# A new infrared regularization for planar $\mathcal{N} = 4$ super Yang-Mills

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based on

F. Alday, J. H., J. Plefka and T. Schuster, arXiv:0908.0684 [hep-th]
J. H., S. Naculich, H. Schnitzer and M. Spradlin arXiv:1001.1358 [hep-th]

Loops and Legs in Quantum Field Theory, April 27th, 2010

# Outline

- Scattering on the Coulomb branch of $\mathcal{N} = 4$ SYM
  - Extended dual conformal symmetry
  - Integral basis at higher loops
  - Exponentiation
  - Regge limit

# Higgs IR regulator for planar $\mathcal{N} = 4$ Super Yang-Mills

- $U(N+M) \to U(N) \times U(M)$

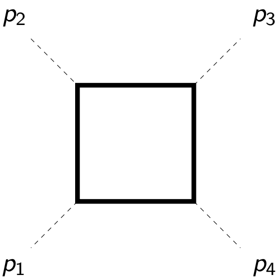
[Alday, Maldacena, 2007; Kawai, Suyama, 2007; Schabinger, 2008; Sever, McGreevy, 2008]

[Alday, J.H., Plefka, Schuster, 2009]

$\to$ leads to massive particles

- scatter massless $U(M)$ particles
- $N \gg M$: only allow loops in $N$-part of $U(N+M)$
  $\to$ renders amplitudes IR finite

- e.g. colour-ordered one-loop amplitude



# One-loop example

**Various interesting limits**

- Regge limit $s \gg t, m^2$

$$M^{(1)} = \log(s/m^2)\alpha(t/m^2) + O(s^0)\,, \qquad \alpha \text{ is Regge trajectory}$$

- large mass limit $m^2 \gg s, t$
- small mass limit $m^2 \ll s, t$ ("mass regulator")

$$M^{(1)} = -\left[\log^2\frac{s}{m^2} + \log^2\frac{t}{m^2}\right] + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{1}{2}\pi^2 + \mathcal{O}(m^2)$$

reminder: in dimensional regularization

$$M^{(1)} = -\left[\frac{1}{2\epsilon^2}\left(\frac{\mu^2}{s}\right)^{\epsilon} + \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{t}\right)^{\epsilon}\right] + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{2}{3}\pi^2 + \mathcal{O}(\epsilon)$$

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# Exponentiation in Higgs regularization

- universal planar structure

$$\log M_4 = D(s) + D(t) + F_4(s/t) + \mathcal{O}(\epsilon)$$

- reminder: dimensional regularization ($\beta = 0$)

$$D(s) = -1/2 \sum a^\ell \left[\Gamma^{(\ell)}_{\rm cusp}/(\ell\epsilon)^2 + \mathcal{G}^{(\ell)}_0/(\ell\epsilon)\right] \left(\mu^2/s\right)^{\ell\epsilon}$$

- finite part $F_4(s/t)$ is also simple! [Anastasiou, Bern, Dixon, Kosower 2002; Bern, Dixon, Smirnov, 2003]

$$F_4 = \frac{1}{2}\Gamma_{\rm cusp}(a)\left[\frac{1}{2}\log^2\frac{s}{t} + \frac{2}{3}\pi^2\right] + c(a)$$

$M^{(2)} - \frac{1}{2}\left(M^{(1)}\right)^2$ interference $1/\epsilon \times O(\epsilon) = O(1)$
$\Rightarrow$ in order to compute $\log M$, need $O(\epsilon)$ terms in $M$

- analog in Higgs regularization [Alday, J. H., Plefka, Schuster, 2009; J. H., Naculich, Schnitzer, Spradlin, 2010]

$$D(s) = -\frac{1}{4}\Gamma_{\rm cusp}(a)\log^2\frac{s}{m^2} - \tilde{G}_0(a)\log\frac{s}{m^2}$$

$F_4$ equal up to scheme-dependent constant
we have $m^2 \times \log m^2 \to 0 \Rightarrow$ can drop all $O(m^2)$ terms

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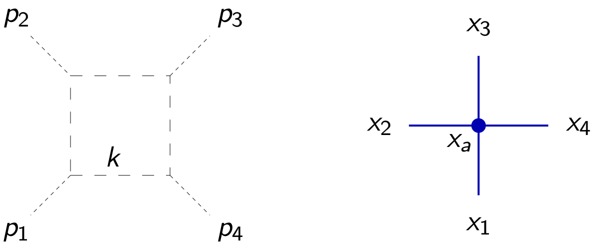
# Dual conformal symmetry (1/2)

- observation: $\mathcal{N} = 4$ SYM loop integrals have a dual conformal symmetry

[Drummond, J.H., Smirnov, Sokatchev, 2006]




- loop integrand has conformal symmetry in dual space

$$x^\mu_{i+1} - x^\mu_i = p_i$$

e.g. inversion symmetry $x^\mu \to x^\mu / x^2$ or special conformal transformations

$$K^\mu = \sum_i \left[ 2 x^\mu_i \, x^\nu_i \frac{\partial}{\partial x_{i\nu}} - x_i^2 \frac{\partial}{\partial x_{i\mu}} \right]$$

