m_{n-1},\cdots,m_1}(z;x_n,\cdots,x_1) = (-1)^n \mathrm{Li}_{m_1,m_2,\cdots,m_n}\left(\frac{x_2}{x_1}, \frac{x_3}{x_2}, \cdots, \frac{z}{x_n}\right) .$$

Particular case of MPL

A particular case of the multiple polylogarithm is the "generalized polylogarithm" defined by

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z) = \sum_{m_n > m_{n-1} > \cdots > m_1 > 0}^{\infty} \frac{z^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

where $|z| < 1$ when all $k_i \geq 1$, or $|z| \leq 1$ when $k_n \leq 2$.

Another particular case is a "multiple polylogarithm of a square root of unity," defined as

$$\mathrm{Li}_{\left(\substack{\sigma_1,\sigma_2,\cdots,\sigma_n \\ s_1,s_2,\cdots,s_n}\right)}(z) = \sum_{m_n > m_{n-1} > \cdots m_1 > 0} z^{m_n} \frac{\sigma_n^{m_n} \cdots \sigma_1^{m_1}}{m_n^{s_n} \cdots m_1^{s_1}} .$$

where $\vec{s} = (s_1, \cdots s_n)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_n)$ are multi-indices and σ_k belongs to the set of the square roots of unity, $\sigma_k = \pm 1$. This particular case of multiple polylogarithms has been analyzed in detail by Remiddi and Vermaseren, 2000.

Multiple Inverse Rational Sums: General case I

M.K. & Kniehl, 2008

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1+j-\frac{p}{q}\right)} S_{a_1}(j-1) S_{a_2}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{z=z(\xi)}$$

$$= \xi^p \sum_{\vec{J},\vec{s}} c_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2-j_3}, \cdots, \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r}\xi\right) ,$$

where

$$\xi = \left(\frac{z}{z-1}\right)^{1/q} , \quad \lambda_q = \exp\left(i\frac{2\pi}{q}\right) .$$

$$1 \leq \{j_m\} \leq q , \quad \sum_{k=1}^{r} s_k = 1+a_1+\cdots+a_p ,$$

p, q are arbitrary integers.

Multiple Inverse Rational Sums: General case II

M.K. & Kniehl, 2008

$$\sum_{j=1}^{\infty} \frac{z^j}{j^c} \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1+j-\frac{p}{q}\right)} S_{a_1}(j-1) S_{a_2}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{z=z(\xi)}$$

$$= \sum_{\vec{J},\vec{s}} \tilde{c}_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2-j_3}, \cdots, \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r} \xi \right) \qquad (c \geq 1) \; ,$$

where

$$1 \leq \{j_m\} \leq q \; , \quad \sum_{k=1}^{r} s_k = 1 + c + a_1 + \cdots + a_p \; .$$

Multiple Rational Sums: General Case I

M.K. & Kniehl, 2008

$$
\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(j+\frac{p}{q}\right)}{\Gamma(j+1)\Gamma\left(1+\frac{p}{q}\right)} S_{a_1}(j-1) S_{a_2}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{z=z(\tau)}
$$

$$
= \sum_{\vec{J},\vec{s},k} \left(c_{\vec{J},\vec{s},k} + d_{\vec{J},\vec{s},k} \tau^{-p} \right) \ln^k \tau
$$

$$
\times \left[\mathrm{Li}_{\vec{s}} \left(\lambda_q^{j_1 - j_2}, \cdots, \lambda_q^{j_r} \tau \right) - \mathrm{Li}_{\vec{s}} \left(\lambda_q^{j_1 - j_2}, \cdots, \lambda_q^{j_r} \right) \right],
$$

where

$$
\tau = (1-z)^{\frac{1}{q}} .
$$

Multiple Rational Sums: General Case II

M.K. & Kniehl, 2008

$$\left.\sum_{j=1}^{\infty} \frac{z^j}{j^c} \frac{\Gamma\left(j+\frac{p}{q}\right)}{\Gamma(j+1)\Gamma\left(1+\frac{p}{q}\right)} S_{a_1}(j-1) S_{a_2}(j-1) \cdots S_{a_k}(j-1)\right|_{z=z(\tau)}$$

$$= \sum_{\vec{J}, \vec{s}, k} \tilde{d}_{\vec{J},\vec{s},k} \ln^k \tau \left[\mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \cdots, \lambda_q^{j_r}\tau\right) - \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \cdots, \lambda_q^{j_r}\right) \right]$$

