NEXT-TO-EIKONAL EXPONENTIATION

Eric Laenen

EL, L. Magnea, G. Stavenga, Phys. Lett.B 669 (2008) 173
EL, G. Stavenga, C. White, JHEP 0903: 054 (2009)
EL, L. Magnea, G. Stavenga, C. White, to appear

Universiteit van Amsterdam
NIKHEF

Loops and Legs, Woerlitz, April 25-30, 2010

OUTLINE

- Introduction
- Extended threshold resummation
- Next-to-eikonal exponentation for matrix elements
  - Path-integral methods
  - Diagrams and induction
- Conclusions

LARGE X BEHAVIOR

- For DY, DIS, Higgs, singular behavior when x→ 1

$$\delta(1-x) \qquad \left[\frac{\ln^i(1-x)}{1-x}\right] \qquad \ln^k(1-x)$$

  - singularity structure for plus distributions is organizable to all orders, perhaps also for divergent logarithms?

- After Mellin transform

$$\text{Constants} \qquad \ln^i(N) \qquad \frac{\ln^k(N)}{N}$$

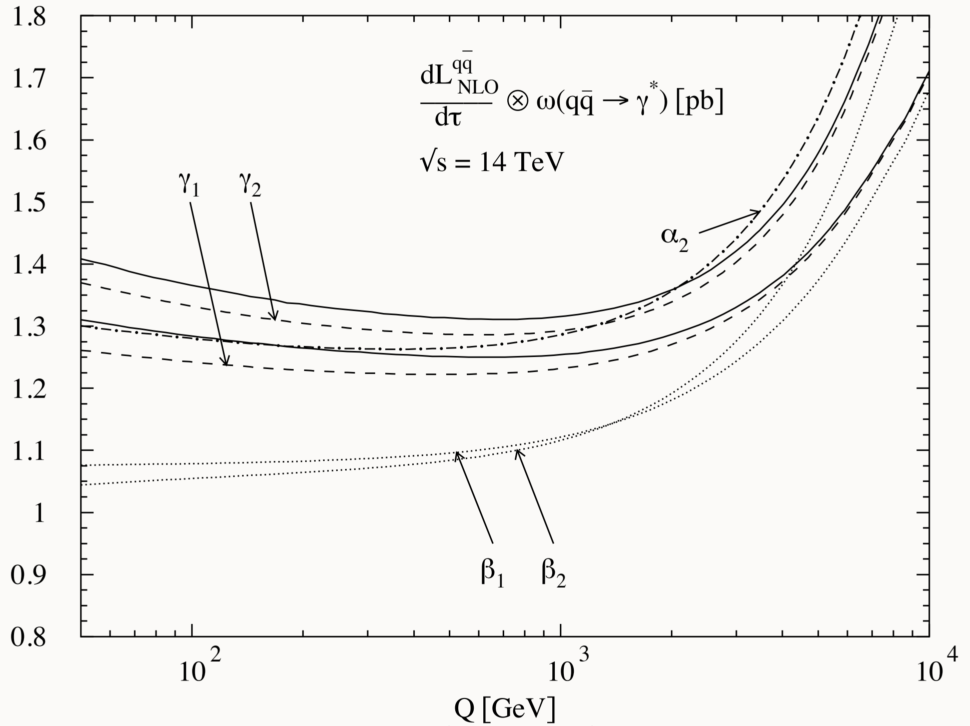
- We know a lot about logs and constants, very little about 1/N