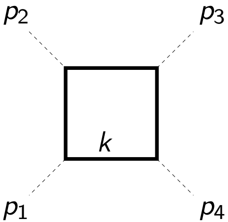
- breaking of symmetry $D = 4 - 2\epsilon$ under control

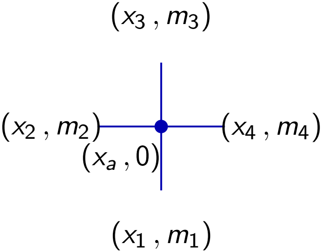
[Drummond, J.H., Korchemsky, Sokatchev, 2007]

# Dual conformal symmetry (2/2)

**refinement:** $U(N+M) \rightarrow U(N) \times U(1)^M$ [Alday, J.H., Plefka, Schuster, 2009]

- on-shell conditions now read $p_i^2 = (m_i - m_{+1})$
- particles in loop have mass $m_i$




- important: in addition to the dual coordinates $x_i$, we can vary the masses $m_i$

$$\hat{K}^\mu = K^\mu + \sum_i \left[ 2x_i^\mu \, m_i \frac{\partial}{\partial m_i} - m_i^2 \frac{\partial}{\partial x_{i\,\mu}} \right]$$

- integral has exact dual conformal symmetry

$$\hat{K}^\mu I = 0$$

- very natural from string theory: $m$ corresponds to radial coordinate of $AdS_5$

# Implications for higher loop integral basis

- basis of loop integrals in $\mathcal{N}=4$ SYM constrained by dual conformal symmetry?

  [Drummond, J.H., Smirnov, Sokatchev, 2006; Bern, Czakon, Dixon, Kosower, Smirnov, 2006; Bern, Carrasco, Johansson, Kosower, 2007]

  [Drummond, Korchemsky, Sokatchev, '07; Nguyen, Spradlin Volovich, '07; Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich, '08]

  [Spradlin, Volovich, Wen, 2008]

- it seems reasonable to speculate that [J.H., Naculich, Schnitzer, Spradlin, 2010]

$$M_n = 1 + \sum_{\mathcal{I}} a^{L(\mathcal{I})}\, c(\mathcal{I})\, \mathcal{I}\,,$$

  where: coupling $a$, loop order $L(\mathcal{I})$
  coefficients $c(\mathcal{I}) \Rightarrow$ compute by (generalized) unitarity
  integrals $\mathcal{I} \Rightarrow$ restricted set of extended dual conformal integrals

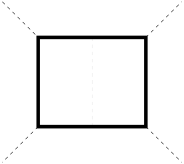
- additional constraints from expected IR structure

$$M_n = \exp\left[-\frac{1}{8}\Gamma_{\rm cusp}(a)\sum_i \log^2\frac{s_i}{m^2} - \frac{1}{2}\tilde{G}_0(a)\sum_i \log\frac{s_i}{m^2} + \mathcal{O}(\log^0 m^2)\right]$$

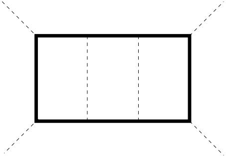
- insights from analytic structure for generic $m^2$, and Regge limit(s)?
- further constraints from the (broken) conventional conformal symmetry?

# Extended dual conformal invariance at higher loops

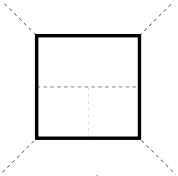
- At 2 loops: Only one integral is allowed by extended dual conformal symmetry:



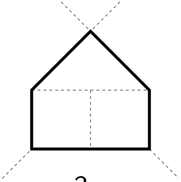
- At 3 loops: four integrals allowed:



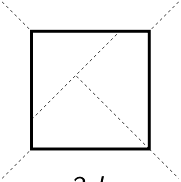
$3a$



$3b$



$3c$



$3d$

Similarly restricts integral basis at higher loops and legs.

$$(-8)M_3 = c_{3a} I_{3a} + c_{3b} I_{3b} + c_{3c} I_{3c} + c_{3d} I_{3d} + \{s \leftrightarrow t\}$$

in dimreg: $c_{3a} = 1$, $c_{3b} = 2$ and $c_{3c} = c_{3d} = 0$

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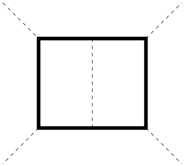

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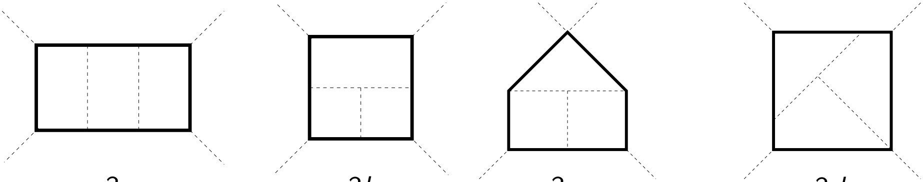

