Five-Loop Anomalous Dimension of Twist-Two Operators

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in collaboration with Tomasz Łukowski and Vitaly Velizhanin, `arXiv:0912.1624`

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Overview

- The $\mathcal{N} = 4$ SYM and asymptotic integrability
- Twist operators and the five-loop result
- Tests
- Conclusions

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The $\mathcal{N} = 4$ SYM and asymptotic integrability

- The $\mathcal{N} = 4$ SYM is a four-dimensional gauge theory with four different supersymmetry generators.

- Beta function vanishes, superconformal symmetry at the quantum level. The symmetry algebra gets extended $\mathfrak{so}(1,3) \oplus \mathfrak{so}(6) \to \mathfrak{psu}(2,2|4)$.

- No asymptotic distances and thus no asymptotic states. Correlation functions are well defined. Interesting observables are ADs of the composite operators

$$\mathcal{O}(x) = \mathrm{Tr}\left(\underbrace{\Phi\,\Psi\;*\;\ldots}_{L}\right),$$

which receive quantum contributions $\Delta(g) = \Delta_0 + \gamma(g)$.

- The full dimensions are eigenvalues of the dilatation operator

$$D\,\mathcal{O}(x) = \Delta_{\mathcal{O}(x)}(g)\,\mathcal{O}(x)\,.$$

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- Huge mixing problem!
- Even more symmetries appear in the planar limit $(N \to \infty\,, g^2 = \frac{g_{\rm YM}^2 N}{16\pi^2} = const)$ [Minahan, Zarembo, 2002]

$$\mathfrak{psu}(2,2|4) \to \mathfrak{psu}(2,2|4) \ltimes \mathfrak{u}(1)^\infty\,.$$

- More precisely, the dilatation operator is a member of an infinite family of commuting charges as long as $\ell < L$.
[N.Beisert '03], [N.Beisert, V.Dippel, M.Staudacher '04],[B.Zwiebel '05], ...
- The mixing problem for $\ell < L$

dilatation operator of the planar $\mathcal{N} = 4$ SYM $\quad = \quad$ Hamiltonian of an integrable spin chain .

- The corresponding spin chain exhibits many novel features like long-rangeness of the interactions, length fluctuations, ...
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Asymptotic All-Loop Bethe Equations

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - g^2/\ x_{1,k}x^+_{4,j}}{1 - g^2/\ x_{1,k}x^-_{4,j}},$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}},$$

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$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i}\, \sigma^2(x_{4,k}, x_{4,j})\right)$$

$$\times \prod_{j=1}^{K_1} \frac{1 - g^2/\ x^-_{4,k}x_{1,j}}{1 - g^2/\ x^+_{4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k} - x_{3,j}}{x^+_{4,k} - x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k} - x_{5,j}}{x^+_{4,k} - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - g^2/\ x^-_{4,k}x_{7,j}}{1 - g^2/\ x^+_{4,k}x_{7,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x^+_{4,j}}{x_{5,k} - x^-_{4,j}},$$

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- The corresponding eigenvalue of the dilatation operator $(D - D_0)$ is given by

$$\gamma(g) = 2\,g^2\,Q_2 = \frac{i}{r-1}\sum_{j=1}^{K_4}\left(\frac{1}{(x^+(u_j))} - \frac{1}{(x^-(u_j))}\right) .$$

- These equations yield the AD of *any* local trace operator up to order $\mathcal{O}(g^{2L})$.
- Recently, adapting the techniques of the Thermodynamic Bethe Ansatz a *complete all-loop set of spectral equations* for the planar $\mathcal{N} = 4$ has been formulated. [G. Arutyunov, S. Frolov, 2007], [G. Arutyunov, S. Frolov, 2009] [D.Bombardielli, D. Fioravanti, R. Tateo '09; N.Gromov, V.Kazakov, A.Kozak, P.Vieira '09; G.Arutyunov, S.Frolov '09]
- It is a an infinite set of coupled non-linear integral equations.
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... but they are still a conjecture

Twist operators and the five-loop result

- A suitable testing ground at weak coupling provide twist operators.
- The twist-two operators (in the $\mathfrak{sl}(2)$ twist equals the length) are the shortest operators in the theory

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{D}^M \mathcal{Z}^2\right) + \ldots .$$

- Interestingly enough closed expressions (as function of M) of the AD can be found to first few orders. At one-loop

$$\gamma(g) = 8\, g^2\, S_1(M)\,.$$

- The harmonic sum S_1 is the simplest of the so called nested harmonic sums

$$S_a(M) = \sum_{i=1}^{M} \frac{(\mathrm{sgn}(a))^i}{i^{|a|}}\,, S_{a_1,\ldots,a_n}(M) = \sum_{i=1}^{M} \frac{(\mathrm{sgn}(a_1))^i}{i^{|a_1|}} S_{a_2,\ldots,a_n}(i)\,.$$

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[Kotikov, Lipatov, Velizhanin, 2003&2004]

$$\frac{\gamma_2(M)}{4} = \mathbf{S}_1(M)$$

$$\frac{\gamma_4(M)}{4} = \mathbf{S_3} + \mathbf{S_{-3}} - 2\left(S_{1,2} + S_{2,1} + S_{1,-2}\right)$$

$$\begin{aligned}\frac{\gamma_6(M)}{8} \quad = \quad & \mathbf{2\,S_5} + \mathbf{2\,S_{-5}} - S_{-3,2} + 2\left(S_{-2,-2,1} + S_{-2,1,-2} + S_{1,-2,-2} + S_{1,-2,2}\right. \\ & \left. - S_{-3,-2} - S_{-2,-3} - S_{3,-2}\right) + 4\left(S_{1,2,2} + S_{2,1,2} + S_{2,2,1} + S_{3,1,1} + S_{1,3,1}\right. \\ & \left. + S_{1,1,3} + S_{1,2,-2} + S_{2,1,-2} - S_{1,4} - S_{4,1} - S_{-4,1}\right) - 5\left(S_{2,3} + S_{3,2}\right) \\ & + 6\,S_{2,-2,1} - 8\left(S_{1,-4} + S_{1,1,-2,1} - S_{1,-3,1}\right) - 9\,S_{2,-3} + 12\,S_{1,1,-3}\end{aligned}$$

Up to this order one can calculate the AD using the asymptotic Bethe equations and there is no need to refer to the full spectral equations...

... at higher orders ABE still "work", but a mismatch with field theory computations is expected.

Four-loop [A. Kotikov, L. Lipatov, A. R., M. Staudacher, V. Velizhanin, 2007]