LN(N)/N TERMS

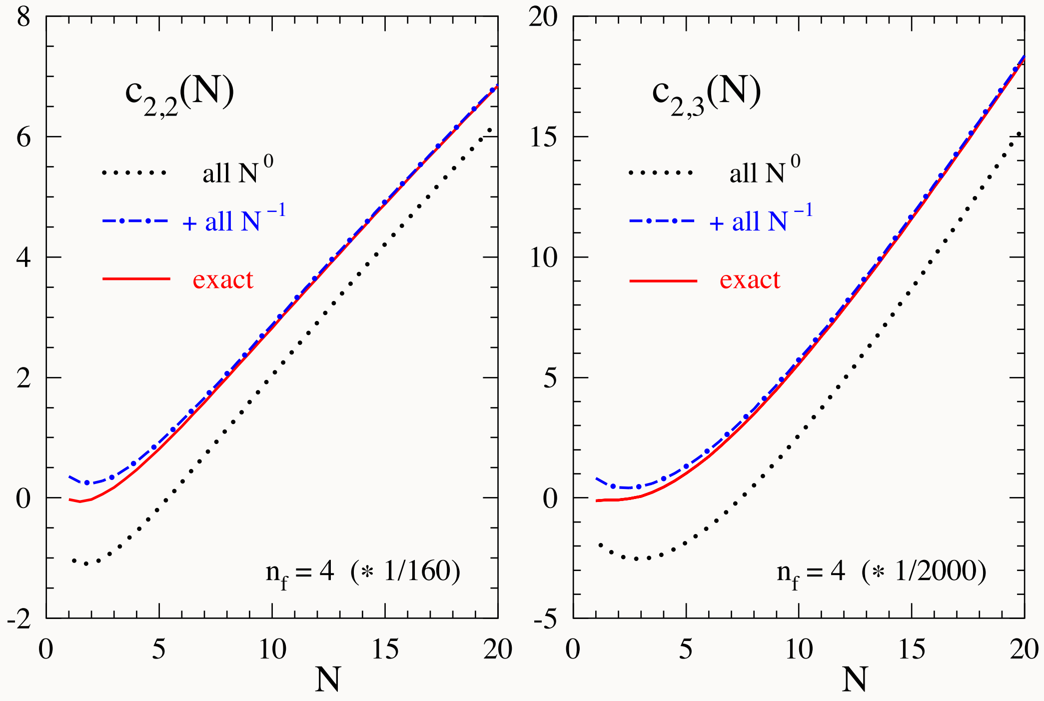
Kraemer, EL, Spira; Catani, De Florian, Grazzini; Kilgore, Harlander

- Can be numerically important

Kraemer, EL, Spira



Moch,Vogt



- We know that the leading series $\ln^i(N)/N$ exponentiates
  - by replacing in resummation formula

$$\frac{1+z^2}{1-z} \longrightarrow \frac{2}{1-z} - 2$$

SUCCESFUL LN(N)/N ORGANIZATION

Dokshitzer, Marchesini, Salam
Basso, Korchemsky

$$\gamma_{qq}(N) = A(\alpha_s)\ln N + B(\alpha_s) + C(\alpha_s)\frac{\ln N}{N} + \ldots$$

- Moch, Vermaseren, Vogt noted an remarkable relation

$$C_2 = A_1^2 \qquad C_3 = 2A_2A_1$$

- DMS reproduced this by changing DGLAP equation

$$\mu^2\frac{\partial}{\partial\mu^2}\psi(x,\mu^2) = \int_x^1 \frac{dz}{z}\,\psi\left(\frac{x}{z}, z\mu^2\right)\mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right)$$

$$\mathcal{P}\left(z,\alpha_s\right) = \frac{A\left(\alpha_s\right)}{(1-z)_+} + B_\delta\left(\alpha_s\right)\delta(1-z) + \mathcal{O}\left((1-z)\right)$$

- Can this be reproduced in threshold resummation?

EXTENDED THRESHOLD RESUMMATION

Modified resummed expression

$$\ln\left[\sigma(N)\right] = \mathcal{F}_{\rm DY}\left(\alpha_s(Q^2)\right) + \int_0^1 dz\, z^{N-1}\left\{\frac{1}{1-z} D\left[\alpha_s\left(\frac{(1-z)^2Q^2}{z}\right)\right]\right.$$
$$\left. + 2\int_{Q^2}^{(1-z)^2Q^2/z} \frac{dq^2}{q^2} P_s\left[z, \alpha_s(q^2)\right]\right\}_+$$

where

$$P_s^{(n)}(z) = \frac{z}{1-z}A^{(n)} + C_\gamma^{(n)}\ln(1-z) + \overline{D}_\gamma^{(n)}$$

(We constructed a similar expression for DIS). Structure:

$$\sigma(N) = \sum_{n=0}^{\infty}\left(g^2\right)^n\left[\sum_{m=0}^{2n} a_{nm}\ln^m N + \sum_{m=0}^{2n-1} b_{nm}\frac{\ln^m N}{N}\right] + \mathcal{O}\left(N^{-2}\right)$$