Lemma A

$$\left(\frac{1}{q}(1-\xi^q)\xi\frac{d}{d\xi}\right)^c \Sigma^{[p,q]}_{\vec{a};-;c;-}(\xi) = \xi^p \sigma^{[p,q]}_{\vec{a};-}(\xi) \; ,$$

Differential form:

$$\left(\frac{1}{q}(1-\xi^q)\xi\frac{d}{d\xi}\right)^{c-j} \Sigma^{[p,q]}_{\vec{a};-;c;-}(\xi) = \Sigma^{[p,q]}_{\vec{a};-;j}(\xi) \; ,$$

Integral form:

$$\left(\frac{1}{q}(1-\xi^q)\xi\frac{d}{d\xi}\right)^{c-j-1} \Sigma^{[p,q]}_{\vec{a};-;c;-}(\xi) = q\int_0^\xi \frac{dt}{(1-t^q)t}\Sigma^{[p,q]}_{\vec{a};-;j}(t) \; , \quad j \geq 1 \; .$$

Lemma A

If, for some integer j, the series $\Sigma^{[p,q]}_{\vec{a};-;j}(\xi)$ is expressible in terms of hyperlogarithms with complex coefficients, then this also holds for the sums $\Sigma^{[p,q]}_{\vec{a};-;j+i}(\xi)$ with positive integers i.

Proposition A

Proposition A

For $c = 0$, the inverse rational sums are expressible in terms of multiple polylogarithms of arguments being powers of q-roots of unity and the variable ξ with complex coefficients $c_{r,\vec{s}}$ times a factor ξ^p, as

$$\Sigma^{[p,q]}_{a_1,\cdots,a_p;\;-;0;-}(z)\Big|_{z=z(\xi)} = \xi^p$$

$$\times \sum_{\vec{J},\vec{s}} c_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2-j_3}, \cdots, \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r}\xi\right) ,$$

where the weights of the l.h.s. and the r.h.s. are equal, i.e. $s_1 + \cdots + s_r = 1 + a_1 + \cdots + a_p$.

where

$$\xi = \left(\frac{z}{z-1}\right)^{1/q} , \quad \lambda_q = \exp\left(i\frac{2\pi}{q}\right) , \quad 1 \le \{j_m\} \le q \, .$$

Corollary A

Substituting expression from Lemma A and performing a trivial splitting of the denominator, we obtain

$$\Sigma^{[p,q]}_{\vec{a};-;1;-}(z)\Big|_{z=z(\xi)}$$

$$= \sum_{\vec{J},\vec{s}} \sum_{j=1}^{q} \lambda_q^{-jp} \int_0^\xi \frac{1}{t - \frac{1}{\lambda_q^j}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2-j_3}, \cdots \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r} t\right)$$

$$= \sum_{\vec{J},\vec{s}} \lambda_q^{-jp}\, c_{\vec{J},\vec{s}} \mathrm{Li}_{1,\vec{s}}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2-j_3}, \cdots, \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r-j_{r+1}}, \lambda_q^{j_{r+1}}\xi\right) .$$

Corollary A:

For $c \geq 1$, the inverse rational sums are expressible in terms of multiple polylogarithms of arguments being powers of q-roots of unity and the variable ξ with complex coefficients $d_{\vec{J},\vec{s}}$, as

$$\Sigma^{[p,q]}_{a_1,\cdots,a_p;\,-;c;-}(z)\Big|_{z=z(\xi)} = \sum_{\vec{J},\vec{s}} d_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \cdots, \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r}\xi\right) \quad (c \geq 1$$