$$
\begin{aligned}
&\mathbf{4\,S_{-7} + 6\,S_{7}} + 2\,(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) + 3\,(-S_{-2,5} \\
&+ S_{-2,3,-2}) + 4\,(S_{-2,1,4} - S_{-2,-2,-2,1} - S_{-2,1,2,-2} - S_{-2,2,1,-2} - S_{1,-2,1,3} \\
&- S_{1,-2,2,2} - S_{1,-2,3,1}) + 5\,(-S_{-3,4} + S_{-2,-2,-3}) + 6\,(-S_{5,-2} \\
&+ S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2}) + 7\,(-S_{-2,-5} + S_{-3,-2,-2} \\
&+ S_{-2,-3,-2} + S_{-2,-2,3}) + 8\,(S_{-4,1,2} + S_{-4,2,1} - S_{-5,-2} - S_{-4,3} \\
&- S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) + 9\,S_{3,-2,-2} - 10\,S_{1,-2,2,-2} + 11\,S_{-3,2,-2} \\
&+ 12\,(-S_{-6,1} + S_{-2,2,-3} + S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1} \\
&- S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} - S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2} \\
&- S_{1,3,1,2} - S_{1,3,2,1} - S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2} \\
&- S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} - S_{3,1,1,2} - S_{3,1,2,1} \\
&- S_{3,2,1,1}) + 13\,S_{2,-2,3} - 14\,S_{2,-2,1,-2} + 15\,(S_{2,3,-2} + S_{3,2,-2}) \\
&+ 16\,(S_{-4,1,-2} + S_{-2,1,-4} - S_{-2,-2,1,2} - S_{-2,-2,2,1} - S_{-2,1,-2,2} - S_{-2,1,1,-3} \\
&- S_{1,-3,1,2} - S_{1,-3,2,1} - S_{1,-2,-2,2} - S_{2,-2,-2,1} + S_{-2,1,1,-2,1} + S_{1,1,-2,1,-2} \\
&+ S_{1,1,-2,1,2} + S_{1,1,-2,2,1}) - 17\,S_{-5,2} + 18\,(-S_{4,-3} - S_{6,1} + S_{1,-3,3}) \\
&+ 20\,(-S_{1,-6} - S_{1,6} - S_{4,3} + S_{-5,1,1} + S_{-4,-2,1} + S_{-3,-2,2} + S_{-2,-4,1} \\
&+ S_{-2,-3,2} + S_{1,3,3} + S_{3,1,3} + S_{3,3,1} - S_{1,1,-2,3} - S_{1,2,-2,-2} - S_{2,1,-2,-2}) \\
&- 21\,S_{3,4} + 22\,(S_{1,-2,-4} + S_{2,2,3} + S_{2,3,2} + S_{3,-2,2} + S_{3,2,2}) + 23\,(-S_{-3,-4} \\
&- S_{5,2} + S_{2,-2,-3}) + 24\,(-S_{-4,-3} + S_{1,-4,-2} - S_{1,-3,1,-2} - S_{1,1,1,4} - S_{1,1,4,1} \\
&- S_{1,3,-2,1} - S_{1,4,1,1} - S_{3,-2,1,1} - S_{3,1,-2,1} - S_{4,1,1,1} + S_{-2,-2,1,1,1} + S_{-2,1,-2,1,1} \\
&+ S_{1,-2,-2,1,1} + S_{1,-2,1,-2,1} + S_{1,1,-2,-2,1} + S_{1,1,1,-2,-2} + S_{1,1,2,-2,1} + S_{1,2,1,-2,1} \\
&+ S_{2,1,1,-2,1}) + 25\,S_{2,-3,-2} + 26\,(-S_{2,5} + S_{1,4,2} + S_{2,4,1} + S_{4,1,2} + S_{4,2,1}) \\
&+ 28\,(S_{1,2,4} + S_{2,1,4} - S_{-3,1,-2,1} - S_{-2,1,-3,1} - S_{1,-2,1,-3}) + 30\,S_{-3,1,-3} \\
&+ 32\,(S_{1,5,1} + S_{5,1,1} - S_{-3,-2,1,1} - S_{-2,-3,1,1} - S_{1,-3,-2,1} - S_{1,-2,-3,1} \\
&- S_{2,2,-2,1} + S_{1,2,-2,1,1} + S_{2,1,-2,1,1} - S_{1,1,1,-2,1,1}) + 36\,(S_{1,1,5} + S_{1,3,-3} \\
&+ S_{3,1,-3} - S_{1,1,-3,-2} - S_{1,1,-2,-3} - S_{1,1,2,-3} - S_{1,2,-2,2} - S_{1,2,1,-3} - S_{2,1,-2,2} \\
&- S_{2,1,1,-3}) + 38\,S_{-3,-3,1} + 40\,(-S_{1,-4,1,1} - S_{2,-3,1,1} + S_{1,1,1,-2,2}) \\
&- 41\,S_{3,-4} + 42\,(-S_{2,-5} + S_{1,-4,2} + S_{1,-3,-3}) + 44\,(S_{1,-5,1} + S_{2,-3,2} + S_{3,-3,1}) \\
&+ 46\,S_{2,2,-3} + 48\,S_{1,1,-3,1,1} + 60\,(S_{1,1,-5} - S_{1,1,-3,2}) + 62\,S_{2,-4,1} + 64\,S_{1,1,1,-3,1} \\
&+ 68\,(S_{1,2,-4} + S_{2,1,-4} - S_{1,2,-3,1} - S_{2,1,-3,1}) - 72\,S_{1,1,1,-4} - 80\,S_{1,1,-4,1} \\
&- \boldsymbol{\zeta(3) S_1 (S_3 - S_{-3} + 2\,S_{-2,1})}.
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&\Big(20480S_{-5}-8192S_{-3}S_{-2}+2048S_5-20480S_{-4,1}-16384S_{-3,2}-\frac{28672}{3}S_{-2,3}\\
&+\frac{32768}{3}S_{-3,1,1}+\frac{16384}{3}S_{-2,1,2}+\frac{16384}{3}S_{-2,2,1}\Big)S_1^4+\Big(20480S_{-3}^2+4096S_3^2+81920S_{-6}\\
&+S_{-2}\big(30720S_{-4}+8192S_4\big)+30720S_6-98304S_{-5,1}-12288S_{-4,-2}-102400S_{-4,2}\\
&-8192S_{-3,-3}-90112S_{-3,3}+S_3\big(24576S_{-3}-16384S_{-2,1}\big)-57344S_{-2,4}+4096S_{4,2}\\
&+16384S_{5,1}+122880S_{-4,1,1}-16384S_{-3,-2,1}+106496S_{-3,1,2}+106496S_{-3,2,1}\\
&-16384S_{-2,-3,1}-8192S_{-2,-2,2}+S_2\Big(-8192S_{-2}^2+49152S_{-4}+8192S_4-\frac{131072}{3}S_{-3,1}\\
&-\frac{81920}{3}S_{-2,2}+\frac{65536}{3}S_{-2,1,1}\Big)+65536S_{-2,1,3}+65536S_{-2,2,2}+65536S_{-2,3,1}\\
&-98304S_{-3,1,1,1}-49152S_{-2,1,1,2}-49152S_{-2,1,2,1}-49152S_{-2,2,1,1}\Big)S_1^3+\Big(\big(12288S_{-3}\\
&+9216S_3\big)S_{-2}^2+\big(53248S_{-5}+24576S_5-61440S_{-4,1}-40960S_{-3,2}-20480S_{-2,3}\\
&+32768S_{-3,1,1}+16384S_{-2,1,2}+16384S_{-2,2,1}\big)S_{-2}+113664S_{-7}+3072S_7-163840S_{-6,1}\\
&-172032S_{-5,2}-174080S_{-4,3}-163840S_{-3,4}+S_2^2\big(36864S_{-3}+12288S_3-24576S_{-2,1}\big)\\
&+\big(-12288S_{-4}-36864S_4\big)S_{-2,1}-118784S_{-2,5}+8192S_{4,3}+8192S_{5,2}-40960S_{6,1}\\
&+253952S_{-5,1,1}+24576S_{-4,-2,1}+24576S_{-4,1,-2}+266240S_{-4,1,2}+266240S_{-4,2,1}\\
&+16384S_{-3,-3,1}-8192S_{-3,-2,2}+16384S_{-3,1,-3}+249856S_{-3,1,3}+8192S_{-3,2,-2}\\
&+258048S_{-3,2,2}+249856S_{-3,3,1}-16384S_{-2,-3,2}-16384S_{-2,-2,3}+S_{-3}\big(14336S_{-4}\\
&+43008S_4-49152S_{-3,1}-24576S_{-2,2}+32768S_{-2,1,1}\big)+S_3\big(52224S_{-4}+12288S_4\\
&-57344S_{-3,1}-40960S_{-2,2}+49152S_{-2,1,1}\big)+172032S_{-2,1,4}+180224S_{-2,2,3}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+180224S_{-2,3,2}+172032S_{-2,4,1}-8192S_{4,1,2}-8192S_{4,2,1}-32768S_{5,1,1}\\
&-368640S_{-4,1,1,1}+32768S_{-3,-2,1,1}-344064S_{-3,1,1,2}-344064S_{-3,1,2,1}-344064S_{-3,2,1,1}\\
&+32768S_{-2,-3,1,1}+16384S_{-2,-2,1,2}+16384S_{-2,-2,2,1}+S_2(92160S_{-5}+S_{-2}(49152S_{-3}\\
&+24576S_3)+30720S_5-122880S_{-4,1}-12288S_{-3,-2}-122880S_{-3,2}-86016S_{-2,3}\\
&+12288S_{4,1}+172032S_{-3,1,1}-24576S_{-2,-2,1}+122880S_{-2,1,2}+122880S_{-2,2,1}\\
&-147456S_{-2,1,1,1})-221184S_{-2,1,1,3}-221184S_{-2,1,2,2}-221184S_{-2,1,3,1}\\
&-221184S_{-2,2,1,2}-221184S_{-2,2,2,1}-221184S_{-2,3,1,1}+393216S_{-3,1,1,1,1}\\
&+196608S_{-2,1,1,1,2}+196608S_{-2,1,1,2,1}+196608S_{-2,1,2,1,1}+196608S_{-2,2,1,1,1})S_1^2\\
&+(2048S_2^4+8192S_{-2}S_2^3+(9216S_{-2}^2+24576S_{-4}+9216S_4-36864S_{-3,1}-30720S_{-2,2}\\
&+49152S_{-2,1,1})S_2^2+(4096S_{-2}^3+(32768S_{-4}+24576S_4-49152S_{-3,1}-24576S_{-2,2}\\
&+32768S_{-2,1,1})S_{-2}+6144S_3^2+53248S_{-6}+6144S_6-90112S_{-5,1}-94208S_{-4,2}\\
&-94208S_{-3,3}+S_3(32768S_{-3}-32768S_{-2,1})-16384S_{-3}S_{-2,1}-77824S_{-2,4}+8192S_{4,2}\\
&-16384S_{5,1}+163840S_{-4,1,1}+16384S_{-3,-2,1}+16384S_{-3,1,-2}+172032S_{-3,1,2}\\
&+172032S_{-3,2,1}-16384S_{-2,-2,2}+139264S_{-2,1,3}+147456S_{-2,2,2}+139264S_{-2,3,1}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&-16384S_{4,1,1} - 294912S_{-3,1,1,1} + 32768S_{-2,-2,1,1} - 245760S_{-2,1,1,2} - 245760S_{-2,1,2,1} \\
&-245760S_{-2,2,1,1} + 393216S_{-2,1,1,1,1}\big)S_2 + 13824S_{-4}^2 + 4608S_4^2 + 16384S_{-3,1}^2 \\
&+14336S_{-2,2}^2 + 57344S_{-8} + S_{-2}^2\big(3072S_{-4} + 12288S_4\big) + 64512S_8 - 98304S_{-7,1} \\
&-30720S_{-6,-2} - 98304S_{-6,2} - 16384S_{-5,-3} - 102400S_{-5,3} - 3072S_{-4,-4} - 98304S_{-4,4} \\
&-98304S_{-3,5} - 92160S_{-2,6} - 15360S_{4,4} - 12288S_{5,3} + 26624S_{6,2} + 36864S_{7,1} \\
&+163840S_{-6,1,1} - 24576S_{-5,-2,1} + 180224S_{-5,1,2} + 180224S_{-5,2,1} - 24576S_{-4,-3,1} \\
&-6144S_{-4,-2,-2} - 18432S_{-4,-2,2} + 184320S_{-4,1,3} + 196608S_{-4,2,2} + 184320S_{-4,3,1} \\
&-8192S_{-3,-4,1} - 4096S_{-3,-3,-2} - 28672S_{-3,-3,2} - 4096S_{-3,-2,-3} + 12288S_{-3,-2,3} \\
&+180224S_{-3,1,4} + 192512S_{-3,2,3} + 192512S_{-3,3,2} + 176128S_{-3,4,1} + 8192S_{-2,-5,1} \\
&-22528S_{-2,-4,2} + 4096S_{-2,-3,3} + 30720S_{-2,-2,4} + S_{-3,1}\big(36864S_{-2,2} - 16384S_{-2,1,1}\big) \\
&-8192S_{-2,2}S_{-2,1,1} + S_{-4}\big(-14336S_{-3,1} - 10240S_{-2,2} + 36864S_{-2,1,1}\big) \\
&+S_4\big(30720S_{-4} - 51200S_{-3,1} - 43008S_{-2,2} + 69632S_{-2,1,1}\big) + 139264S_{-2,1,5} \\
&+S_{-2,1}\big(-4096S_{-5} - 20480S_5 + 24576S_{-4,1} + 36864S_{-3,2} + 28672S_{-2,3} - 16384S_{-3,1,1} \\
&-8192S_{-2,1,2} - 8192S_{-2,2,1}\big) + 145408S_{-2,2,4} + 147456S_{-2,3,3} + 143360S_{-2,4,2} \\
&+131072S_{-2,5,1} - 8192S_{4,1,3} - 8192S_{4,2,2} - 8192S_{4,3,1} - 16384S_{5,1,2} - 16384S_{5,2,1} \\
&-294912S_{-5,1,1,1} - 319488S_{-4,1,1,2} - 319488S_{-4,1,2,1} - 319488S_{-4,2,1,1} + 49152S_{-3,-3,1,1} \\
&+8192S_{-3,-2,-2,1} + 16384S_{-3,-2,1,2} + 16384S_{-3,-2,2,1} - 16384S_{-3,1,1,-3} - 311296S_{-3,1,1,3} \\
&-327680S_{-3,1,2,2} - 311296S_{-3,1,3,1} - 16384S_{-3,2,-2,1} - 327680S_{-3,2,1,2} - 327680S_{-3,2,2,1} \\
&-311296S_{-3,3,1,1} + 73728S_{-2,-4,1,1} + 8192S_{-2,-3,-2,1} + 40960S_{-2,-3,1,2} + 40960S_{-2,-3,2,1}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+8192S_{-2,-2,-3,1}+4096S_{-2,-2,-2,2}+16384S_{-2,-2,1,3}+16384S_{-2,-2,2,2}+16384S_{-2,-2,3,1}\\
&-24576S_{-2,1,1,-4}+S_{-3}\big(40960S_{-5}+16384S_{5}-28672S_{-4,1}-22528S_{-3,2}-22528S_{-2,3}\\
&+4096S_{4,1}+49152S_{-3,1,1}-8192S_{-2,-2,1}+36864S_{-2,1,2}+36864S_{-2,2,1}\\
&-49152S_{-2,1,1,1}\big)+S_{3}\big(40960S_{-5}+8192S_{5}-53248S_{-4,1}-51200S_{-3,2}-\frac{112640S_{-2,3}}{3}\\
&+\frac{212992}{3}S_{-3,1,1}+\frac{143360}{3}S_{-2,1,2}+\frac{143360}{3}S_{-2,2,1}-49152S_{-2,1,1,1}\big)-221184S_{-2,1,1,4}\\
&-8192S_{-2,1,2,-3}-237568S_{-2,1,2,3}-237568S_{-2,1,3,2}-221184S_{-2,1,4,1}-16384S_{-2,2,-3,1}\\
&-8192S_{-2,2,-2,2}-8192S_{-2,2,1,-3}+S_{-2}\big(4096S_{-3}^{2}+8192S_{3}^{2}+56320S_{-6}+25600S_{6}\\
&-32768S_{-5,1}-26624S_{-4,2}-28672S_{-3,3}+S_{3}\big(20480S_{-3}-8192S_{-2,1}\big)-24576S_{-2,4}\\
&+2048S_{4,2}+8192S_{5,1}+36864S_{-4,1,1}-8192S_{-3,-2,1}+36864S_{-3,1,2}+36864S_{-3,2,1}\\
&-8192S_{-2,-3,1}-4096S_{-2,-2,2}+24576S_{-2,1,3}+24576S_{-2,2,2}+24576S_{-2,3,1}\\
&-49152S_{-3,1,1,1}-24576S_{-2,1,1,2}-24576S_{-2,1,2,1}-24576S_{-2,2,1,1}\big)-237568S_{-2,2,1,3}\\
&-245760S_{-2,2,2,2}-237568S_{-2,2,3,1}-16384S_{-2,3,-2,1}-237568S_{-2,3,1,2}-237568S_{-2,3,2,1}\\
&-221184S_{-2,4,1,1}+24576S_{4,1,1,2}+24576S_{4,1,2,1}+24576S_{4,2,1,1}+98304S_{5,1,1,1}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+491520S_{-4,1,1,1,1}-98304S_{-3,-2,1,1,1}-32768S_{-3,1,-2,1,1}+491520S_{-3,1,1,1,2}\\
&+491520S_{-3,1,1,2,1}+491520S_{-3,1,2,1,1}+491520S_{-3,2,1,1,1}-98304S_{-2,-3,1,1,1}\\
&-49152S_{-2,-2,1,1,2}-49152S_{-2,-2,1,2,1}-49152S_{-2,-2,2,1,1}-32768S_{-2,1,-3,1,1}\\
&-16384S_{-2,1,-2,1,2}-16384S_{-2,1,-2,2,1}+327680S_{-2,1,1,1,3}+327680S_{-2,1,1,2,2}\\
&+327680S_{-2,1,1,3,1}+327680S_{-2,1,2,1,2}+327680S_{-2,1,2,2,1}+327680S_{-2,1,3,1,1}\\
&-16384S_{-2,2,-2,1,1}+327680S_{-2,2,1,1,2}+327680S_{-2,2,1,2,1}+327680S_{-2,2,2,1,1}\\
&+327680S_{-2,3,1,1,1}-655360S_{-3,1,1,1,1,1}-327680S_{-2,1,1,1,1,2}-327680S_{-2,1,1,1,2,1}\\
&-327680S_{-2,1,1,2,1,1}-327680S_{-2,1,2,1,1,1}-327680S_{-2,2,1,1,1,1}\Big)S_1+512S_3^3-7168S_{-9}\\
&+7168S_9-18432S_{-8,1}-2048S_{-2,-7}+S_3^2\big(3072S_{-3}-2048S_{-2,1}\big)+S_2^3\big(1024S_{-3}\\
&+1024S_3-2048S_{-2,1}\big)+S_{-2}\big(3072S_{-3}S_4-6144S_{-2,1}S_4+S_3\big(3072S_{-4}+6144S_4\\
&-4096S_{-3,1}-2048S_{-2,2}\big)\big)-8192S_{1,-8}+8192S_{1,8}-16384S_{2,-7}+16384S_{2,7}\\
&-3072S_{3,-6}+3072S_{3,6}-13824S_{4,-5}+4608S_{4,5}-34816S_{5,-4}-2048S_{5,4}-35328S_{6,-3}\\
&-4608S_{6,3}+10240S_{7,-2}+9216S_{7,2}+16384S_{8,1}+26624S_{-7,1,1}-27648S_{-6,-2,1}\\
&-6144S_{-6,1,-2}+12288S_{-6,1,2}+12288S_{-6,2,1}-18432S_{-5,-3,1}-2048S_{-5,-2,-2}\\
&-4096S_{-5,-2,2}-18432S_{-5,1,-3}-4096S_{-5,2,-2}+26624S_{-4,-4,1}+44032S_{-4,-3,-2}\\
&+51200S_{-4,-3,2}+70656S_{-4,-2,-3}+12288S_{-4,-2,3}+13312S_{-4,1,-4}+17408S_{-4,1,4}\\
&+7168S_{-4,2,-3}-1024S_{-4,3,-2}+44032S_{-4,4,1}-10240S_{-3,-5,1}+45056S_{-3,-4,-2}\\
&+51200S_{-3,-4,2}+157696S_{-3,-3,-3}+33792S_{-3,-3,3}+73728S_{-3,-2,-4}+8192S_{-3,-2,4}
\end{aligned}
$$