	C_F^2		C_AC_F		n_fC_F	
b_{23}	4	4	0	0	0	0
b_{22}	$\frac{7}{2}$	4	$\frac{11}{6}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$8\zeta_2 - 11$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$	$-\frac{11}{9}$	$-\frac{11}{9}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{275}{216}$	$\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$	$-\frac{19}{27}$	$-\frac{2}{3}\zeta_2 + \frac{7}{27}$

EXTENDED THRESHOLD RESUMMATION

DIS

	C_F^2		C_AC_F		n_fC_F	
d_{23}	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
d_{22}	$\frac{39}{16}$	$\frac{55}{16}$	$\frac{11}{48}$	$\frac{11}{48}$	$-\frac{1}{24}$	$-\frac{1}{24}$
d_{21}	$\frac{7}{4}\zeta_2 - \frac{49}{32}$	$-\frac{1}{4}\zeta_2 - \frac{105}{32}$	$-\frac{5}{4}\zeta_2 + \frac{1333}{288}$	$-\frac{1}{4}\zeta_2 + \frac{565}{288}$	$-\frac{107}{144}$	$-\frac{47}{144}$
d_{20}	$\frac{15}{4}\zeta_3 - \frac{47}{16}\zeta_2 - \frac{431}{64}$	$-\frac{3}{4}\zeta_3 + \frac{53}{16}\zeta_2 - \frac{21}{64}$	$-\frac{11}{4}\zeta_3 + \frac{13}{48}\zeta_2 - \frac{17579}{1728}$	$\frac{5}{4}\zeta_3 + \frac{7}{16}\zeta_2 - \frac{953}{1728}$	$\frac{1}{24}\zeta_2 - \frac{1699}{864}$	$-\frac{1}{8}\zeta_2 + \frac{73}{864}$

Almost works, but not quite. Similar at 3 loop.

Even more general approach by Grunberg, Ravindran. Does not work fully either.

Other approach, talk by A. Vogt

We must go beyond the eikonal approximation

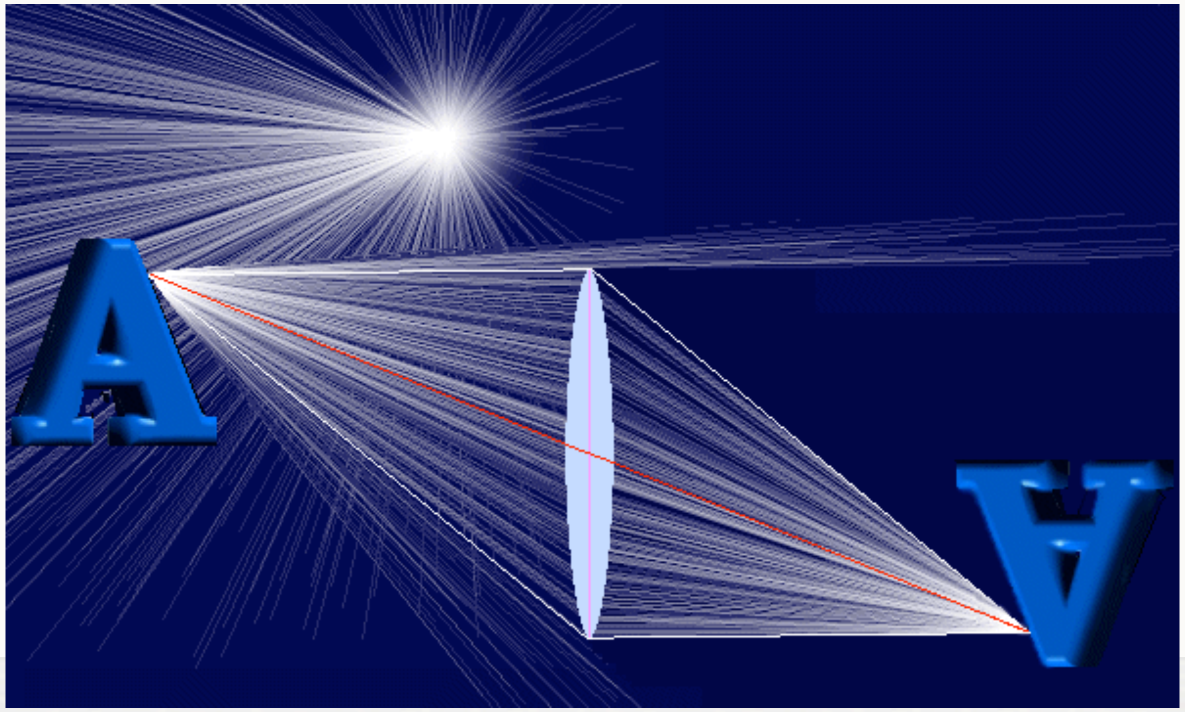
Investigate in more detail, for amplitudes

