Large-x results for coefficient and splitting functions

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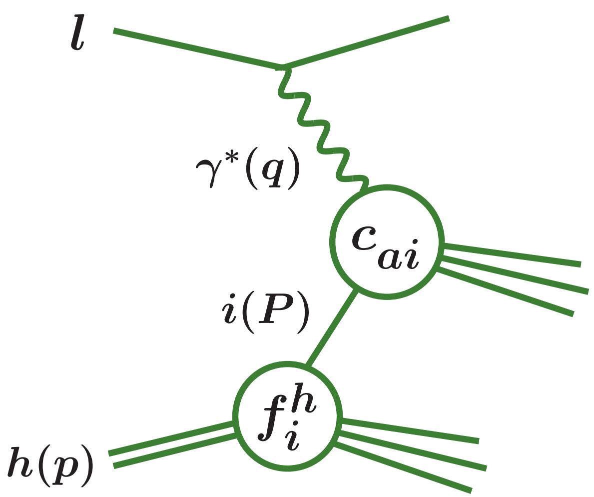
Partly with Sven Moch (DESY), Gary Soar (UoL), Jos Vermaseren (NIKHEF)

- Hard lepton-hadron processes in higher-order perturbative QCD
  large-x / large-N splitting functions P_{ik} and coefficient functions $C_{a,i}$
- Non-singlet physical evolution kernels, $\ln^n(1-x)$ behaviour
  $\Rightarrow$ highest (two or three) logarithms in $C_{a,i}$ to all orders in α_s
- Singlet physical kernels for the systems (F_2, F_ϕ) and (F_2, F_L)
  $\Rightarrow$ leading three powers of $\ln(1-x)$ of P_{ik} and $C_{L,i}$ at fourth order
- D-dimensional structure of leading-logarithmic large-x amplitudes
  $\Rightarrow$ All-order off-diagonal splitting functions and coefficient functions

MV, arXiv: 0902.2342, 0909.2124 (JHEP); SMVV, 0912.0369 (NPB); A.V., to appear; . . .

Hard lepton-hadron processes in pQCD (I)

Inclusive deep-inelastic scattering (DIS), semi-incl. l^+l^- annihilation (SIA)



Left $\longrightarrow$ right: DIS, q spacelike, $Q^2 = -q^2$

$P = \xi p\,,\ f_i^h =$ **parton distributions**

Top $\longrightarrow$ bottom: l^+l^-, q timelike, $Q^2 = q^2$

$p = \xi P\,,$ **fragmentation distributions**

Drell-Yan (DY) l^+l^- production: bottom $\longrightarrow$ top, 2$^{\text{nd}}$ hadron from right ($\{\ldots\}$)

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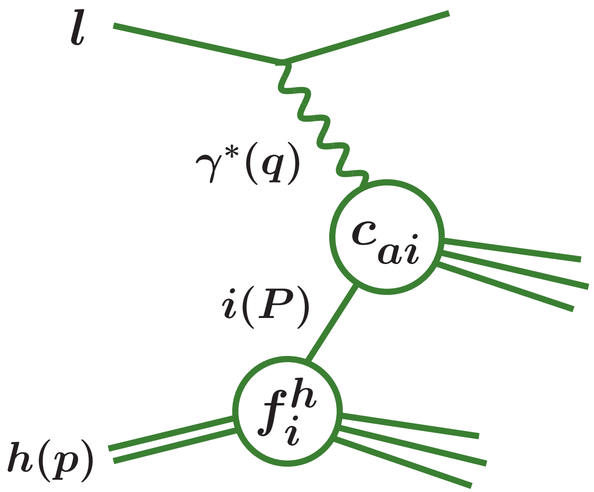

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Structure functions/normalized cross sections F_a: coefficient functions

$$F_a(x, Q^2) \; = \; \Big[C_{a,i\{j\}}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2) \{ \otimes f_j^{h'}(\mu^2) \} \Big](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables: $x = Q^2/(2p\cdot q)$ in DIS etc. μ: renorm./mass-fact. scale

Hard lepton-hadron processes in pQCD (II)

Parton/fragmentation distributions f_i : (renorm. group) evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) = \left[P_{ik}^{(S,T)}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

$\otimes$ = Mellin convolution. Initial conditions incalculable in perturbative QCD

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Expansion in α_s: splitting functions P, coefficient fct's c_a of observables

$$\begin{aligned} P \;&=\; \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \ldots \\ C_a \;&=\; \underbrace{\alpha_s^{n_a} \Big[c_a^{(0)} + \alpha_s c_a^{(1)}}_{} + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \ldots \Big] \end{aligned}$$

NLO: first real prediction of size of cross sections

NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate of pQCD predictions
N^{3}LO: for high precision (α_s from DIS), slow convergence (Higgs in $pp/p\bar{p}$)