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# Two- and three-loop exponentiation

- analog of BDS in Higgs regularization:

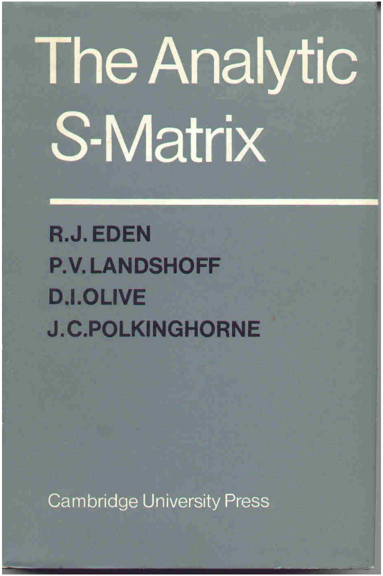
[Alday, J. H., Plefka, Schuster, 2009; J. H., Naculich, Schnitzer, Spradlin, 2010]

$$\log M_4 = -\frac{1}{4}\Gamma_{\rm cusp}(a)\left[\log^2\frac{s}{m^2}+\log^2\frac{t}{m^2}\right]-\tilde{G}_0(a)\left[\log\frac{s}{m^2}+\log\frac{t}{m^2}\right]$$
$$+\frac{1}{4}\Gamma_{\rm cusp}(a)\left[\log^2\frac{s}{t}+\pi^2\right]+\tilde{c}(a)+O(m^2)$$

- verified by computing dual conformal integrals up to $O(m^2)$
  - at two loops [Alday, J. H., Plefka, Schuster, 2009]
  - at three loops [J. H., Naculich, Schnitzer, Spradlin, 2010]

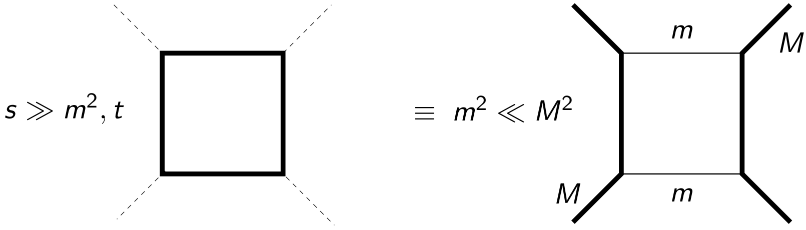
- method used: MB representation of all integrals, asymptotic expansion, numerical integration (Mathematica packages *MBasymptotics, MB*; *CUBA*)

# The analytic S-matrix



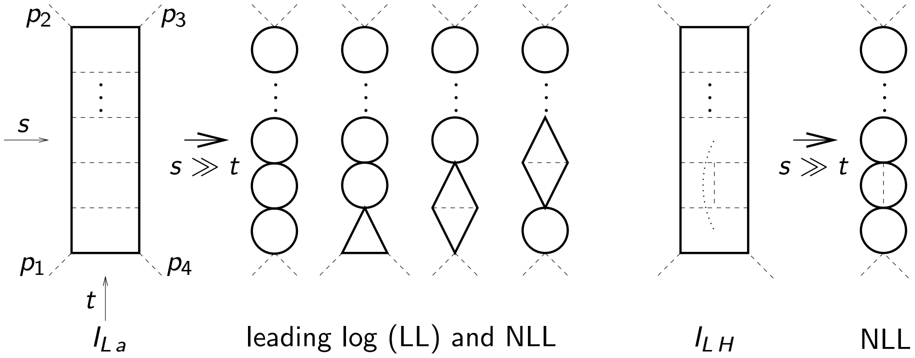
# Regge limits for amplitudes on the Coulomb branch

- take Regge limit $s = (p_1 + p_2)^2 \to \infty$ [J. H., Naculich, Schnitzer, Spradlin, 2010]
  expect
  $$\beta(t/m^2) \left(\frac{s}{m^2}\right)^{\alpha(t/m^2)} + \mathcal{O}(m^2)$$
  trajectory $\alpha(t/m^2) = -\frac{1}{2}\Gamma_{\rm cusp}(a) \log(t/m^2) - \tilde{\mathcal{G}}_0(a)$
- Regge limit is simpler here compared to dimensional regularization
- dual conformal symmetry implies:
  Regge limit $s \gg m^2, t$ equivalent to $m^2 \ll M^2$ in "Bhabha-type" scattering




- determine leading Regge behavior of integrals [Eden et al, The analytic S-matrix]

# LL and NLL Regge limit to all loop orders




**Regge limit to all loop orders:**

- LL : ladder integrals
- NLL and LL : ladders and ladders with one H-shaped insertion

  [J. H., Naculich, Schnitzer, Spradlin, to appear]

- in contrast, in dimensional regularization, many different diagrams contribute

# Summary

- Higgs IR regulator for planar $\mathcal{N} = 4$ SYM
  - makes dual conformal symmetry exact
  - restricts integral basis
  - exponentiation of amplitude easier to compute
  - Regge limit: LL and NLL computed to all loop orders