HISTORY OF EIKONAL APPROXIMATION

- “Eikon” originally from Greek εικεναι [to resemble]
  - leading to εικον [icon, image]
- Predates quantum mechanics, and even Maxwell
  - also known in optics as “ray optics”
    - Rays are straight lines, perpendicular to wave fronts

RAY OPTICS

- Can describe formation of images/eikons
  - wavelength << size of scatterer
- Cannot describe diffraction, polarization etc
  - these are wave phenomena

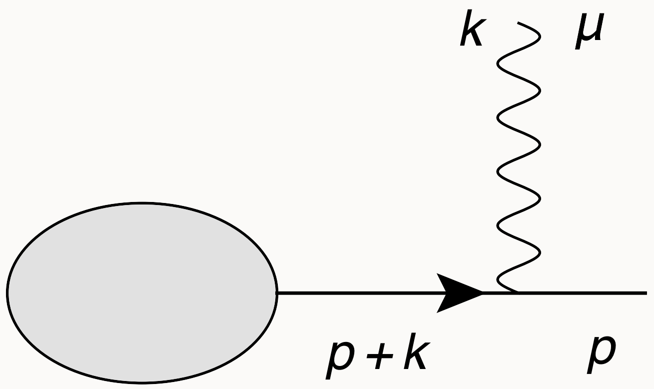


EIKONAL APPROXIMATION IN QFT

- At amplitude level
  - Reveals new symmetries, new structures in gauge theory
  - Intuitive interpretation
  - Practical
    - Coherence, resummation, EFT, ...

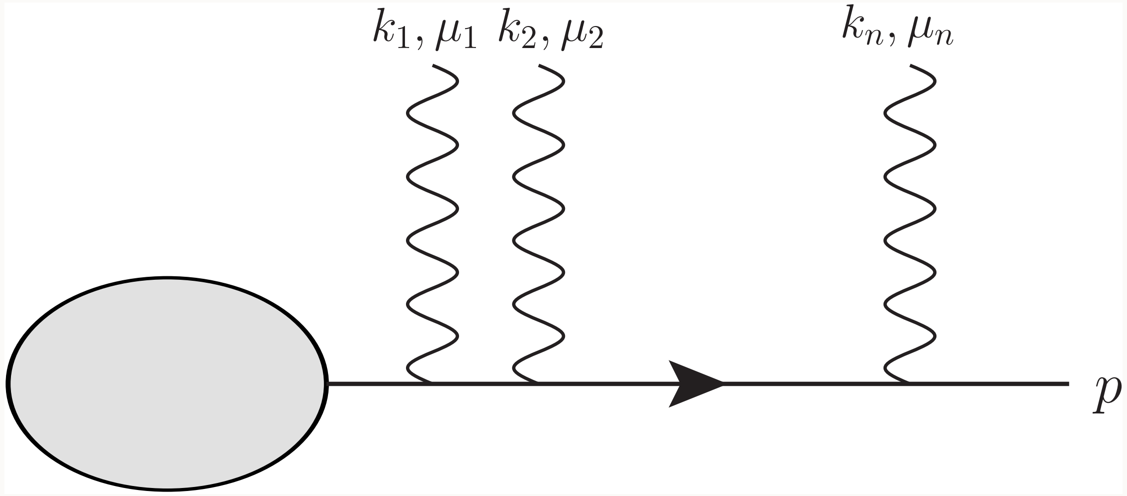
BASICS, QED

- Soft emission by charged particle
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order



$$\frac{(p+k)^\mu + p^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k}$$

BASICS QED, CONT'D



Exact:
$$\frac{1}{(p+K_1)^2}(2p+K_2+K_1)^{\mu_1} \dots \frac{1}{(p+K_n)^2}(2p+K_n)^{\mu_n}, \quad K_i = \sum_{m=i}^{n} k_m.$$

Approx:
$$\frac{1}{2pK_1}2p^{\mu_1} \dots \frac{1}{2pK_n}2p^{\mu_n}$$

Eikonal identity:
$$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2} + \frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1} = \frac{1}{p\cdot k_1\,p\cdot k_2}$$

