Exponential suppression with 4 legs and an infinity of loops

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Abstract: The L-loop 4-point ladder diagram of massless ϕ^3 theory is finite when all 4 legs are off-shell and is given in terms of polylogarithms with orders ranging from L to $2L$. We obtain the exact solution of the linear Dyson–Schwinger equation that sums these ladder diagrams and show that this sum vanishes exponentially fast at strong coupling.

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Results for two-loop 3-point and 4-point ladder diagrams were obtained in [1]. In [2], results were found for an arbitrary number of loops, L. These were confirmed in [3], using Gegenbauer-polynomial methods. Here we shall sum the 4-point ladders of Figure 1b.

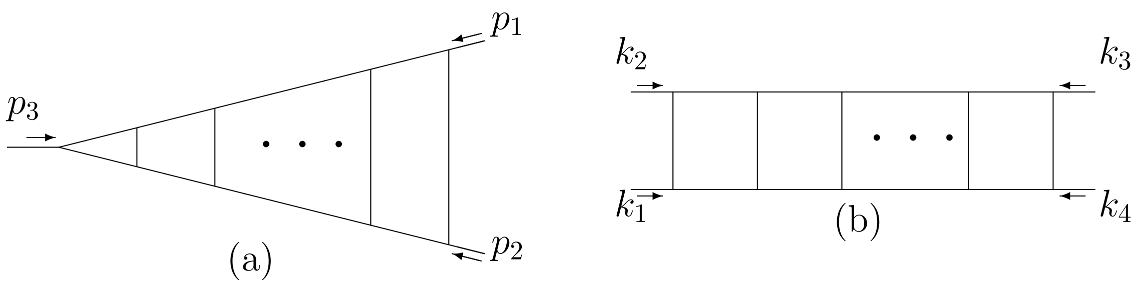

Figure 1: (a) 3-point, and (b) 4-point, L-loop diagrams in ϕ^3 theory

With massless internal propagators, each diagram gives a finite real contribution for positive values of the 6 kinematic invariants k_1^2, k_2^2, k_3^2, k_4^2, $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$. We shall investigate the strong-coupling limit of the exact solution to a linear Dyson-Schwinger equation, of the schematic form $\mathcal{D} = \mathcal{T} + g^2 \int \mathrm{d}^4k\ \mathcal{T} \cdot \mathcal{D}$, which sums these ladder diagrams. Here, $\mathcal{T}$ is the t-channel tree-diagram, which we normalize to $1/t$, and the dot indicates convolution under the 4-dimensional integration that adds another loop.

We offer two motivations for this investigation. First, it was shown in [3] that the ladder diagrams contributing to the derivative of the self-energy energy of massless ϕ^3 theory sum to give the constant $-\frac{1}{2}\zeta(-1) = \frac{1}{24}$ at infinite coupling. As far back as 1993, we suspected that by including the tree-diagram $\mathcal{T}$ in $\mathcal{D}$ we would obtain zero for the sum of 4-point ladder diagrams at infinite coupling. It has taken us 17 years to prove that this indeed is the case. Secondly, and more recently, we have noted that ladder approximations are of interest to workers in $N = 4$ super Yang–Mills theory, whose strong coupling limit may be governed by an AdS/CFT correspondence. We hope that it may be of interest to colleagues to see the explicit form of a 4-point ladder sum, as a function of the 6 kinematic invariants and the coupling g^2, which also has the dimensions of $(\text{mass})^2$ in ϕ^3 theory.

It was a pleasant surprise to us to obtain, eventually, the solution to this toy problem as a single integral of elementary functions that manifestly vanishes exponentially fast as the dimensionless coupling $g^2/(4\pi^2 s)$ tends to infinity. The polylogarithmic complexity of perturbation theory is in marked contrast to the simplicity of the all-orders result that we shall now derive.