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&-8192S_{-3,1,-5}+61440S_{-3,1,5}+14336S_{-3,2,-4}+20480S_{-3,2,4}-3072S_{-3,3,-3}\\
&+10240S_{-3,4,-2}+45056S_{-3,4,2}+90112S_{-3,5,1}-13312S_{-2,-6,1}+1024S_{-2,-5,-2}\\
&-4096S_{-2,-5,2}+68608S_{-2,-4,-3}+12288S_{-2,-4,3}+70656S_{-2,-3,-4}+8192S_{-2,-3,4}\\
&+15360S_{-2,-2,-5}+7168S_{-2,-2,5}-7168S_{-2,1,-6}+21504S_{-2,1,6}-10240S_{-7,-2}\\
&-13312S_{-7,2}+16896S_{-6,-3}-5632S_{-6,3}+5120S_{-5,-4}+1024S_{-5,4}+3584S_{-4,-5}\\
&-27136S_{-4,5}+9216S_{-3,-6}-23552S_{-3,6}-4096S_{-2,2,-5}+28672S_{-2,2,5}\\
&+1024S_{-2,3,4}+8192S_{-2,4,-3}+11264S_{-2,4,3}+13312S_{-2,5,-2}+40960S_{-2,5,2}\\
&+35840S_{-2,6,1}+40960S_{1,-7,1}-11264S_{1,-6,-2}+8192S_{1,-6,2}-32768S_{1,-5,-3}\\
&+4096S_{1,-5,3}+18432S_{1,-4,-4}+23552S_{1,-4,4}-10240S_{1,-3,-5}+71680S_{1,-3,5}\\
&-11264S_{1,-2,-6}+25600S_{1,-2,6}+32768S_{1,1,-7}-32768S_{1,1,7}+8192S_{1,2,-6}-8192S_{1,2,6}\\
&+4096S_{1,3,-5}+35840S_{1,4,-4}-6144S_{1,4,4}+83968S_{1,5,-3}+18432S_{1,5,3}+17408S_{1,6,-2}\\
&+22528S_{1,6,2}-32768S_{1,7,1}+14336S_{2,-6,1}-20480S_{2,-5,-2}-8192S_{2,-5,2}\\
&+22528S_{2,-4,-3}+1024S_{2,-4,3}+32768S_{2,-3,-4}+30720S_{2,-3,4}-6144S_{2,-2,-5}\\
&+38912S_{2,-2,5}+8192S_{2,1,-6}-8192S_{2,1,6}-4096S_{2,2,-5}+16384S_{2,2,5}-1024S_{2,3,-4}\\
&-5120S_{2,3,4}+43008S_{2,4,-3}+9216S_{2,4,3}+32768S_{2,5,-2}+40960S_{2,5,2}+6144S_{2,6,1}\\
&+2048S_{3,-5,1}-3072S_{3,-4,-2}-3072S_{3,-4,2}+12288S_{3,-3,-3}+1024S_{3,-3,3}+5120S_{3,-2,-4}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+7168S_{3,-2,4}+4096S_{3,1,-5}-1024S_{3,2,-4}-5120S_{3,2,4}+3072S_{3,3,-3}+9216S_{3,4,-2}\\
&+9216S_{3,4,2}+8192S_{3,5,1}+39936S_{4,-4,1}-6144S_{4,-3,-2}+31744S_{4,-3,2}-6144S_{4,-2,-3}\\
&+15360S_{4,-2,3}+32768S_{4,1,-4}-6144S_{4,1,4}+36864S_{4,2,-3}+9216S_{4,2,3}+8192S_{4,3,-2}\\
&+9216S_{4,3,2}-6144S_{4,4,1}+86016S_{5,-3,1}+8192S_{5,-2,-2}+36864S_{5,-2,2}+81920S_{5,1,-3}\\
&+18432S_{5,1,3}+32768S_{5,2,-2}+40960S_{5,2,2}+18432S_{5,3,1}+50176S_{6,-2,1}+20480S_{6,1,-2}\\
&+22528S_{6,1,2}+22528S_{6,2,1}-18432S_{7,1,1}-24576S_{-6,1,1,1}+8192S_{-5,-2,1,1}\\
&+28672S_{-5,1,-2,1}+8192S_{-5,1,1,-2}-102400S_{-4,-3,1,1}-88064S_{-4,-2,-2,1}\\
&-53248S_{-4,-2,1,-2}-59392S_{-4,-2,1,2}-59392S_{-4,-2,2,1}-55296S_{-4,1,-3,1}\\
&-34816S_{-4,1,-2,-2}-43008S_{-4,1,-2,2}-14336S_{-4,1,1,-3}-2048S_{-4,1,2,-2}-12288S_{-4,2,-2,1}\\
&-2048S_{-4,2,1,-2}-102400S_{-3,-4,1,1}-188416S_{-3,-3,-2,1}-126976S_{-3,-3,1,-2}\\
&-155648S_{-3,-3,1,2}-155648S_{-3,-3,2,1}-180224S_{-3,-2,-3,1}-24576S_{-3,-2,-2,-2}\\
&-90112S_{-3,-2,-2,2}-155648S_{-3,-2,1,-3}-36864S_{-3,-2,1,3}-65536S_{-3,-2,2,-2}\\
&-81920S_{-3,-2,2,2}-36864S_{-3,-2,3,1}-61440S_{-3,1,-4,1}-102400S_{-3,1,-3,-2}\\
&-122880S_{-3,1,-3,2}-159744S_{-3,1,-2,-3}-30720S_{-3,1,-2,3}-28672S_{-3,1,1,-4}\\
&-40960S_{-3,1,1,4}-12288S_{-3,1,2,-3}+2048S_{-3,1,3,-2}-98304S_{-3,1,4,1}-61440S_{-3,2,-3,1}\\
&-40960S_{-3,2,-2,-2}-49152S_{-3,2,-2,2}-12288S_{-3,2,1,-3}+4096S_{-3,3,-2,1}+2048S_{-3,3,1,-2}\\
&-90112S_{-3,4,1,1}+8192S_{-2,-5,1,1}-83968S_{-2,-4,-2,1}-53248S_{-2,-4,1,-2}-59392S_{-2,-4,1,2}\\
&-59392S_{-2,-4,2,1}-169984S_{-2,-3,-3,1}-24576S_{-2,-3,-2,-2}-83968S_{-2,-3,-2,2}\\
&-151552S_{-2,-3,1,-3}-36864S_{-2,-3,1,3}-65536S_{-2,-3,2,-2}-81920S_{-2,-3,2,2}\\
&-36864S_{-2,-3,3,1}-75776S_{-2,-2,-4,1}-24576S_{-2,-2,-3,-2}-79872S_{-2,-2,-3,2}
\end{aligned}
$$