The 2010 frontier: α_s^4 for DIS, α_s^3 for SIA (and DY) **Baikov, Chetyrkin; MV**

$\overline{\text{MS}}$ splitting functions at large x / large N

Mellin trf. $f(N) = \int_0^1 dx\, (x^{N-1}\{-1\})\, f(x)_{\{+\}}$: **M-convolutions → products**

$$\frac{\ln^n(1-x)}{(1-x)_+} \stackrel{M}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \ldots\,, \quad \ln^n(1-x) \stackrel{M}{=} \frac{(-1)^n}{N} \ln^n N + \ldots$$

Diagonal splitting functions: no higher-order enhancement at N^0, N^{-1}

$$P^{(l-1)}_{\mathrm{qq/gg}}(N) = A^{(l)}_{\mathrm{q/g}} \ln N + B^{(l)}_{\mathrm{q/g}} + C^{(l)}_{\mathrm{q/g}} \frac{1}{N} \ln N + \ldots\,, \quad A_{\mathrm{g}} = C_A / C_F\, A_{\mathrm{q}}$$

. . . ; Korchemsky (89); Dokshitzer, Marchesini, Salam (05)

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Off-diagonal: double-log behaviour, colour structure with $C_{AF} = C_A - C_F$

$$C_F^{-1} P^{(l)}_{\text{gq}} / n_f^{-1} P^{(l)}_{\text{qg}} = \frac{1}{N} \ln^{2l} N \; \# C^l_{AF}$$
$$+ \frac{1}{N} \ln^{2l-1} N \, (\# C_{AF} + \# C_F + \# n_f) \, C^{l-1}_{AF} + \ldots$$

Double logs $\ln^n N$, $l+1 \le n \le 2l$ **vanish for** $C_F = C_A$ **(→ SUSY case)**

Aim: obtain, at least, these (next-to) leading terms to all orders l **in** α_{s}

$\overline{\text{MS}}$ coefficient functions at large x / large N

'Diagonal' [$\mathcal{O}(1)$] coeff. fct's for $F_{2,3,\phi}$ in DIS, $F_{T,A,\phi}$ in SIA, $F_{\rm DY} = \frac{1}{\sigma_0}\frac{d\sigma}{dQ^2}$

$$C^{(l)}_{2,\mathrm{q}/\phi,\mathrm{g}/\ldots} \;=\; \# \ln^{2l} N + \ldots + N^{-1}(\# \ln^{2l-1} N + \ldots) \,+\, \ldots$$

N^0 parts: threshold exponentiation   Sterman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log (N^{3}LL) accuracy - mod. $A^{(4)}$

$\Rightarrow$ highest seven (DIS), six (SIA, DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05)
(+ more papers, esp. using SCET, from 2006), SIA: Blümlein, Ravindran (06); MV (09)

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'Off-diagonal' [$\mathcal{O}(\alpha_s)$] quantities: leading N^{-1} double logarithms

$$C^{(l)}_{\phi,\mathrm{q}/2,\mathrm{g}/\ldots} \;=\; N^{-1}(\#\ \ln^{2l-1} N + \#\ \ln^{2l-2} N + \ldots) + \ldots$$

Longitudinal DIS/SIA structure functions [recall: l = order in α_s – 1]

$$C^{(l)}_{\mathrm{L,q}} \;=\; N^{-1}(\#\ \ln^{2l} N + \ldots) + \ldots\,, \quad C^{(l)}_{\mathrm{L,g}} \;=\; N^{-2}(\#\ \ln^{2l} N + \ldots) + \ldots$$

Aim: predict highest N^{-1} [N^{-2} for $C_{\mathrm{L,g}}$] double logarithms to all orders

Non-singlet (NS) physical evolution kernels

Eliminate parton densities from scaling violations of observables

$$\frac{dF_a}{d\ln Q^2} = \frac{d\,C_a}{d\ln Q^2}\,q + C_a P\,q = \Big(\beta(a_s)\,\frac{d\,C_a}{da_s} + C_a P\Big) C_a^{-1} F_a$$

$$= \Big(P_a + \beta(a_s)\,\frac{d\ln C_a}{da_s}\Big) F_a = K_a F_a \equiv \sum_{l=0} a_s^{\,l+1} K_{a,l}\, F_a$$

K_a: **physical kernel of the NS quantity** F_a **at** $\mu = Q$. **For** $c_{a,0} = 1$**:**

$$K_a = a_s P_{a,0} + \sum_{l=1} a_s^{\,l+1}\Big(P_{a,l} - \sum_{k=0}^{l-1} \beta_k\, \tilde{c}_{a,l-k}\Big), \qquad a_s \equiv \alpha_s/(4\pi)$$

with

$$\tilde{c}_{a,1} = c_{a,1} \;, \quad \tilde{c}_{a,2} = 2\,c_{a,2} - c_{a,1}^2$$

$$\tilde{c}_{a,3} = 3\,c_{a,3} - 3\,c_{a,2}\,c_{a,1} + c_{a,1}^3$$

$$\tilde{c}_{a,4} = 4\,c_{a,4} - 4\,c_{a,3}\,c_{a,1} - 2\,c_{a,2}^2 + 4\,c_{a,2}\,c_{a,1}^2 - c_{a,1}^4 \;, \quad \ldots$$