Depth 0 sums

$$\Sigma^{[p,q]}_{-;-;c;-}(\xi) = \sum_{j=1}^{\infty} \frac{z^j}{j^c} \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1+j-\frac{p}{q}\right)} \, .$$

$$\left(\frac{1}{q}(1-\xi^q)\xi\frac{d}{d\xi}\right)^c \Sigma^{[p,q]}_{-;-;c;-}(\xi) = \xi^p \sigma^{[p,q]}_{-;-}(\xi) \, ,$$

$$\frac{d}{d\xi}\sigma^{[p,q]}_{-;-}(\xi) = \sum_{j=1}^{q} \lambda_q^{jp} \frac{1}{\xi - \frac{1}{\lambda_q^j}} \, .$$

$$\sigma^{[p,q]}_{-;-}(\xi) = -\sum_{j=1}^{q} \lambda_q^{jp} \mathrm{Li}_1\left(\lambda_q^j \xi\right) \, , \quad \Sigma^{[p,q]}_{-;-;0;-}(\xi) = -\xi^p \sum_{j=1}^{q} \lambda_q^{jp} \mathrm{Li}_1\left(\lambda_q^j \xi\right) \, ,$$

$$\Sigma^{[p,q]}_{-;-;1;-}(\xi) = -\sum_{k,j_1=1}^{q} \lambda_j^{(k-j_1)p} \mathrm{Li}_{1,1}\left(\lambda_q^{k-j_1}, \lambda_q^{j_1}\xi\right) \, .$$

Depth 1 sums

$$\Sigma^{[p,q]}_{a_1;-;c;-}(\xi) = \sum_{j=1}^{\infty} \frac{z^j}{j^c} \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1+j-\frac{p}{q}\right)} S_{a_1}(j-1)$$

$$\left(\xi \frac{d}{d\xi}\right)^c \Sigma^{[p,q]}_{a_1;-;c;-}(\xi) = \xi^p \sigma^{[p,q]}_{a_1;-}(\xi) \; ,$$

$$\frac{d}{d\xi}\sigma^{[p,q]}_{a_1;-}(\xi) = -q \frac{\xi^{q-p-1}}{1-\xi^q} \Sigma^{[p,q]}_{-;-;a_1-1;-}(\xi) \; .$$

$$\Sigma^{[p,q]}_{a_1;-;0;-}(\xi) = \xi^p \sigma^{[p,q]}_{a_1;-}(\xi) \; , \quad c = 0$$

- $a_1 = 1.$

$$\sigma_{1;0}(\xi) = \sum_{j_1,j_2=1}^{q} \lambda_q^{j_1 p} \mathrm{Li}_{1,1}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2}\xi\right)$$

- $a_1 \neq 2.$

Mathematical Induction I

- Let us assume that **Proposition A** is valid for multiple inverse rational sums of depth k,

$$\Sigma^{[p,q]}_{a_1,\cdots,a_k;-;0;-}(z) = \xi^p \sum_{\vec{s},1\leq\{j_m\}\leq q} c_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2},\cdots,\lambda_q^{j_{r-1}-j_r},\lambda_q^{j_r}\xi\right) ,$$

- Then for $c \geq 1$, **Corollary A** also holds for multiple inverse rational sums of depth k,

$$\Sigma^{[p,q]}_{a_1,\cdots,a_k;-;c;-}(z) = \sum_{\vec{s},1\leq\{j_m\}\leq q} d_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2},\cdots,\lambda_q^{j_{r-1}-j_r},\lambda_q^{j_r}\xi\right) .$$

Mathematical Induction II

- For the sum of depth $k+1$, the coefficients of the non-homogeneous part may be expressed as linear combinations of sums of depth j $(j = 0, \cdots, k)$

$$\frac{d}{d\xi}\sigma^{[p,q]}_{a_1,\cdots,a_{k+1};-}(\xi) = -q\frac{\xi^{q-p-1}}{1-\xi^q}\sum_{j=1}^{\infty} z^j \frac{\Gamma(1+j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1+j-\frac{p}{q}\right)}$$

$$\times \sum_{p=0}^{k} \sum_{(i_1,\cdots,i_{k+1})} \frac{1}{p!(k+1-p)!} \frac{S_{i_1}(j-1)\cdots S_{i_p}(j-1)}{j^{i_{p+1}+\cdots i_{k+1}}},$$

1. If $i_{p+1}+\cdots i_{k+1} \geq 2$, the r.h.s. of this equation is expressible in terms of multiple polylogarithms of weight k with complex coefficients.
2. If $i_{p+1}+\cdots i_{k+1} = 1$, the r.h.s. of this equation is expressible in terms of multiple polylogarithms of weight k with a common factor ξ^p.