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&-24576S_{-2,-2,-2,-3} - 22528S_{-2,-2,-2,3} - 69632S_{-2,-2,1,-4} - 8192S_{-2,-2,1,4} \\
&-73728S_{-2,-2,2,-3} - 18432S_{-2,-2,2,3} - 16384S_{-2,-2,3,-2} - 18432S_{-2,-2,3,2} \\
&-8192S_{-2,-2,4,1} + 12288S_{-2,1,-5,1} - 38912S_{-2,1,-4,-2} - 43008S_{-2,1,-4,2} \\
&-157696S_{-2,1,-3,-3} - 30720S_{-2,1,-3,3} - 71680S_{-2,1,-2,-4} - 8192S_{-2,1,-2,4} \\
&+8192S_{-2,1,1,-5} + S_{-4}\big(4608S_{-5} + 1536S_{5} - 9216S_{-4,1} - 9216S_{-3,2} - 9216S_{-2,3} \\
&+18432S_{-3,1,1} + 18432S_{-2,1,2} + 18432S_{-2,2,1} - 36864S_{-2,1,1,1}\big) + S_{4}\big(4608S_{-5} + 1536S_{5} \\
&-9216S_{-4,1} - 9216S_{-3,2} - 9216S_{-2,3} + 18432S_{-3,1,1} + 18432S_{-2,1,2} + 18432S_{-2,2,1} \\
&-36864S_{-2,1,1,1}\big) + S_{2}^{2}\big(3072S_{-5} + 1024S_{5} - 6144S_{-4,1} - 6144S_{-3,2} + S_{-2}\big(2048S_{-3} \\
&+4096S_{3} - 4096S_{-2,1}\big) - 6144S_{-2,3} + 12288S_{-3,1,1} + 12288S_{-2,1,2} + 12288S_{-2,2,1} \\
&-24576S_{-2,1,1,1}\big) + S_{-2,2}\big(- 3072S_{-5} - 1024S_{5} + 6144S_{-4,1} + 6144S_{-3,2} + 6144S_{-2,3} \\
&-12288S_{-3,1,1} - 12288S_{-2,1,2} - 12288S_{-2,2,1} + 24576S_{-2,1,1,1}\big) + S_{-3,1}\big(- 6144S_{-5} \\
&-2048S_{5} + 12288S_{-4,1} + 12288S_{-3,2} + 12288S_{-2,3} - 24576S_{-3,1,1} - 24576S_{-2,1,2} \\
&-24576S_{-2,2,1} + 49152S_{-2,1,1,1}\big) - 57344S_{-2,1,1,5} - 8192S_{-2,1,2,-4} - 14336S_{-2,1,2,4} \\
&+4096S_{-2,1,3,-3} - 12288S_{-2,1,4,-2} - 43008S_{-2,1,4,2} - 90112S_{-2,1,5,1} - 20480S_{-2,2,-4,1}
\end{aligned}
$$