We write the perturbation series of ladder diagrams as

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{t}\left\{1 + \sum_{L=1}^{\infty}\left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X, Y)\right\} \tag{1}$$

with dimensionless ratios

$$X \equiv \frac{k_1^2 k_3^2}{st}, \quad Y \equiv \frac{k_2^2 k_4^2}{st}, \quad \kappa^2 \equiv \frac{g^2}{4\pi^2 s} \tag{2}$$

that we assume to be positive. As shown in [2], the L-loop term

$$\Phi^{(L)}(X, Y) = -\frac{1}{L!\,(L-1)!}\int_0^1 \frac{\mathrm{d}\xi}{Y\xi^2 + (1 - X - Y)\xi + X} \times \left(\ln\xi \ln\frac{Y}{X} + \ln^2\xi\right)^{L-1}\left(\ln\frac{Y}{X} + 2\ln\xi\right) \tag{3}$$

depends only on the cross ratios X and Y. The origin of this simplification was elucidated in [3], which gave the conformal transformation that relates Figure 1b to Figure 1a. When scaled by an appropriate power of p_3^2, the latter depends only on the ratios $x = p_1^2/p_3^2$ and $y = p_2^2/p_3^2$ and is given by $\Phi^{(L)}(x, y)$.

The integral (3) may be evaluated in terms of polylogarithms [2]. Here, we shall consider the case where the Källen function

$$\mu = \sqrt{4XY - (X + Y - 1)^2} \tag{4}$$

is real and positive. Then we are comfortably outside the region that contains Landau singularities and hence may define the geometrical angle [2]

$$\phi = \arccos\left(\frac{X + Y - 1}{\sqrt{4XY}}\right) \tag{5}$$

with $\pi > \phi > 0$. In this region, the L-loop term [2]

$$\Phi^{(L)}(X,Y) = \frac{2}{\mu L!} \sum_{j=L}^{2L} \frac{j!}{(j-L)!\,(2L-j)!} \left(\ln\frac{X}{Y}\right)^{2L-j} \Im \mathrm{Li}_j\left(\sqrt{\frac{Y}{X}}\,\exp(\mathrm{i}\phi)\right) \tag{6}$$

is given in terms of products of powers $\ell \equiv \ln(X/Y)$ and the imaginary parts of polylogarithms Li_j with orders running from $j = L$ to $j = 2L$. The symmetry $\Phi^{(L)}(X,Y) = \Phi^{(L)}(Y,X)$ is ensured by the inversion formula for polylogarithms, given in [6].

In the first instance we omit the tree term and use the integral representation (3) to sum the series

$$\sum_{L=1}^{\infty}\left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y) = \frac{\kappa}{2}\int_0^1 \frac{d\xi}{X+(1-X-Y)\xi+Y\xi^2}\left(\ln\frac{Y}{X}+2\ln\xi\right) \times \frac{1}{\sqrt{\ln\xi\left(\ln\frac{Y}{X}+\ln\xi\right)}} J_1\left(\kappa\sqrt{\ln\xi\left(\ln\frac{Y}{X}+\ln\xi\right)}\right) \quad (7)$$

where J_1 is a Bessel function. Substituting $\xi = e^{-\eta}$ and denoting $\ell \equiv \ln\frac{X}{Y}$, we obtain

$$\sum_{L=1}^{\infty}\left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y) = -\frac{\kappa}{2}\int_0^\infty \frac{e^{-\eta}\,d\eta}{X+(1-X-Y)e^{-\eta}+Ye^{-2\eta}} \times \frac{2\eta+\ell}{\sqrt{\eta(\ell+\eta)}} J_1\left(\kappa\sqrt{\eta(\ell+\eta)}\right). \quad (8)$$

The denominator in (8) may be re-written as

$$\begin{aligned} X + (1 - X - Y)e^{-\eta} + Ye^{-2\eta} &= e^{-\eta}\left[1 - X - Y + 2\sqrt{XY}\cosh\left(\eta + \frac{\ell}{2}\right)\right] \\ &= -2\sqrt{XY}e^{-\eta}\left[\cos\phi - \cosh\left(\eta + \frac{\ell}{2}\right)\right]. \end{aligned} \tag{9}$$