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NNLO/N^{3}LO observation: all K_a singly enhanced at all powers of $N^{-1}/1-x$

Conjecture: Single-log behaviour of K_a persists to all higher orders in α_s

$\Leftrightarrow$ exponentiation of the coefficient functions beyond soft-gluon N^0 terms

Higher-order non-singlet predictions

DIS/SIA leading terms, with $p_{\mathrm{qq}}(x) = 2/(1-x)_+ - 1 - x$: $K_{a,0}(x) = 2\, C_F\, p_{\mathrm{qq}}(x)$

$$K_{a,1}(x) \;=\; \ln(1-x)\, p_{\mathrm{qq}}(x) \left[-2\, C_F \beta_0 \mp 8\, C_F^2 \,\ln x \right]$$

$$K_{a,2}(x) \;=\; \ln^2(1-x)\, p_{\mathrm{qq}}(x) \left[\;2\, C_F \beta_0^2 \pm 12\, C_F^2\, \beta_0 \,\ln x + \mathcal{O}(\ln^2 x) \right]$$

$$K_{a,3}(x) \;=\; \ln^3(1-x)\, p_{\mathrm{qq}}(x) \left[-2\, C_F \beta_0^3 \mp 44/3\; C_F^2\, \beta_0^2 \,\ln x + \mathcal{O}(\ln^2 x) \right]$$

First term: leading large n_f, all orders via C_2 of Mankiewicz, Maul, Stein (97)

Proof of N^{-1} conjecture $\Leftrightarrow$ next-to-leading large-n_f calculation to all orders in α_s

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$$K_{a,4}(x):\quad \underbrace{\tilde{c}_{a,4}}_{\rm SL} \;=\; \underbrace{4\, c_{a,4}}_{\text{DL, new}} \;\underbrace{-\,4\, c_{a,3}\, c_{a,1} - 2\, c_{a,2}^{\,2} + 4\, c_{a,2}\, c_{a,1}^{\,2} - c_{a,1}^{\,4}}_{\text{DL, known for DIS/SIA}}$$

$\Rightarrow$ coefficients of highest three double logarithms from fourth order in $\alpha_{\rm s}$, i.e., $\ln^{7,6,5}(1-x)$ at order $\alpha_{\rm s}^4$ for $F_{1,2,3}$ in DIS and $F_{\rm T,I,A}$ in SIA

Leading terms: $K_1 = K_2$, $K_{\rm T} = K_{\rm I}$ [total ('integrated') fragmentation fct.]

$\Rightarrow$ also three logarithms for space- and timelike F_L: $\ln^{6,5,4}(1-x)$ at $\alpha_{\rm s}^4$ etc

Alternative derivation: physical kernels for F_L, agreement non-trivial check

Example: α_s^4 coefficient function for F_1 in DIS

$$
\begin{aligned}
c^{(4)}_{1,\mathrm{ns}}(x) \;=\; & \Big(\ln^7(1-x)\; 8/3\, C_F^4 - \ln^6(1-x)\; 14/3\, C_F^3\, \beta_0 + \ln^5(1-x)\; 8/3\, C_F^2\, \beta_0^2 \Big)\, p_{\mathrm{qq}}(x) \\
& + \ln^6(1-x)\; \Big[C_F^4 \,\{ p_{\mathrm{qq}}(x)\,(-14 - 68/3\, \mathrm{H}_0) + 4 + 8\, \mathrm{H}_0 - (1-x)(6 + 4\, \mathrm{H}_0) \} \Big] \\
& + \ln^5(1-x)\; \Big[C_F^4 \Big\{ p_{\mathrm{qq}}(x)\,(-9 - 8\, \widetilde{\mathrm{H}}_{1,0} + 448/3\, \mathrm{H}_{0,0} + 84\, \mathrm{H}_0 - 64\, \zeta_2) + 48\, \widetilde{\mathrm{H}}_{1,0} \\
& \quad -22 - 96\, \mathrm{H}_{0,0} - 104\, \mathrm{H}_0 - (1-x)(13 + 24\, \widetilde{\mathrm{H}}_{1,0} - 48\, \mathrm{H}_{0,0} - 84\, \mathrm{H}_0 - 16\, \zeta_2) \Big\} \\
& \quad + C_F^3\, \beta_0 \,\{ p_{\mathrm{qq}}(x)\,(41 + 316/9\, \mathrm{H}_0) - 10 - 32/3\, \mathrm{H}_0 + (1-x)(41/3 + 16/3\, \mathrm{H}_0) \} \\
& \quad + C_F^3\, C_A \Big\{ p_{\mathrm{qq}}(x)\,(16 + 8\, \widetilde{\mathrm{H}}_{1,0} + 8\, \mathrm{H}_{0,0} - 24\, \zeta_2) + 4 + (1-x)(28 - 8\, \zeta_2) \Big\} \\
& \quad + C_F^3\, (C_A - 2\, C_F)\, p_{\mathrm{qq}}(-x)\,(16\, \widetilde{\mathrm{H}}_{-1,0} - 8\, \mathrm{H}_{0,0}) \Big] \;+\; \mathcal{O}\big(\ln^4(1-x)\big)
\end{aligned}
$$