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&-43008S_{-2,2,-3,-2}-49152S_{-2,2,-3,2}-79872S_{-2,2,-2,-3}-12288S_{-2,2,-2,3}\\
&-8192S_{-2,2,1,-4}+S_{-3}\big(7680S_{-6}+2560S_{6}-12288S_{-5,1}-12288S_{-4,2}-12288S_{-3,3}\\
&-9216S_{-2,4}+18432S_{-4,1,1}+18432S_{-3,1,2}+18432S_{-3,2,1}+12288S_{-2,1,3}+12288S_{-2,2,2}\\
&+12288S_{-2,3,1}-24576S_{-3,1,1,1}-12288S_{-2,1,1,2}-12288S_{-2,1,2,1}-12288S_{-2,2,1,1}\big)\\
&+S_{3}\big(2560S^{2}_{-3}-6144S_{-2,1}S_{-3}+2048S^{2}_{-2,1}+7680S_{-6}+2560S_{6}-12288S_{-5,1}\\
&-12288S_{-4,2}-12288S_{-3,3}-9216S_{-2,4}+18432S_{-4,1,1}+18432S_{-3,1,2}+18432S_{-3,2,1}\\
&+12288S_{-2,1,3}+12288S_{-2,2,2}+12288S_{-2,3,1}-24576S_{-3,1,1,1}-12288S_{-2,1,1,2}\\
&-12288S_{-2,1,2,1}-12288S_{-2,2,1,1}\big)+S_{-2,1}\big(-15360S_{-6}-5120S_{6}+24576S_{-5,1}\\
&+24576S_{-4,2}+24576S_{-3,3}+18432S_{-2,4}-36864S_{-4,1,1}-36864S_{-3,1,2}\\
&-36864S_{-3,2,1}-24576S_{-2,1,3}-24576S_{-2,2,2}-24576S_{-2,3,1}+49152S_{-3,1,1,1}\\
&+24576S_{-2,1,1,2}+24576S_{-2,1,2,1}+24576S_{-2,2,1,1}\big)-14336S_{-2,2,1,4}\\
&-51200S_{-2,2,4,1}+2048S_{-2,3,-3,1}-2048S_{-2,3,-2,-2}-2048S_{-2,3,-2,2}+4096S_{-2,3,1,-3}\\
&-4096S_{-2,4,-2,1}-12288S_{-2,4,1,-2}-38912S_{-2,4,1,2}-38912S_{-2,4,2,1}\\
&-81920S_{-2,5,1,1}-16384S_{1,-6,1,1}+40960S_{1,-5,-2,1}+24576S_{1,-5,1,-2}-83968S_{1,-4,-3,1}\\
&-51200S_{1,-4,-2,-2}-59392S_{1,-4,-2,2}-28672S_{1,-4,1,-3}+2048S_{1,-4,1,3}-4096S_{1,-4,2,-2}\\
&+2048S_{1,-4,3,1}-96256S_{1,-3,-4,1}-129024S_{1,-3,-3,-2}-155648S_{1,-3,-3,2}\\
&-165888S_{1,-3,-2,-3}-36864S_{1,-3,-2,3}-51200S_{1,-3,1,-4}-59392S_{1,-3,1,4}\\
&-40960S_{1,-3,2,-3}+8192S_{1,-3,3,-2}-96256S_{1,-3,4,1}+8192S_{1,-2,-5,1}-51200S_{1,-2,-4,-2}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&-73728S_{2,-3,1,-3}+4096S_{2,-3,1,3}-16384S_{2,-3,2,-2}+4096S_{2,-3,3,1}-55296S_{2,-2,-4,1}\\
&-69632S_{2,-2,-3,-2}-81920S_{2,-2,-3,2}-86016S_{2,-2,-2,-3}-18432S_{2,-2,-2,3}\\
&-30720S_{2,-2,1,-4}-32768S_{2,-2,1,4}-28672S_{2,-2,2,-3}+6144S_{2,-2,3,-2}-49152S_{2,-2,4,1}\\
&+16384S_{2,1,-5,1}-2048S_{2,1,-4,-2}+4096S_{2,1,-4,2}-110592S_{2,1,-3,-3}+4096S_{2,1,-3,3}\\
&-34816S_{2,1,-2,-4}-28672S_{2,1,-2,4}+8192S_{2,1,1,-5}-32768S_{2,1,1,5}-36864S_{2,1,4,-2}\\
&-40960S_{2,1,4,2}-65536S_{2,1,5,1}-16384S_{2,2,-3,-2}-8192S_{2,2,-3,2}-65536S_{2,2,-2,-3}\\
&-49152S_{2,2,4,1}+8192S_{2,3,-3,1}+10240S_{2,3,-2,-2}+8192S_{2,3,-2,2}-49152S_{2,4,-2,1}\\
&-36864S_{2,4,1,-2}-40960S_{2,4,1,2}-40960S_{2,4,2,1}-81920S_{2,5,1,1}+6144S_{3,-4,1,1}\\
&-22528S_{3,-3,-2,1}-2048S_{3,-3,1,-2}-4096S_{3,-3,1,2}-4096S_{3,-3,2,1}-26624S_{3,-2,-3,1}\\
&-18432S_{3,-2,-2,-2}-18432S_{3,-2,-2,2}-10240S_{3,-2,1,-3}+2048S_{3,-2,1,3}-2048S_{3,-2,2,-2}\\
&+2048S_{3,-2,3,1}-2048S_{3,1,-4,1}+10240S_{3,1,-3,-2}+4096S_{3,1,-3,2}-14336S_{3,1,-2,-3}\\
&-4096S_{3,1,-2,3}+2048S_{3,1,1,-4}+10240S_{3,1,1,4}-14336S_{3,1,4,1}+8192S_{3,2,-3,1}\\
&+10240S_{3,2,-2,-2}+8192S_{3,2,-2,2}-6144S_{3,3,-2,1}-18432S_{3,4,1,1}-63488S_{4,-3,1,1}\\
&+8192S_{4,-2,-2,1}+4096S_{4,-2,1,-2}-38912S_{4,-2,1,2}-38912S_{4,-2,2,1}-65536S_{4,1,-3,1}\\
&+8192S_{4,1,-2,-2}-24576S_{4,1,-2,2}-73728S_{4,1,1,-3}-18432S_{4,1,1,3}-32768S_{4,1,2,-2}\\
&-40960S_{4,1,2,2}-18432S_{4,1,3,1}-40960S_{4,2,-2,1}-32768S_{4,2,1,-2}-40960S_{4,2,1,2}\\
&-40960S_{4,2,2,1}-18432S_{4,3,1,1}-73728S_{5,-2,1,1}-98304S_{5,1,-2,1}-65536S_{5,1,1,-2}\\
&-81920S_{5,1,1,2}-81920S_{5,1,2,1}-81920S_{5,2,1,1}-45056S_{6,1,1,1}+118784S_{-4,-2,1,1,1}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, ’09]

$$
\begin{aligned}
&+86016S_{-4,1,-2,1,1} + 24576S_{-4,1,1,-2,1} + 4096S_{-4,1,1,1,-2} + 311296S_{-3,-3,1,1,1}\\
&+180224S_{-3,-2,-2,1,1} + 180224S_{-3,-2,1,-2,1} + 131072S_{-3,-2,1,1,-2} + 163840S_{-3,-2,1,1,2}\\
&+163840S_{-3,-2,1,2,1} + 163840S_{-3,-2,2,1,1} + 245760S_{-3,1,-3,1,1} + 196608S_{-3,1,-2,-2,1}\\
&+122880S_{-3,1,-2,1,-2} + 147456S_{-3,1,-2,1,2} + 147456S_{-3,1,-2,2,1} + 122880S_{-3,1,1,-3,1}\\
&+81920S_{-3,1,1,-2,-2} + 98304S_{-3,1,1,-2,2} + 24576S_{-3,1,1,1,-3} + 24576S_{-3,1,2,-2,1}\\
&+98304S_{-3,2,-2,1,1} + 24576S_{-3,2,1,-2,1} + 118784S_{-2,-4,1,1,1} + 167936S_{-2,-3,-2,1,1}\\
&+172032S_{-2,-3,1,-2,1} + 131072S_{-2,-3,1,1,-2} + 163840S_{-2,-3,1,1,2} + 163840S_{-2,-3,1,2,1}\\
&+163840S_{-2,-3,2,1,1} + 159744S_{-2,-2,-3,1,1} + 24576S_{-2,-2,-2,-2,1}\\
&+24576S_{-2,-2,-2,1,-2} + 77824S_{-2,-2,-2,1,2} + 77824S_{-2,-2,-2,2,1} + 163840S_{-2,-2,1,-3,1}\\
&+24576S_{-2,-2,1,-2,-2} + 81920S_{-2,-2,1,-2,2} + 147456S_{-2,-2,1,1,-3} + 36864S_{-2,-2,1,1,3}\\
&+65536S_{-2,-2,1,2,-2} + 81920S_{-2,-2,1,2,2} + 36864S_{-2,-2,1,3,1} + 81920S_{-2,-2,2,-2,1}\\
&+65536S_{-2,-2,2,1,-2} + 81920S_{-2,-2,2,1,2} + 81920S_{-2,-2,2,2,1} + 36864S_{-2,-2,3,1,1}\\
&+86016S_{-2,1,-4,1,1} + 192512S_{-2,1,-3,-2,1} + 122880S_{-2,1,-3,1,-2}\\
&+147456S_{-2,1,-3,1,2} + 147456S_{-2,1,-3,2,1} + 176128S_{-2,1,-2,-3,1} + 24576S_{-2,1,-2,-2,-2}\\
&+86016S_{-2,1,-2,-2,2} + 155648S_{-2,1,-2,1,-3} + 36864S_{-2,1,-2,1,3} + 65536S_{-2,1,-2,2,-2}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+81920S_{-2,1,-2,2,2}+36864S_{-2,1,-2,3,1}+40960S_{-2,1,1,-4,1}+86016S_{-2,1,1,-3,-2}\\
&+98304S_{-2,1,1,-3,2}+159744S_{-2,1,1,-2,-3}+24576S_{-2,1,1,-2,3}+16384S_{-2,1,1,1,-4}\\
&+28672S_{-2,1,1,1,4}+102400S_{-2,1,1,4,1}+32768S_{-2,1,2,-3,1}+28672S_{-2,1,2,-2,-2}\\
&+32768S_{-2,1,2,-2,2}-8192S_{-2,1,3,-2,1}+86016S_{-2,1,4,1,1}+98304S_{-2,2,-3,1,1}\\
&+102400S_{-2,2,-2,-2,1}+57344S_{-2,2,-2,1,-2}+65536S_{-2,2,-2,1,2}+65536S_{-2,2,-2,2,1}\\
&+32768S_{-2,2,1,-3,1}+28672S_{-2,2,1,-2,-2}+32768S_{-2,2,1,-2,2}+4096S_{-2,3,-2,1,1}\\
&-8192S_{-2,3,1,-2,1}+77824S_{-2,4,1,1,1}+118784S_{1,-4,-2,1,1}+49152S_{1,-4,1,-2,1}\\
&+8192S_{1,-4,1,1,-2}+311296S_{1,-3,-3,1,1}+192512S_{1,-3,-2,-2,1}+139264S_{1,-3,-2,1,-2}\\
&+163840S_{1,-3,-2,1,2}+163840S_{1,-3,-2,2,1}+221184S_{1,-3,1,-3,1}+118784S_{1,-3,1,-2,-2}\\
&+147456S_{1,-3,1,-2,2}+81920S_{1,-3,1,1,-3}+8192S_{1,-3,1,2,-2}+73728S_{1,-3,2,-2,1}\\
&+8192S_{1,-3,2,1,-2}+118784S_{1,-2,-4,1,1}+184320S_{1,-2,-3,-2,1}+131072S_{1,-2,-3,1,-2}\\
&+163840S_{1,-2,-3,1,2}+163840S_{1,-2,-3,2,1}+184320S_{1,-2,-2,-3,1}+24576S_{1,-2,-2,-2,-2}\\
&+94208S_{1,-2,-2,-2,2}+155648S_{1,-2,-2,1,-3}+36864S_{1,-2,-2,1,3}+65536S_{1,-2,-2,2,-2}\\
&+81920S_{1,-2,-2,2,2}+36864S_{1,-2,-2,3,1}+81920S_{1,-2,1,-4,1}+118784S_{1,-2,1,-3,-2}\\
&+147456S_{1,-2,1,-3,2}+159744S_{1,-2,1,-2,-3}+36864S_{1,-2,1,-2,3}+40960S_{1,-2,1,1,-4}\\
&+53248S_{1,-2,1,1,4}+24576S_{1,-2,1,2,-3}-4096S_{1,-2,1,3,-2}+94208S_{1,-2,1,4,1}\\
&+90112S_{1,-2,2,-3,1}+53248S_{1,-2,2,-2,-2}+65536S_{1,-2,2,-2,2}+24576S_{1,-2,2,1,-3}\\
&-4096S_{1,-2,3,1,-2}+94208S_{1,-2,4,1,1}-32768S_{1,1,-5,1,1}+77824S_{1,1,-4,-2,1}\\
&+12288S_{1,1,-4,1,-2}+8192S_{1,1,-4,1,2}+8192S_{1,1,-4,2,1}+278528S_{1,1,-3,-3,1}\\
&+139264S_{1,1,-3,-2,-2}+163840S_{1,1,-3,-2,2}+147456S_{1,1,-3,1,-3}-8192S_{1,1,-3,1,3}
\end{aligned}
$$