Sum over all perm's:
$$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Independent, uncorrelated emissions, Poisson process

NON-ABELIAN EIKONAL APPROXIMATION

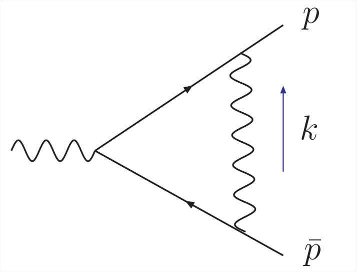
- Same methods as for QED, but organization harder: SU(3) generator at every vertex
  - no obvious decorrelation

Order the T_a according to λ

- Key "object": Wilson line $\Phi_n(\lambda_2, \lambda_1) = \overset{\swarrow}{P} \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A^a(\lambda n)\, T_a\right]$
  - Order by order in "g", it generates QCD eikonal Feynman rules

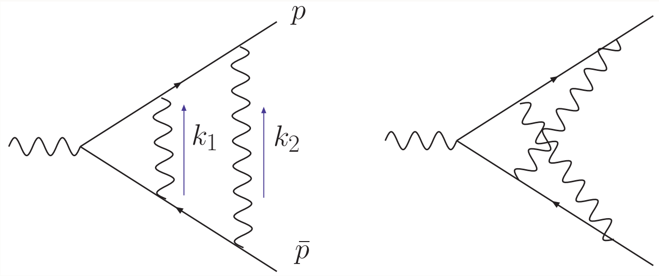
EXPONENTIATON

One loop vertex correction, in eikonal approximation



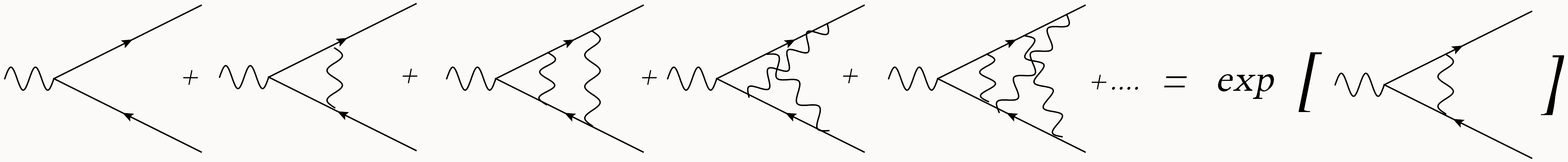
$$\mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation



$$\mathcal{A}_0 \, \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

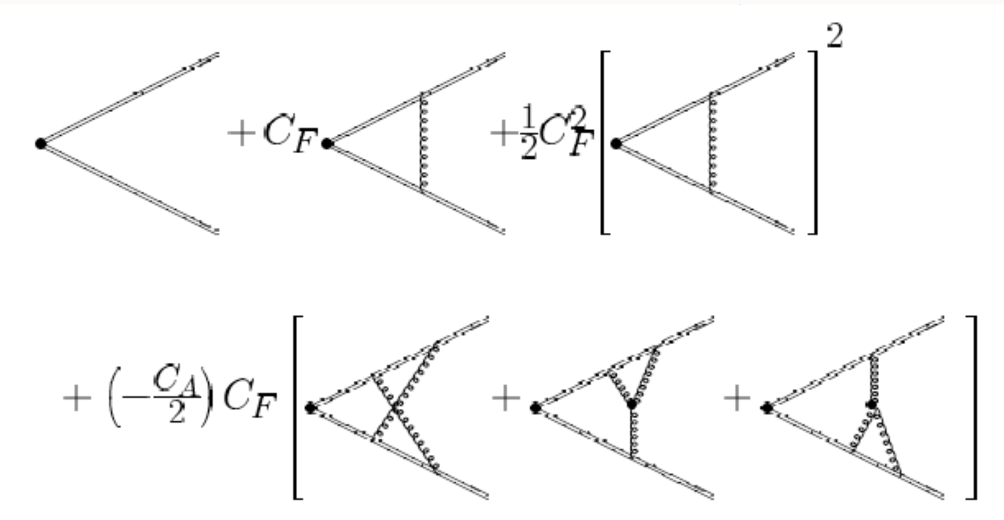
Exponential series



NON-ABELIAN EXPONENTIATION: WEBS

Gatheral; Frenkel, Taylor; Sterman

- Take quark - antiquark line, connect with soft gluons in all possible ways, use eikonal approximation
- Exponentiation still occurs, without path ordering!
  - A selection of diag d color weights: “webs”



  - Prove by induction; recursive definition of color weights
- How can we extend this to include next-to-eikonal terms?