In this way, we arrived at

$$\sum_{L=1}^{\infty}\left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y) = \frac{\kappa}{4\sqrt{XY}}\int_0^{\infty}\frac{\mathrm{d}\eta}{\cos\phi - \cosh\left(\eta + \frac{\ell}{2}\right)}\,\frac{2\eta + \ell}{\sqrt{\eta(\ell + \eta)}}J_1\left(\kappa\sqrt{\eta(\ell + \eta)}\right) \tag{10}$$

and obtained, in 1993, an explicit summation of all 4-point ladder diagrams with loop numbers $L > 0$. Yet we could find no way of investigating our hunch that inclusion of the tree diagram, with $L = 0$, might give an exponentially vanishing result at infinitely strong coupling.

The first break-through came from noticing that

$$\frac{2\eta+\ell}{\sqrt{\eta(\ell+\eta)}}\, J_1\left(\kappa\sqrt{\eta(\ell+\eta)}\right) = -\frac{2}{\kappa}\,\frac{\mathrm{d}}{\mathrm{d}\eta} J_0\left(\kappa\sqrt{\eta(\ell+\eta)}\right). \tag{11}$$

Then, integrating by parts, we found that the Dyson–Schwinger solution is

$$\mathcal{D}(k_1^2,k_2^2,k_3^2,k_4^2,s,t) = \frac{1}{2t\sqrt{XY}} \int_0^\infty \mathrm{d}\eta\, \frac{\sinh\left(\eta+\frac{\ell}{2}\right) J_0\left(\kappa\sqrt{\eta(\ell+\eta)}\right)}{\left[\cosh\left(\eta+\frac{\ell}{2}\right)-\cos\phi\right]^2} \tag{12}$$

where the tree-term $1/t$ is precisely included by the surface term of the partial integration enabled by (11). Our hopes had increased: the full Dyson-Schwinger solution (12) is now presented in a form that looks more promising for confirmation of our guess of exponential suppression at strong coupling.

Next, we shift the integration variable η and obtain

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{2t\sqrt{XY}} \int\limits_{\ell/2}^{\infty} \mathrm{d}\eta \, \frac{\sinh\eta \; J_0\Big(\kappa\sqrt{\eta^2 - \frac{1}{4}\ell^2}\Big)}{(\cosh\eta - \cos\phi)^2}. \tag{13}$$

The $X \longleftrightarrow Y$ symmetry of expression (13) is now quite easy to understand. If we were to interchange X and Y, then the only thing that would change is the lower limit of integration: $\ell/2 \to -\ell/2$, since $\phi \equiv \arccos((X+Y-1)/\sqrt{4XY})$ is symmetric in (X, Y). The integral between $-\ell/2$ and $\ell/2$ is zero, since the integrand is an odd function of η and an even function of $\ell \equiv \ln(X/Y)$. Hence we may take $\frac{1}{2}|\ell| = \frac{1}{2}|\ln X - \ln Y|$ as the lower limit of integration in (13).

We re-write (13) as

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{2t\sqrt{XY}} \int\limits_0^\infty \frac{\mathrm{d}\eta\ \sinh\eta}{(\cosh\eta - \cos\phi)^2}\ J_0\Big(\kappa\sqrt{\eta^2 - \tfrac{1}{4}\ell^2}\Big)\, \vartheta\left(\eta^2 - \tfrac{1}{4}\ell^2\right), \quad (14)$$

where $\vartheta(x)$ is the Heaviside function, with $\vartheta(x) = 1$, for $x > 1$, and $\vartheta(x) = 0$, otherwise. Now, let us use the integral representation

$$\int\limits_0^\infty \mathrm{d}\tau\ \sin(\kappa\eta\cosh\tau)\ \cos\left(\tfrac{1}{2}\ell\kappa\sinh\tau\right) = \frac{\pi}{2}\ J_0\Big(\kappa\sqrt{\eta^2 - \tfrac{1}{4}\ell^2}\Big)\ \vartheta\left(\eta^2 - \tfrac{1}{4}\ell^2\right) \quad (15)$$

which may be obtained from Equation (2.5.25.9) of [4] (with the substitutions $x = \kappa\sinh\tau$, $y = \kappa$, $c = \eta$, and $b = \frac{1}{2}\ell$). The key point is that we are rid of the integration limit $\ell/2$.