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+32768S_{1,1,-3,2,-2}-8192S_{1,1,-3,3,1}+110592S_{1,1,-2,-4,1}+139264S_{1,1,-2,-3,-2}\\
&+163840S_{1,1,-2,-3,2}+172032S_{1,1,-2,-2,-3}+36864S_{1,1,-2,-2,3}+61440S_{1,1,-2,1,-4}\\
&+65536S_{1,1,-2,1,4}+57344S_{1,1,-2,2,-3}-12288S_{1,1,-2,3,-2}+98304S_{1,1,-2,4,1}\\
&-32768S_{1,1,1,-5,1}+4096S_{1,1,1,-4,-2}-8192S_{1,1,1,-4,2}+221184S_{1,1,1,-3,-3}\\
&-8192S_{1,1,1,-3,3}+69632S_{1,1,1,-2,-4}+57344S_{1,1,1,-2,4}-16384S_{1,1,1,1,-5}+65536S_{1,1,1,1,5}\\
&+73728S_{1,1,1,4,-2}+81920S_{1,1,1,4,2}+131072S_{1,1,1,5,1}+32768S_{1,1,2,-3,-2}\\
&+16384S_{1,1,2,-3,2}+131072S_{1,1,2,-2,-3}+98304S_{1,1,2,4,1}-16384S_{1,1,3,-3,1}\\
&-20480S_{1,1,3,-2,-2}-16384S_{1,1,3,-2,2}+98304S_{1,1,4,-2,1}+73728S_{1,1,4,1,-2}\\
&+81920S_{1,1,4,1,2}+81920S_{1,1,4,2,1}+163840S_{1,1,5,1,1}-8192S_{1,2,-4,1,1}+163840S_{1,2,-3,-2,1}\\
&+57344S_{1,2,-3,1,-2}+16384S_{1,2,-3,1,2}+16384S_{1,2,-3,2,1}+147456S_{1,2,-2,-3,1}\\
&+73728S_{1,2,-2,-2,-2}+81920S_{1,2,-2,-2,2}+90112S_{1,2,-2,1,-3}-8192S_{1,2,-2,1,3}\\
&+24576S_{1,2,-2,2,-2}-8192S_{1,2,-2,3,1}+32768S_{1,2,1,-3,-2}+16384S_{1,2,1,-3,2}\\
&+131072S_{1,2,1,-2,-3}+98304S_{1,2,1,4,1}+81920S_{1,2,4,1,1}-8192S_{1,3,-3,1,1}+28672S_{1,3,-2,-2,1}\\
&+8192S_{1,3,-2,1,2}+8192S_{1,3,-2,2,1}-16384S_{1,3,1,-3,1}-20480S_{1,3,1,-2,-2}-16384S_{1,3,1,-2,2}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+61440S_{1,4,-2,1,1}+90112S_{1,4,1,-2,1}+65536S_{1,4,1,1,-2}+81920S_{1,4,1,1,2}+81920S_{1,4,1,2,1}\\
&+81920S_{1,4,2,1,1}+163840S_{1,5,1,1,1}+8192S_{2,-4,1,1,1}+163840S_{2,-3,-2,1,1}\\
&+114688S_{2,-3,1,-2,1}+32768S_{2,-3,1,1,-2}+163840S_{2,-2,-3,1,1}+98304S_{2,-2,-2,-2,1}\\
&+73728S_{2,-2,-2,1,-2}+81920S_{2,-2,-2,1,2}+81920S_{2,-2,-2,2,1}+131072S_{2,-2,1,-3,1}\\
&+65536S_{2,-2,1,-2,-2}+81920S_{2,-2,1,-2,2}+57344S_{2,-2,1,1,-3}+8192S_{2,-2,1,2,-2}\\
&+491520S_{2,-2,2,-2,1}+8192S_{2,-2,2,1,-2}-8192S_{2,1,-4,1,1}+163840S_{2,1,-3,-2,1}\\
&+57344S_{2,1,-3,1,-2}+16384S_{2,1,-3,1,2}+16384S_{2,1,-3,2,1}+147456S_{2,1,-2,-3,1}\\
&+73728S_{2,1,-2,-2,-2}+81920S_{2,1,-2,-2,2}+90112S_{2,1,-2,1,-3}-8192S_{2,1,-2,1,3}\\
&+24576S_{2,1,-2,2,-2}-8192S_{2,1,-2,3,1}+32768S_{2,1,1,-3,-2}+16384S_{2,1,1,-3,2}\\
&+131072S_{2,1,1,-2,-3}+98304S_{2,1,1,4,1}+81920S_{2,1,4,1,1}+16384S_{2,2,-3,1,1}\\
&+98304S_{2,2,-2,-2,1}+32768S_{2,2,-2,1,-2}+16384S_{2,2,-2,1,2}+16384S_{2,2,-2,2,1}\\
&-16384S_{2,3,-2,1,1}+81920S_{2,4,1,1,1}+8192S_{3,-3,1,1,1}+36864S_{3,-2,-2,1,1}\\
&+16384S_{3,-2,1,-2,1}+4096S_{3,-2,1,1,-2}-8192S_{3,1,-3,1,1}+28672S_{3,1,-2,-2,1}\\
&+8192S_{3,1,-2,1,2}+8192S_{3,1,-2,2,1}-16384S_{3,1,1,-3,1}-20480S_{3,1,1,-2,-2}-16384S_{3,1,1,-2,2}\\
&-16384S_{3,2,-2,1,1}+77824S_{4,-2,1,1,1}+49152S_{4,1,-2,1,1}+81920S_{4,1,1,-2,1}+65536S_{4,1,1,1,-2}\\
&+81920S_{4,1,1,1,2}+81920S_{4,1,1,2,1}+81920S_{4,1,2,1,1}+81920S_{4,2,1,1,1}+163840S_{5,1,1,1,1}\\
&-327680S_{-3,-2,1,1,1,1}-294912S_{-3,1,-2,1,1,1}-196608S_{-3,1,1,-2,1,1}-49152S_{-3,1,1,1,-2,1}\\
&-327680S_{-2,-3,1,1,1,1}-155648S_{-2,-2,-2,1,1,1}-163840S_{-2,-2,1,-2,1,1}\\
&-163840S_{-2,-2,1,1,-2,1}-131072S_{-2,-2,1,1,1,-2}-163840S_{-2,-2,1,1,1,2}-163840S_{-2,-2,1,1,2,1}\\
&-163840S_{-2,-2,1,2,1,1}-163840S_{-2,-2,2,1,1,1}-294912S_{-2,1,-3,1,1,1}-172032S_{-2,1,-2,-2,1,1}
\end{aligned}
$$