PATH INTEGRAL METHOD

EL, Stavenga, White

Represent propagator as particle path integral, between coord. and momentum states

$$\tilde{\Delta}_F(p_f^2) = \frac{1}{2}\int_0^\infty dT\, \frac{\langle p_f|U(T)|x_i\rangle}{\langle p_f|x_i\rangle} = -\frac{i}{p_f^2+m^2-i\varepsilon}$$

where

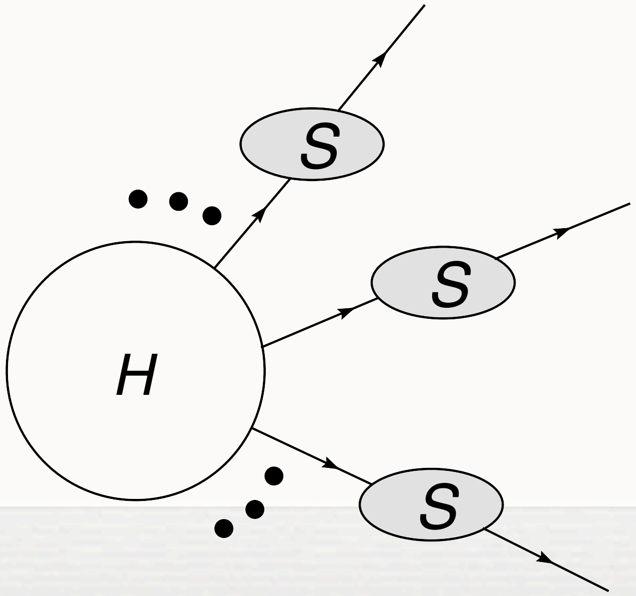
$$\langle p_f|U(T)|x_i\rangle = e^{-ip_f x_i - i\frac{1}{2}(p_f^2+m^2)T}\int_{x(0)=0}^{p(T)=0}\mathcal{D}p\mathcal{D}x\, e^{i\int_0^T dt(p\dot{x}-\frac{1}{2}p^2)}$$

with a gauge field

$$\langle p_f|U(T)|x_i\rangle = \int_{x(0)=x_i}^{p(T)=p_f}\mathcal{D}p\mathcal{D}x\, \exp\Big[-ip(T)x(T)+i\int_0^T dt(p\dot{x}-\frac{1}{2}(p^2+ m^2)+p\cdot {A}+\frac{i}{2}\partial\cdot {A}-\frac{1}{2}{A}^2)\Big]$$

n-point Green’s function

$$G(p_1,\ldots,p_n) = \int \mathcal{D}A_s^\mu\, H(x_1,\ldots,x_n)$$
$$\times \langle p_1|((p-A_s)^2-i\varepsilon)^{-1}|x_1\rangle\ldots\langle p_n|(p-A_s)^2-i\varepsilon)^{-1}|x_n\rangle$$