First line includes identity of coefficients of leading $\ln^k(1-x)$ and $\dfrac{\ln^k(1-x)}{x-1}$ terms

Conjectured by Krämer, Laenen, Spira (97)

Modified basis $\widetilde{\mathrm{H}}_{m_1,m_2,\ldots} \equiv \widetilde{\mathrm{H}}_{m_1,m_2,\ldots}(x)$ of harmonic polylogarithms, e.g.,

$$\widetilde{\mathrm{H}}_{1,0} = \mathrm{H}_{1,0} + \zeta_2 \;, \quad \widetilde{\mathrm{H}}_{1,1,0} = \mathrm{H}_{1,1,0} - \zeta_2 \ln(1-x) - \zeta_3$$

All $\ln(1-x)$ terms and ζ-functions taken out of expansions to all orders in $1-x$

All-order exponentiation of the $1/N$ terms (I)

For $F_{1,2,3}$, $F_{\mathrm{T,I,A}}$ and F_{DY}, up to terms of order $1/N^2$, with $L \equiv \ln N$

$$C_a(N) - C_a\Big|_{N^0L^k} \;=\; \frac{1}{N}\left(\left[d_{a,1}^{(1)}L + d_{a,0}^{(1)}\right]a_{\mathrm{s}} + \left[\widetilde{d}_{a,1}^{(2)}L + d_{a,0}^{(2)}\right]a_{\mathrm{s}}^2 + \ldots\right)$$
$$\exp\left\{Lh_1(a_{\mathrm{s}}L) + h_2(a_{\mathrm{s}}L) + a_{\mathrm{s}}h_3(a_{\mathrm{s}}L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_{\mathrm{s}}L) \;\equiv\; \sum_{n=1} h_{kn}(a_{\mathrm{s}}L)^n$

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Coefficients for DIS/SIA (upper/lower sign) relative to $N^0 L^k$ resummation

$g_{lk} =$ coefficients in soft-gluon exponentiation

$$\begin{aligned}
h_{1k} &= g_{1k} \\
h_{21} &= g_{21} + \frac{1}{2}\beta_0 \pm 6\, C_F \\
h_{22} &= g_{22} + \frac{5}{24}\beta_0^2 \pm \frac{17}{9}\beta_0\, C_F - 18\, C_F^2 \\
h_{23} &= g_{23} + \frac{1}{8}\beta_0^3 \pm \left(\frac{\xi_{\mathrm{K}_4}}{8} - \frac{53}{18}\right)\beta_0^2 C_F - \frac{34}{3}\beta_0\, C_F^2 \pm 72\, C_F^3
\end{aligned}$$

ξ_{K_4}: next-to-leading large-n_f coefficient at fourth order – should be feasible

First term of h_3 also known, but non-universal within DIS and SIA ($\Leftrightarrow F_L$)

All-order exponentiation of the $1/N$ terms (II)

For space-like (-) and time-like (+) structure/fragmentation functions F_L

$$C_L^{(\pm)}(N) \;=\; N^{-1}(d_1^{(\pm)} a_s + d_2^{(\pm)} a_s^2 + \ldots)\,\exp\{L h_1(a_s L) + h_2(a_s L) + \ldots\}$$

with

$$h_{11} \;=\; 2\,C_F \;\;,\quad h_{12} \;=\; \frac{2}{3}\,\beta_0\,C_F \;\;,\quad h_{13} \;=\; \frac{1}{3}\,\beta_0^2\,C_F$$

$$h_{21} \;=\; \beta_0 \,+\, 4\,\gamma_e C_F \,-\, C_F \,+\, (4-4\,\zeta_2)(C_A - 2C_F)$$

$$h_{22} \;=\; \underbrace{\frac{1}{2}\,(\,\beta_0\,h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} \,-\, \underbrace{8\,(C_A - 2C_F)^2(1 - 3\,\zeta_2 + \zeta_3 + \zeta_2^2\,)}_{\text{Who ordered THIS?}}$$

as g_{22} in soft-gluon exp. Who ordered THIS?