By this device, we obtain

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{\pi t\sqrt{XY}} \int\limits_0^\infty \frac{\mathrm{d}\eta \ \sinh\eta}{(\cosh\eta - \cos\phi)^2} \int\limits_0^\infty \mathrm{d}\tau \ \sin(\kappa\eta\cosh\tau) \ \cos\left(\tfrac{1}{2}\ell\kappa\sinh\tau\right) \tag{16}$$

as a double integral. Next, the substitution $z = \kappa\cosh\tau$ gives $\kappa\sinh\tau = \sqrt{z^2-\kappa^2}$ and $\mathrm{d}\tau = \mathrm{d}z/\sqrt{z^2-\kappa^2}$. Hence we obtain

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{\pi t\sqrt{XY}} \int\limits_0^\infty \frac{\mathrm{d}\eta \ \sinh\eta}{(\cosh\eta - \cos\phi)^2} \int\limits_\kappa^\infty \frac{\mathrm{d}z \ \sin(\eta z)}{\sqrt{z^2-\kappa^2}} \ \cos\left(\tfrac{1}{2}\ell\sqrt{z^2-\kappa^2}\right). \tag{17}$$

Now we reverse the order of the integrations, obtaining

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{\pi t\sqrt{XY}} \int\limits_{\kappa}^{\infty} \frac{\mathrm{d}z}{\sqrt{z^2-\kappa^2}} \; \cos\left(\tfrac{1}{2}\ell\sqrt{z^2-\kappa^2}\right) \int\limits_{0}^{\infty} \frac{\mathrm{d}\eta \; \sinh\eta \; \sin(\eta z)}{(\cosh\eta - \cos\phi)^2} \; . \tag{18}$$

From Equation (2.5.48.18) of [4] (with $t = \pi - \phi$, $c = 1$, $b = z$), we obtain

$$\int\limits_{0}^{\infty} \frac{\mathrm{d}\eta \; \sinh\eta \; \sin(\eta z)}{(\cosh\eta - \cos\phi)^2} = \frac{\pi z}{\sin\phi} \; \frac{\sinh\left[(\pi-\phi)z\right]}{\sinh(\pi z)} . \tag{19}$$

Recalling that $\mu = 2\sqrt{XY}\sin\phi$, we obtain

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{2}{t\mu}\int_\kappa^\infty \frac{z\,\mathrm{d}z}{\sqrt{z^2-\kappa^2}}\,\frac{\sinh\left[(\pi-\phi)z\right]}{\sinh(\pi z)}\,\cos\left(\tfrac{1}{2}\ell\sqrt{z^2-\kappa^2}\right)\,. \tag{20}$$

This is our final solution to the Dyson-Schwinger equation that sums all L-loop 4-point ladder diagrams, including (most crucially) the tree-diagram, with $L = 0$ loops. The sum manifestly vanishes, exponentially fast, as the dimensionless coupling $\kappa = g/(2\pi\sqrt{s})$ tends to infinity, since the ratio of sinh functions in the integrand of (20) satisfies

$$\frac{\sinh\left[(\pi-\phi)z\right]}{\sinh(\pi z)} \le \frac{\sinh\left[(\pi-\phi)\kappa\right]}{\sinh(\pi\kappa)} = O(\exp(-\kappa\phi)) \tag{21}$$

with $\pi > \phi > 0$.

So we are done, 17 years after conjecturing such an exponential suppression.

Our actual route to this final answer bears scant relation to the more coherent explanation offered here. After many fruitful exchanges of ideas, the first author guessed the final result, by means far too involved to be recounted here, and then the second author neatly devised a process of reverse-engineering that resulted in the proof presented here, via formulas presented in [4, 5].

It is not clear to either of us whether our explicit all-orders summation of 4-point ladder diagrams may still hold some interest for the loops-and-legs community that has nurtured our efforts. Yet we hope that it might. In any case, it was fun to achieve.

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