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$-180224S_{-2,1,-2,1,-2,1} - 131072S_{-2,1,-2,1,1,-2} - 163840S_{-2,1,-2,1,1,2} - 163840S_{-2,1,-2,1,2,1}$$
$$-163840S_{-2,1,-2,2,1,1} - 196608S_{-2,1,1,-3,1,1} - 204800S_{-2,1,1,-2,-2,1} - 114688S_{-2,1,1,-2,1,-2}$$
$$-131072S_{-2,1,1,-2,1,2} - 131072S_{-2,1,1,-2,2,1} - 65536S_{-2,1,1,1,-3,1} - 57344S_{-2,1,1,1,-2,-2}$$
$$-65536S_{-2,1,1,1,-2,2} + S_2\big(\big(1024S_{-3} + 4096S_3\big)S_{-2}^2 + \big(11264S_{-5} + 5120S_5 - 8192S_{-4,1}$$
$$-6144S_{-3,2} - 8192S_{-2,3} + 2048S_{4,1} + 12288S_{-3,1,1} - 4096S_{-2,-2,1} + 12288S_{-2,1,2}$$
$$+12288S_{-2,2,1} - 24576S_{-2,1,1,1}\big)S_{-2} + 8192S_{-7} + 9216S_7 - 16384S_{-6,1} - 6144S_{-5,-2}$$
$$-16384S_{-5,2} - 1024S_{-4,-3} - 17408S_{-4,3} - 15360S_{-3,4} - 18432S_{-2,5} - 5120S_{4,3}$$
$$+4096S_{5,2} + 6144S_{6,1} + 32768S_{-5,1,1} - 6144S_{-4,-2,1} + 36864S_{-4,1,2} + 36864S_{-4,2,1}$$
$$-4096S_{-3,-3,1} - 2048S_{-3,-2,-2} - 4096S_{-3,-2,2} + 36864S_{-3,1,3} + 40960S_{-3,2,2}$$
$$+36864S_{-3,3,1} + 2048S_{-2,-4,1} - 8192S_{-2,-3,2} + 10240S_{-2,-2,3} + S_{-2,1}\big(- 4096S_{-4}$$
$$-8192S_4 + 12288S_{-3,1} + 16384S_{-2,2} - 8192S_{-2,1,1}\big) + S_{-3}\big(6144S_{-4} + 3072S_4$$
$$-6144S_{-3,1} - 4096S_{-2,2} + 12288S_{-2,1,1}\big) + S_3\big(10240S_{-4} + 3072S_4 - \frac{47104S_{-3,1}}{3}$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&-\frac{40960S_{-2,2}}{3} + \frac{69632}{3}S_{-2,1,1}) + 34816S_{-2,1,4} + 36864S_{-2,2,3} + 36864S_{-2,3,2} \\
&+32768S_{-2,4,1} - 4096S_{4,1,2} - 4096S_{4,2,1} - 73728S_{-4,1,1,1} - 81920S_{-3,1,1,2} \\
&-81920S_{-3,1,2,1} - 81920S_{-3,2,1,1} + 24576S_{-2,-3,1,1} + 4096S_{-2,-2,-2,1} + 8192S_{-2,-2,1,2} \\
&+8192S_{-2,-2,2,1} - 8192S_{-2,1,1,-3} - 73728S_{-2,1,1,3} - 81920S_{-2,1,2,2} - 73728S_{-2,1,3,1} \\
&-8192S_{-2,2,-2,1} - 81920S_{-2,2,1,2} - 81920S_{-2,2,2,1} - 73728S_{-2,3,1,1} + 24576S_{4,1,1,1} \\
&+163840S_{-3,1,1,1,1} - 49152S_{-2,-2,1,1,1} - 16384S_{-2,1,-2,1,1} + 163840S_{-2,1,1,1,2} \\
&+163840S_{-2,1,1,2,1} + 163840S_{-2,1,2,1,1} + 163840S_{-2,2,1,1,1} - 327680S_{-2,1,1,1,1,1}) \\
&-65536S_{-2,1,2,-2,1,1} - 131072S_{-2,2,-2,1,1,1} - 65536S_{-2,2,1,-2,1,1} - 327680S_{1,-3,-2,1,1,1} \\
&-294912S_{1,-3,1,-2,1,1} - 147456S_{1,-3,1,1,-2,1} - 16384S_{1,-3,1,1,1,-2} - 327680S_{1,-2,-3,1,1,1} \\
&-188416S_{1,-2,-2,-2,1,1} - 180224S_{1,-2,-2,1,-2,1} - 131072S_{1,-2,-2,1,1,-2} \\
&-163840S_{1,-2,-2,1,1,2} - 163840S_{1,-2,-2,1,2,1} - 163840S_{1,-2,-2,2,1,1} - 294912S_{1,-2,1,-3,1,1} \\
&-188416S_{1,-2,1,-2,-2,1} - 131072S_{1,-2,1,-2,1,-2} - 163840S_{1,-2,1,-2,1,2} - 163840S_{1,-2,1,-2,2,1} \\
&-180224S_{1,-2,1,1,-3,1} - 106496S_{1,-2,1,1,-2,-2} - 131072S_{1,-2,1,1,-2,2} - 49152S_{1,-2,1,1,1,-3} \\
&-49152S_{1,-2,1,2,-2,1} - 131072S_{1,-2,2,-2,1,1} - 49152S_{1,-2,2,1,-2,1} - 16384S_{1,1,-4,1,1,1} \\
&-327680S_{1,1,-3,-2,1,1} - 229376S_{1,1,-3,1,-2,1} - 65536S_{1,1,-3,1,1,-2} - 327680S_{1,1,-2,-3,1,1} \\
&-196608S_{1,1,-2,-2,-2,1} - 147456S_{1,1,-2,-2,1,-2} - 163840S_{1,1,-2,-2,1,2} - 163840S_{1,1,-2,-2,2,1} \\
&-262144S_{1,1,-2,1,-3,1} - 131072S_{1,1,-2,1,-2,-2} - 163840S_{1,1,-2,1,-2,2} - 114688S_{1,1,-2,1,1,-3}
\end{aligned}
$$

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Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&-16384S_{1,1,-2,1,2,-2} - 98304S_{1,1,-2,2,-2,1} - 16384S_{1,1,-2,2,1,-2} + 16384S_{1,1,1,-4,1,1}\\
&-327680S_{1,1,1,-3,-2,1} - 114688S_{1,1,1,-3,1,-2} - 32768S_{1,1,1,-3,1,2} - 32768S_{1,1,1,-3,2,1}\\
&-294912S_{1,1,1,-2,-3,1} - 147456S_{1,1,1,-2,-2,-2} - 163840S_{1,1,1,-2,-2,2} - 180224S_{1,1,1,-2,1,-3}\\
&+16384S_{1,1,1,-2,1,3} - 49152S_{1,1,1,-2,2,-2} + 16384S_{1,1,1,-2,3,1} - 65536S_{1,1,1,1,-3,-2}\\
&-32768S_{1,1,1,1,-3,2} - 262144S_{1,1,1,1,-2,-3} - 196608S_{1,1,1,1,4,1} - 163840S_{1,1,1,4,1,1}\\
&-32768S_{1,1,2,-3,1,1} - 196608S_{1,1,2,-2,-2,1} - 65536S_{1,1,2,-2,1,-2} - 32768S_{1,1,2,-2,1,2}\\
&-32768S_{1,1,2,-2,2,1} + 32768S_{1,1,3,-2,1,1} - 163840S_{1,1,4,1,1,1} - 32768S_{1,2,-3,1,1,1}\\
&-163840S_{1,2,-2,-2,1,1} - 131072S_{1,2,-2,1,-2,1} - 49152S_{1,2,-2,1,1,-2} - 32768S_{1,2,1,-3,1,1}\\
&-196608S_{1,2,1,-2,-2,1} - 65536S_{1,2,1,-2,1,-2} - 32768S_{1,2,1,-2,1,2} - 32768S_{1,2,1,-2,2,1}\\
&-16384S_{1,3,-2,1,1,1} + 32768S_{1,3,1,-2,1,1} - 163840S_{1,4,1,1,1,1} - 163840S_{2,-2,-2,1,1,1}\\
&-163840S_{2,-2,1,-2,1,1} - 98304S_{2,-2,1,1,-2,1} - 16384S_{2,-2,1,1,1,-2} - 32768S_{2,1,-3,1,1,1}\\
&-163840S_{2,1,-2,-2,1,1} - 131072S_{2,1,-2,1,-2,1} - 49152S_{2,1,-2,1,1,-2} - 32768S_{2,1,1,-3,1,1}\\
&-196608S_{2,1,1,-2,-2,1} - 65536S_{2,1,1,-2,1,-2} - 32768S_{2,1,1,-2,1,2} - 32768S_{2,1,1,-2,2,1}\\
&-32768S_{2,2,-2,1,1,1} - 16384S_{3,1,-2,1,1,1} + 32768S_{3,1,1,-2,1,1} - 163840S_{4,1,1,1,1,1}\\
&+327680S_{-2,-2,1,1,1,1,1} + 327680S_{-2,1,-2,1,1,1,1} + 262144S_{-2,1,1,-2,1,1,1}
\end{aligned}
$$

NEW!