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Remarks/questions

- **Less predictive than resum. of $N^0 L^k$ terms: nothing new, but A_2, in g_{22}**
- **Full NLL accuracy – complete $g_2(a_s L)$ – should be feasible for $F_{1,2,3}$ etc**
- **Full NNLL for $F_{1,2,3}$ etc, NLL for F_L: a log too far? h_{23} for F_L, anyone?**

Singlet physical evolution kernel for (F_2, F_ϕ)

F_ϕ: Higgs-exchange DIS in heavy-top limit, to order α_s^2 also by

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (09)

As in the non-singlet case above, but with 2-vectors$/2\times 2$ matrices P_{ij} and

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,\mathrm{q}} & C_{2,\mathrm{g}} \\ C_{\phi,\mathrm{q}} & C_{\phi,\mathrm{g}} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}$$

Furmanski, Petronzio (81); ...

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Furmanski, Petronzio (81); . . .

$$\frac{dF}{d\ln Q^2} = \frac{dC}{d\ln Q^2}\, q + CP\, q = \Big(\beta(a_s)\, \frac{dC}{da_s} + CP\Big) C^{-1} F$$

$$= \Big(\underbrace{\beta(a_s)\, \frac{d\ln C}{da_s}}_{\text{DL (ns + ps)}} + \underbrace{[C, P]\, C^{-1} + P}_{\text{DL (singlet only)}} \Big) F = KF$$

Singlet physical evolution kernel for (F_2, F_ϕ)

F_ϕ: Higgs-exchange DIS in heavy-top limit, to order α_s^2 also by

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (09)

As in the non-singlet case above, but with 2-vectors$/2\times 2$ matrices P_{ij} and

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,\mathrm{q}} & C_{2,\mathrm{g}} \\ C_{\phi,\mathrm{q}} & C_{\phi,\mathrm{g}} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}$$

Furmanski, Petronzio (81); ...

$$\frac{dF}{d\ln Q^2} = \frac{dC}{d\ln Q^2}\, q + CP\, q = \Big(\beta(a_\mathrm{s})\, \frac{dC}{da_\mathrm{s}} + CP\Big) C^{-1} F$$

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DL (ns + ps) DL (singlet only)

Observation at NLO, NNLO: single-log enhancement to all powers of $1-x$

$$K_{ab}^{(n)} \sim \ln^n(1-x) + \ldots\,, \quad \text{leading } K_{22/\phi\phi}^{(n)} \text{ same as NS}/C_F = 0$$

Conjecture: this behaviour persists to $\mathrm{N^3LO}$

$\Rightarrow$ prediction of $\ln^{6,5,4}(1-x)$ of $P^{(3)}_{\mathrm{qg,gq}}$ [and $\ln^{5,4,3}(1-x)$ of $P^{(3)}_{\mathrm{ps,gg}|C_F}$]

Example: α_s^4 splitting function $P_{qg}^{(3)}(x)$

For brevity: only $(1-x)^0$ part shown – known to all powers, $C_{AF} \equiv C_A - C_F$

$$
\begin{aligned}
P_{qg}^{(3)}(x) \quad = \quad & \ln^6(1-x) \cdot 0 \\
& + \ln^5(1-x) \left[\frac{22}{27} C_{AF}^3 n_f - \frac{14}{27} C_{AF}^2 C_F n_f + \frac{4}{27} C_{AF}^2 n_f^2 \right] \\
& + \ln^4(1-x) \left[\left(\frac{293}{27} - \frac{80}{9} \zeta_2 \right) C_{AF}^3 n_f + \left(\frac{4477}{16} - 8\zeta_2 \right) C_{AF}^2 C_F n_f \right. \\
& \quad \left. - \frac{13}{81} C_{AF} C_F^2 n_f - \frac{116}{81} C_{AF}^2 n_f^2 + \frac{17}{81} C_{AF} C_F n_f^2 - \frac{4}{81} C_{AF} n_f^3 \right] \\
& + \mathcal{O}\left(\ln^3(1-x) \right)
\end{aligned}
$$

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\end{aligned}
$$

- Vanishing of the coefficient of the leading term at order α_s^4 :
  accidental (??) cancellation of contributions, for all four splitting fct’s
- Remaining terms vanish in the supersymmetric case $C_A = C_F (= n_f)$
  Nontrivial check: same as for $P_{qg}^{(2)}$, not obvious from above construction

(published MV, SMVV papers to here)

Singlet physical evolution kernel for (F_2, F_L)