Multiple Inverse Binomial Sums: $q = 2$

M.K. Ward, Yost, 2007

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=-\frac{(1-y)^2}{y}}$$

$$= \frac{1-y}{1+y} \sum_{p,\vec{s}} c_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y)$$

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=-\frac{(1-y)^2}{y}}$$

$$= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y) \; , \quad c \geq 2 \; .$$

where $S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a}$, is harmonic sum

Multiple Binomial Sums: $q = 2$

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$$\sum_{j=1}^{\infty} \binom{2j}{j} u^j S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=\frac{\chi}{(1+\chi)^2}}$$

$$= \sum_{p,\vec{s}} \left[\frac{c_{p,\vec{s}}}{1-\chi} + d_{p,\vec{s}} \right] \ln^p \chi \ \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ ,$$

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=\frac{\chi}{(1+\chi)^2}}$$

$$= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p \chi \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ , \quad c \geq 1$$

where $S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a}$.

Special Values of Argument

It is evident that some (or all, if the basis is complete) of the alternating or non-alternating multiple Euler-Zagier sums (or multiple zeta values) can be written in terms of multiple (inverse) binomial sums of special values of arguments. Two arguments where such a representation is possible are trivially obtained by setting the arguments of the harmonic polylogarithms y, χ to ± 1:

$$u = 4 \,, \quad y = -1 \,,$$
$$u = \frac{1}{4} \,, \quad \chi = 1 \,.$$

Another such point is "golden ratio",

$$u = -1 \,, \quad y = \frac{3-\sqrt{5}}{2}$$

has been discussed intensively in the context of Apéry-like expressions for Riemann zeta functions. For two other points

$$u = 1 \,, \quad y = \exp\left(i\frac{\pi}{3}\right) \,,$$
$$u = 2 \,, \quad y = i \,,$$

Zeta Function and Inverse Binomial Sums: Borwein et al, 2005

$$\zeta(4n+3) = \sum_{j=0}^{n} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4j+3} \binom{2k}{k}} \sum_{\vec{a}} c_{\vec{a}} \sum_{n_1=1}^{k-1} \frac{1}{n_1^{4\vec{a}}} \,,$$

where $\vec{a}$ is multi index, $\vec{a} = a_1, a_2, \ldots, \quad a_1 + a_2 + \ldots = n - j$ and $c_{\vec{a}}$ are rational numbers.

$$\zeta(2n+2) = \sum_{j=0}^{n} \sum_{k=1}^{\infty} \frac{1}{k^{2r} \binom{2k}{k}} \Pi_{i=1}^{N} c_i \sum_{n_i=1}^{k-1} \frac{1}{n_i^{2a_i}} \,,$$

where $r + \sum_{i=1}^{N} a_i = n + 1$ and c_i are rational numbers.

Conclusion

- The generating function approach is quite general, however demand an extra analysis of many different sums.
- The analytical results for more general (arbitrary) sums can be deduced from consideration of Horm-type hypergeometric functions and their ε-expansion via proper differential equations.

Horn-type Hypergeometric Functions: Horn-type series

In accordance with Horn definition, a formal (Laurent) power series in r variables,

$$\Phi(\vec{x}) = \sum C(\vec{m}) \vec{x}^m \equiv \sum_{m_1, m_2, \cdots, m_r} C(m_1, m_2, \cdots, m_r) x_1^{m_1} \cdots x_r^{m_r},$$

is called hypergeometric if for each $i = 1, \cdots, r$ the ratio

$$\frac{C(\vec{m} + e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}.$$

is a rational function in the index of summation: $\vec{m} = (m_1, \cdots, m_r)$, where $\vec{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)$, is unit vector with unity in the j^{th} place.