Five-loop [T. Łukowski, A. R., V. Velizhanin, '09]

$$
\begin{aligned}
&+131072 S_{-2,1,1,1,-2,1,1} + 327680 S_{1,-2,-2,1,1,1,1} + 327680 S_{1,-2,1,-2,1,1,1} \\
&+262144 S_{1,-2,1,1,-2,1,1} + 98304 S_{1,-2,1,1,1,-2,1} + 327680 S_{1,1,-2,-2,1,1,1} \\
&+327680 S_{1,1,-2,1,-2,1,1} + 196608 S_{1,1,-2,1,1,-2,1} + 32768 S_{1,1,-2,1,1,1,-2} + 65536 S_{1,1,1,-3,1,1,1} \\
&+327680 S_{1,1,1,-2,-2,1,1} + 262144 S_{1,1,1,-2,1,-2,1} + 98304 S_{1,1,1,-2,1,1,-2} + 65536 S_{1,1,1,1,-3,1,1} \\
&+393216 S_{1,1,1,1,-2,-2,1} + 131072 S_{1,1,1,1,-2,1,-2} + 65536 S_{1,1,1,1,-2,1,2} + 65536 S_{1,1,1,1,-2,2,1} \\
&+65536 S_{1,1,2,-2,1,1,1} + 65536 S_{1,2,1,-2,1,1,1} + 65536 S_{2,1,1,-2,1,1,1} - 131072 S_{1,1,1,1,-2,1,1,1} \\
&+512 \Big(4 S_{-2,1} S_{-3} - S_{-3}^2 + S_3^2 - 4 S_{-2,1}^2 + S_1^2 \big(2 S_{-2}^2 - 4 S_{-4} + 6 S_4 + 16 S_{-3,1} + 12 S_{-2,2} \\
&-16 S_{-2,1,1} \big) + S_1 \big(-2 S_{-5} - 4 S_{-3} S_2 + 4 S_{-2} S_3 + 4 S_2 S_3 + 6 S_5 + 8 S_{-4,1} - 4 S_{-3,-2} \\
&+12 S_{-3,2} + 8 S_{-2} S_{-2,1} + 8 S_2 S_{-2,1} + 8 S_{-2,3} + 4 S_{4,1} - 24 S_{-3,1,1} - 8 S_{-2,-2,1} - 24 S_{-2,1,2} \\
&-24 S_{-2,2,1} + 48 S_{-2,1,1,1} \big) \Big) \zeta(3) \\
&+2560\, S_1 (S_3 - S_{-3} + 2\, S_{-2,1}) \,\zeta(5)
\end{aligned}
$$

- The remaining contributions may be calculated by
  - Exploiting the putative spectral equations
  - Lüscher corrections adapted to AdS/CFT

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- Tedious algebra, computations with high precision (app. 1000 significant numbers) and EZ-Face allowed to find

$$
\begin{aligned}
\Delta_w = {} & 13440\,\zeta(7)S_1^2 - 1536\,\zeta(3)^2 S_1^3 + 2560\,\zeta(5)S_1(3\,S_1(2\,S_{-2}+S_2) - S_1^3 + S_{-3} + S_3 - 2\,S_{-2,1}) \\
& +1024\,\zeta(3)S_1(-2\,S_1^3 S_{-2} + 2\,S_1^2(2\,S_{-3}+3\,S_3) + S_1(4\,S_{-2}^2 + 6\,S_2 S_{-2} + 3\,S_{-4} - S_4 \\
& -2\,(S_{-3,1} - 2\,S_{-2,-2} + S_{-2,2} + S_{3,1} - 2\,S_{-2,1,1})) + 2\,S_{-2}(S_{-3} + S_3 - 2\,S_{-2,1})) \\
& -1024\,S_1((S_1(3\,S_2 + 2\,S_{-2}) + S_{-3} + S_3 - 2\,S_{-2,1} - S_1^3)(S_{-5} - S_5 + 2\,S_{-2,-3} - 2\,S_{3,-2} \\
& +2\,S_{4,1} - 4\,S_{-2,-2,1}) + 2\,S_1^2(2S_{-6} - 2\,S_6 - S_{-4,-2} + 2\,S_{-3,-3} + 3\,S_{-2,-4} + S_{-2,4} \\
& -2\,S_{3,-3} - 2\,S_{4,-2} + S_{4,2} + 4\,S_{5,1} - 4\,S_{-3,-2,1} - 4\,S_{-2,-3,1} - 2\,S_{-2,-2,-2} - 2\,S_{-2,-2,2}) \\
& +S_1(5\,S_{-7} - 5\,S_7 - 4\,S_{-6,1} + 4\,S_{-5,-2} - S_{-5,2} + 3\,S_{-4,-3} + S_{-3,-4} - S_{-3,4} + 8\,S_{-2,-5} \\
& -6\,S_{-2,5} - 4\,S_{3,-4} + 2\,S_{3,4} - 8\,S_{4,-3} + 3\,S_{4,3} - 6\,S_{5,-2} + S_{5,2} + 6\,S_{6,1} + 2\,S_{-5,1,1} \\
& -6\,S_{-4,-2,1} - 2\,S_{-3,-3,1} + 2\,S_{-3,-2,-2} - 2\,S_{-3,1,-3} - 8\,S_{-2,-4,1} + 6\,S_{-2,-3,-2} - 2\,S_{-2,-3,2} \\
& +14\,S_{-2,-2,-3} - 6\,S_{-2,-2,3} - 2\,S_{-2,1,-4} + 2\,S_{-2,1,4} - 2\,S_{-2,2,-3} - 4\,S_{-2,3,-2} + 10\,S_{-2,4,1} \\
& +2\,S_{3,-3,1} - 4\,S_{3,-2,-2} + 2\,S_{3,-2,2} + 2\,S_{3,1,-3} + 2\,S_{3,2,-2} + 10\,S_{4,-2,1} + 6\,S_{4,1,-2} - 2\,S_{4,1,2} \\
& -2\,S_{4,2,1} - 2\,S_{5,1,1} + 4\,S_{-3,1,-2,1} + 4\,S_{-2,-3,1,1} - 20\,S_{-2,-2,-2,1} - 8\,S_{-2,-2,1,-2} \\
& +4\,S_{-2,-2,1,2} + 4\,S_{-2,-2,2,1} + 4\,S_{-2,1,-3,1} - 4\,S_{-2,1,-2,-2} + 4\,S_{-2,1,1,-3} + 4\,S_{-2,2,-2,1} \\
& -4\,S_{3,-2,1,1} - 4\,S_{3,1,1,-2} + 4\,S_{4,1,1,1} - 8\,S_{-2,-2,1,1,1} - 8\,S_{-2,1,1,-2,1}))\,.
\end{aligned}
$$

Tests

- Adding up these two contributions should provide complete answer. How to check its veracity?
- The BFKL equation! It predicts the leading poles at $M = -1 + \omega$ at *any* loop order

$$
\begin{aligned}
\gamma \;=\; & \left(2 + 0\,\omega + \mathcal{O}(\omega^2)\right)\left(\tfrac{-4\,g^2}{\omega}\right) - \left(0 + 0\,\omega + \mathcal{O}(\omega^2)\right)\left(\tfrac{-4\,g^2}{\omega}\right)^2 \\
& + \left(0 + \zeta(3)\,\omega + \mathcal{O}(\omega^2)\right)\left(\tfrac{-4\,g^2}{\omega}\right)^3 \\
& - (4\,\zeta(3) + 5/4\,\zeta(4)\,\omega + \mathcal{O}(\omega^2))\,(\tfrac{-4\,g^2}{\omega})^4 \\
& - (0 + \left(2\,\zeta(2)\,\zeta(3) + 16\,\zeta(5)\right)\omega + \mathcal{O}(\omega^2))\,(\tfrac{-4g^2}{\omega})^5 \pm \ldots
\end{aligned}
$$

- Upon analytic continuation of the five-loop result to $M = -1 + \omega$ we found perfect agreement!
- Analytic and numerical analysis of the spectral equations reproduce the above result. [Arutyunov, Frolov, Suzuki, 2010], [Bajnok, Hegedus, 2010]

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&+ \Big(0 + \zeta(3)\,\omega + \mathcal{O}(\omega^2)\Big)\left(\frac{-4\,g^2}{\omega}\right)^3 \\
&- \big(4\,\zeta(3) + 5/4\,\zeta(4)\,\omega + \mathcal{O}(\omega^2)\big)\left(\frac{-4\,g^2}{\omega}\right)^4 \\
&- \Big(0 + \Big(2\,\zeta(2)\,\zeta(3) + 16\,\zeta(5)\Big)\,\omega + \mathcal{O}(\omega^2)\Big)\left(\frac{-4g^2}{\omega}\right)^5 \pm \ldots
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\begin{aligned}
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& +\Big(0+\,\zeta(3)\,\omega+\mathcal{O}(\omega^2)\Big)\left(\frac{-4\,g^2}{\omega}\right)^3 \\
& -(4\,\zeta(3)+5/4\,\zeta(4)\,\omega+\mathcal{O}(\omega^2))\,(\tfrac{-4\,g^2}{\omega})^4 \\
& -(0+\Big(2\,\zeta(2)\,\zeta(3)+16\,\zeta(5)\Big)\,\omega+\mathcal{O}(\omega^2))\,(\tfrac{-4g^2}{\omega})^5\pm\ldots
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Conclusions

- The recently proposed spectral equations for the planar $\mathcal{N} = 4$ SYM theory, if supplemented by appropriate analytic properties, provide the full solution to the spectral problem!
- There is no need for Feynman diagram computations, as long as the ADs are concerned.
- Their veracity needs thus to be extensively tested!
- Motivated by this we have calculated the five-loop anomalous dimension of twist-two operators. It has been found to satisfy all known constraints.

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