As above, but with $F_\phi \rightarrow \widehat{F}_L = F_L / a_s c_{L,q}^{(0)}$, hence $\widehat{c}_{L,\mathrm{q/g}}^{(n)} \sim \{1/\frac{1}{N}\} \ln^{2n} N$

$$F = \begin{pmatrix} F_2 \\ \widehat{F}_L \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1 & \widehat{c}_{L,\mathrm{g}}^{(0)} \end{pmatrix} + \sum_{n=1} a_s^n \begin{pmatrix} c_{2,\mathrm{q}}^{(n)} & c_{2,\mathrm{g}}^{(n)} \\ \widehat{c}_{L,\mathrm{q}}^{(n)} & \widehat{c}_{L,\mathrm{g}}^{(n)} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\mathrm{L}} \\ K_{\mathrm{L}2} & K_{\mathrm{LL}} \end{pmatrix}$$

Catani (96), Blümlein, Ravindran, van Neerven (00) [different normalization]

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Observation: single-log enhancement of N^0 part of K at NLO and NNLO

N^{3}LO conjecture + above $P^{(3)}_{\mathrm{qg}}$: prediction of three double logs in $c^{(3)}_{L,\mathrm{q/g}}$, e.g.

$$\begin{aligned} N^2 c^{(3)}_{L,\mathrm{g}}(N) \quad = \quad & \ln^6 N \, \frac{32}{3} C_A^3 n_f \\ & + \ln^5 N \left[\frac{1504}{9} C_A^3 n_f - \frac{64}{9} C_A^2 n_f^2 - \frac{104}{3} C_A^2 n_f C_F - \frac{40}{3} n_f C_F^2 \right] \\ & + \ln^4 N \Big[\text{known coefficients} \Big] + \mathcal{O}\left(\ln^3 N\right) \end{aligned}$$

Agrees with/extends results [NS-like $C_F = 0$ part of $C_{L,\mathrm{g}}$ only] of MV (02/09)

Off-diagonal leading logs before factorization

Phys. kernel results: expect iterative structure of unfactorized amplitudes

$$T_{a,j} = \widetilde{C}_{a,i}\, Z_{ij}\ , \quad -\gamma = P = \frac{dZ}{d\ln Q^2}\, Z^{-1}\ , \quad \frac{da_\mathrm{s}}{d\ln Q^2} = -\varepsilon a_\mathrm{s} + \beta_{D=4}$$

$\widetilde{C}_a$ **(terms with** $\varepsilon^k,\ k \geq 0$**):** $D = 4-2\varepsilon$ **dimensional coefficient functions**

$$Z\big|_{a_\mathrm{s}^n} \;=\; \frac{1}{\varepsilon^n}\frac{\gamma_0^n}{n!} + \ldots + \frac{1}{\varepsilon^2}\left(\frac{\gamma_0\gamma_{n-2}}{n(n-1)} + \frac{\gamma_{n-2}\gamma_0}{n} + \ldots\right) + \frac{1}{\varepsilon}\frac{\gamma_{n-1}}{n}$$

$\varepsilon^{-n}\ldots\varepsilon^{-2}$**: lower-order terms,** ε^{-1}**:** n**-loop splitting functions + ...,**
ε^0**:** n**-loop coefficient fct's + ...,** $\varepsilon^k, 0 < k < l$**: required for order** $n+l$

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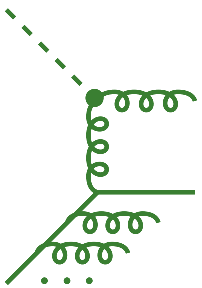
Leading-log (LL) $1/N$ terms of $T_{\phi,q}^{(n)}$ and $T_{2,g}^{(n)}$, with $L \equiv \ln N$:

$$\frac{1}{C_F}T_{\phi,q}^{(n)} = \frac{1}{n_f}T_{2,g}^{(n)} = \frac{L^{n-1}}{N\varepsilon^n}\sum_{k=0}^{\infty}(\varepsilon L)^k \mathcal{L}_{n,k}\left(C_F^{n-1} + C_F^{n-2}C_A + \ldots + C_A^{n-1}\right)$$

to all orders in ε (calc. + D-dim. structure), with same coefficients $\mathcal{L}_{n,k}$

$\Rightarrow$ **all-order relation for one colour structure of either amplitude sufficient**

All-order off-diagonal leading-log amplitudes



$$T^{(n)}_{\phi,\mathrm{q}}\Big|_{C_F\ \mathrm{only}} \overset{\mathrm{LL}}{=} \frac{1}{n}\, T^{(1)}_{\phi,\mathrm{q}} \underbrace{T^{(n-1)}_{2,\mathrm{q}}}_{\frac{1}{(n-1)!}\,(T^{(1)}_{2,\mathrm{q}})^{n-1}} \overset{\mathrm{LL}}{=} \frac{1}{n!}\, T^{(1)}_{\phi,\mathrm{q}}\, (T^{(1)}_{2,\mathrm{q}})^{n-1}$$