Horn-type Hypergeometric Functions: Solution

Ore[1930], Sato[1990] found the general form of coefficients

$$C(\vec{m}) = C(\vec{m}) = \Pi_{i=1}^{r} \lambda_i^{m_i} R(\vec{m}) \left(\frac{\Pi_{j=1}^{N} \Gamma(\mu_j(\vec{m}) + \gamma_j)}{\Pi_{k=1}^{M} \Gamma(\nu_k(\vec{m}) + \delta_k)} \right) ,$$

where $N, M \geq 0$, $\lambda_j, \delta_j, \gamma_j \in \mathbb{C}$ are arbitrary complex numbers, $\mu_j, \nu_k : \mathbb{Z}^r \to \mathbb{Z}$ are arbitrary integer-valued linear maps, and R is an arbitrary rational function.

The Horn type hypergeometric function satisfies the following system of equation

$$Q_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} \Phi(\vec{x}) = P_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \Phi(\vec{x}) \, .$$

Horn-type Hypergeometric Functions: Differential Reduction

Let us consider the series

$$\Phi(\vec{\gamma};\vec{\sigma};\vec{x}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\Pi_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\Pi_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \,,$$

The sequences $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively. Two functions with sets of parameters shifted by a unit, $\Phi(\vec{\gamma}+\vec{e_c};\vec{\sigma};\vec{x})$ and $\Phi(\vec{\gamma};\vec{\sigma};\vec{x})$, are related by a linear differential operator:

$$\Phi(\vec{\gamma}+\vec{e_c};\vec{\sigma};\vec{x}) = \left(\sum_{a=1}^{r} \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{x})$$

$$\Phi(\vec{\gamma};\vec{\sigma}-\vec{e_c};\vec{x}) = \left(\sum_{b=1}^{r} \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{x}) \,.$$

Horn-type Hypergeometric Functions: Takayama

The inverse differential operators can be constructed:

$$\Phi(\vec{\gamma} - \vec{e_c}; \vec{\sigma}; \vec{x}) = \sum_a S_a(\vec{x}, \vec{\partial_x}) \Phi(\vec{\gamma}; \vec{\sigma}; \vec{x})$$

$$\Phi(\vec{\gamma}; \vec{\sigma} + \vec{e_c}; \vec{x}) = \sum_b L_b(\vec{x}, \vec{\partial_x}) \Phi(\vec{\gamma}; \vec{\sigma}; \vec{x}) \; .$$

In this way, the Horn-type structure provides an opportunity to reduce hypergeometric functions to a set of basis functions with parameters differing from the original values by integer shifts:

$$P_0(\vec{x}) \Phi(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{x}) = \sum_{m_1, \cdots, m_p = 0} P_{m_1, \cdots, m_r}(\vec{x}) \left(\frac{\partial}{\partial \vec{x}} \right)^{\vec{m}} \Phi(\vec{\gamma}; \vec{\sigma}; \vec{x}) \; ,$$

where $P_0(\vec{x})$ and $P_{m_1, \cdots, m_p}(\vec{x})$ are polynomials with respect to $\vec{\gamma}, \vec{\sigma}$ and $\vec{x}$ and $\vec{k}, \vec{l}$ are lists of integers.

Differential Reduction of Hypergeometric Functions

Consider the ring

$$\mathbb{R} = \mathbb{C}(x_1, \cdots x_n)[\partial/\partial x_1, \cdots, \partial/\partial x_n]$$

of linear partial differential operators and it maximal left-ideal I_λ parametrized by complex number λ. Denote by S_λ the collection of functions annihilated by I_λ.

Theorem (Takayama):

If we have step-up operators,

$$H_\lambda^+ : S_\lambda \rightarrow S_{\lambda+1} \; ,$$

then the step-down operators

$$B_\lambda^- : S_{\lambda+1} \rightarrow S_\lambda \; ,$$

could be constructed by solving the equation

$$B_{\lambda+1}^- H_\lambda^+ \equiv 1 \quad (mod \quad I_\lambda)$$