Three-loop diagram calculation + $P^{(3)}_{\mathrm{gq}} \overset{\mathrm{LL}}{=} 0$ + general mass factorization:

first four powers in ε known at any order. Rest → higher-order predictions

$$T_{\phi,\mathrm{q}}\Big|_{C_F\ \mathrm{only}} \overset{\mathrm{LL}}{=} T^{(1)}_{\phi,\mathrm{q}}\, \frac{\exp(a_{\mathrm{s}} T^{(1)}_{2,\mathrm{q}}) - 1}{T^{(1)}_{2,\mathrm{q}}}$$

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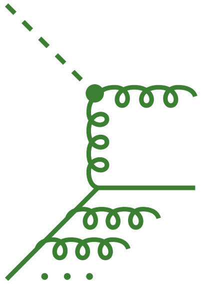

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Exact D-dimensional leading-log expressions for the one-loop amplitudes

$$T^{(1)}_{\phi,\mathrm{q}} \stackrel{\mathrm{LL}}{=} -2C_F\, \frac{1}{\varepsilon}(1-x)^{-\varepsilon} \stackrel{\mathrm{M}}{=} -\frac{2C_F}{N}\, \frac{1}{\varepsilon}\exp(\varepsilon \ln N)$$

$$T^{(1)}_{2,\mathrm{q}} \stackrel{\mathrm{LL}}{=} -4C_F\, \frac{1}{\varepsilon}(1-x)^{-1-\varepsilon} + \text{virtual} \stackrel{\mathrm{M}}{=} 4C_F\, \frac{1}{\varepsilon^2}(\exp(\varepsilon \ln N) - 1)$$

$\Rightarrow$ **leading-log expression for $T_{\phi,\mathrm{q}}$ and $T_{2,\mathrm{g}}$ completely determined**

Leading-log splitting and coefficient functions

Expansions and iterative mass factorization to 'any' order [done in FORM]

$\Rightarrow$ **All-order expressions for LL off-diagonal splitting and coefficient fct's**

$$P_{\mathrm{gq}}^{\mathrm{LL}}(N,\alpha_{\mathrm{s}}) \;=\; \frac{C_F}{N}\,\frac{\alpha_{\mathrm{s}}}{2\pi}\sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,\tilde{a}_{\mathrm{s}}^{\,n}\;, \qquad \tilde{a}_{\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{\pi}\,(C_F - C_A)\ln^2 N$$

Bernoulli numbers B_n: zero for odd $n \geq 3 \;\Rightarrow\; P_{\mathrm{gq}}^{(3)}(N) \stackrel{\mathrm{LL}}{=} 0$ not accidental

$$B_0 = 1\,,\; B_1 = -\frac{1}{2}\,,\; B_2 = \frac{1}{6}\,,\; B_4 = -\frac{1}{30}\,,\; B_6 = \frac{1}{42}\,,\;\ldots,\; B_{12} = -\frac{691}{2730}\,,\;\ldots$$

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$$NC_{\phi,{\rm q}}^{\rm LL}(N) = \sum_{n=1}^{\infty}\left(\frac{\alpha_{\rm s}}{2\pi}\right)^{n}\ln^{2n-1}N\sum_{a=1}^{n}C_F^{a}C_A^{n-a}\sum_{j=0}^{n}\frac{2^{j-1}B_j}{(j!)^2}\,\frac{(-1)^{j+a}}{(n-j)!}\binom{j-1}{a-1}$$

$P_{\rm qg}^{\rm LL}, C_{2,{\rm g}}^{\rm LL}$: same functions but with $C_F \rightarrow n_f$ once, then $C_F \leftrightarrow C_A$ in rest

$$\Rightarrow\quad C_{\phi,{\rm q}}^{\rm LL}(N) \overset{C_A=0}{=} \frac{1}{2N\ln N}\left(\mathcal{B}_0(\tilde{a}_{\rm s}) - e^{\frac{1}{2}\tilde{a}_{\rm s}}\right), \qquad \mathcal{B}_0(x) = \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}\,x^n$$

First properties of the new (?) $\mathcal{B}$-functions

Relation between even-n Bernoulli numbers and the Riemann ζ-function

$$\mathcal{B}_0(x) = 1 - \frac{x}{2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\,\zeta_{2n}\left(\frac{x}{2\pi}\right)^{2n}$$

$\mathcal{B}_0(2\pi i)$ **numerically known (Wolfram MathWorld, Sloane's A093721), no closed form**

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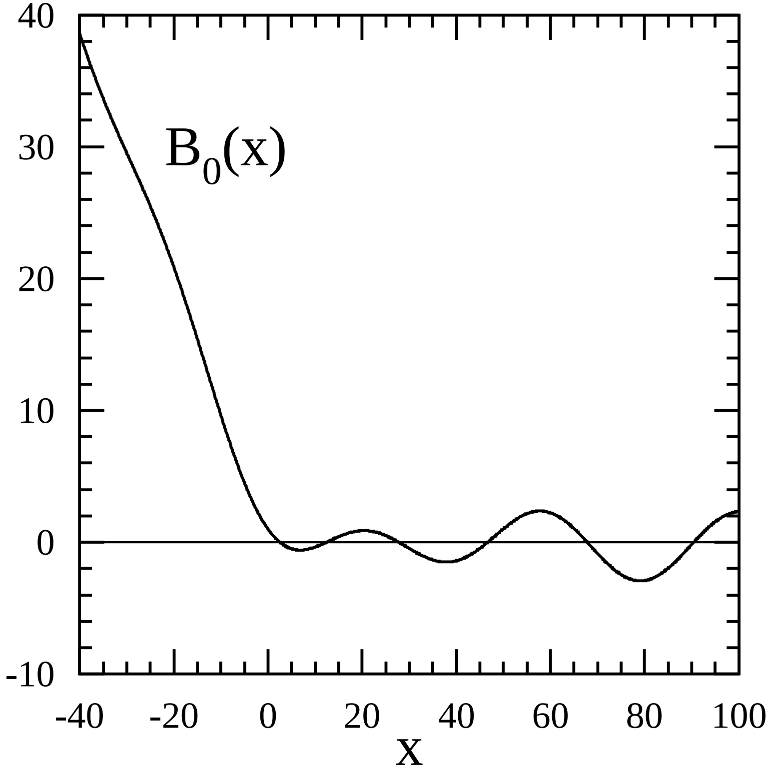

Further $\mathcal{B}$-functions for later use

$$\mathcal{B}_1(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+1)!}\,x^n$$

$$\mathcal{B}_{-1}(x) = \sum_{n=1}^{\infty} \frac{B_n}{n!(n-1)!}\,x^n$$

Relations to $\mathcal{B}_0(x)$

$$\frac{d}{dx}(x\mathcal{B}_1) = \mathcal{B}_0\,, \quad \frac{d}{dx}\mathcal{B}_0 = \frac{1}{x}\mathcal{B}_{-1}$$

The evolution kernel for (F_2, F_ϕ) revisited

Off-diagonal N^{-1} leading-logarithmic physical kernels: $K = CPC^{-1}$ with

$$C^{-1} = \frac{1}{C_{2,\mathrm{q}}\, C_{\phi,\mathrm{g}}} \begin{pmatrix} C_{\phi,\mathrm{g}} & -C_{2,\mathrm{g}} \\ -C_{\phi,q} & C_{2,q} \end{pmatrix}, \quad P^{(n>0)} = \begin{pmatrix} 0 & P^{(n)}_{\mathrm{qg}} \\ P^{(n)}_{\mathrm{gq}} & 0 \end{pmatrix}$$

$$\Rightarrow \quad C_{2,\mathrm{q}}\, K_{qg} \overset{\mathrm{LL}}{=} C_{\phi,\mathrm{g}}\, P_{\mathrm{gq}} + C_{\phi,\mathrm{q}}\, \alpha_{\mathrm{s}} (P^{(0)}_{\mathrm{qq}} - P^{(0)}_{\mathrm{gg}}) + C_{2,q}\, \alpha_{\mathrm{s}} P^{(0)}_{\mathrm{gq}}$$

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Amplitude-based results on p. 16: right-hand sides vanish at all orders $n > 0$

$\Rightarrow$ Singlet physical-kernel conjecture proven for leading double logarithms

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Amplitude-based results on p. 16: right-hand sides vanish at all orders $n > 0$

$\Rightarrow$ **Singlet physical-kernel conjecture proven for leading double logarithms**

$\Rightarrow$ Closed expression for complete LL off-diagonal coefficient functions

$$C_{\phi,\mathrm{q}}^{\mathrm{LL}}(N) = \frac{1}{2N \ln N} \frac{C_F}{C_F - C_A} \left\{ \exp(2C_A a_{\mathrm{s}} \ln^2 N)\, \mathcal{B}_0(\tilde{a}_{\mathrm{s}}) - \exp(2C_F a_{\mathrm{s}} \ln^2 N) \right\}$$

$\exp(\ldots)$: LL soft-gluon exponentials. $C_{2,\mathrm{g}}^{\mathrm{LL}}$ by colour-factor replacement

Summary and outlook

- Non-singlet physical kernels for nine observables in DIS, SIA and DY:
  single-log large-x enhancement at NNLO$/$N^{3}LO to all orders in $1-x$
  All-order conjecture $\Rightarrow$ leading three (DY: two) logs of higher-order C_a
- Exponentiation in N-space, less predictive than threshold resummation
  For progress: subleading large-n_f; top-down studies (E. Laenen's talk)

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- Limited phenomenol. relevance now: assess relevance of NS $1/N$ terms
- Near$/$mid future: combine with other results, esp. fixed-N calculations:
  (close to) feasible now (K. Chetyrkin's talk)
- Far future: use to check all-$N/$all-x fourth-order diagram calculations