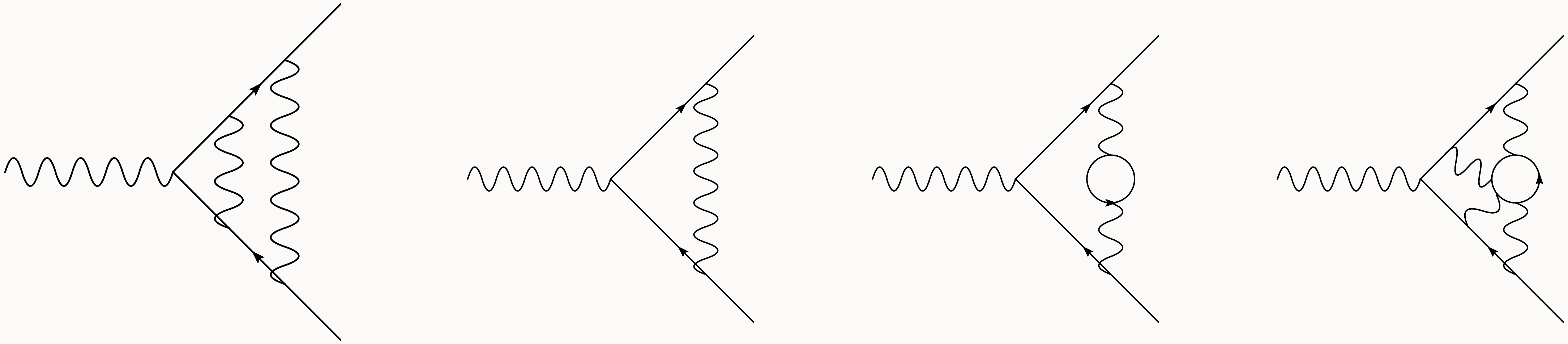
PATH INTEGRAL METHOD

Truncate external lines for S-matrix element $i(p_f^2+m^2)\langle p_f| - i((p-A)^2 - i\varepsilon)^{-1}|x_i\rangle = e^{-ip_f x_i} f(\infty)$

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A_s^\mu H(x_1,\ldots,x_n)\, e^{-ip_1x_1} f_1(\infty) \ldots e^{-ip_nx_n} f_n(\infty)\, e^{iS[A_s]}$$

$$f(\infty) = \int_{x(0)=0} \mathcal{D}x\, e^{i\int_0^\infty dt\left(\frac{1}{2}\dot{x}^2 + (p_f+\dot{x})\cdot A(x_i+p_ft+x(t)) + \frac{i}{2}\partial\cdot A(x_i+p_ft+x)\right)}$$

Eikonal vertices act as sources for gauge bosons along path



Disconnected

Connected

QED: exponentiation now textbook result:
all diagrams = exp (connected diagrams)

REPLICA TRICK

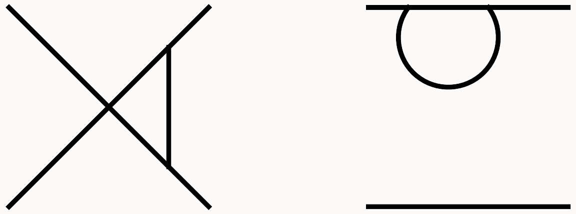
- Can relate exponentiation of soft gauge fields to that of connected diagrams in QFT. Proof: replica trick (from stat. mech.)
- Consider a N copies of a scalar theory

$$Z[J]^N = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_N e^{iS[\phi_1]+..+iS[\phi_N]+J\phi_1+..J\phi_N}$$

  - If Z is exponential, find out what contributes to log Z

$$Z^N = 1 + N \log Z + \mathcal{O}(N^2)$$

  - Amounts to diagrams that allow only one replica → connected!



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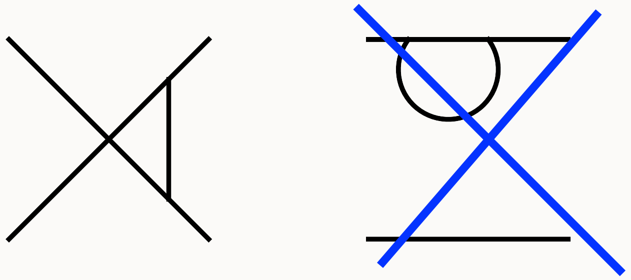
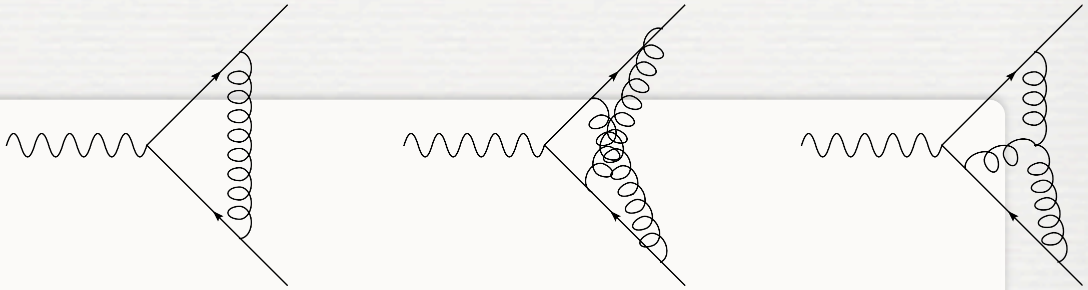


APPLICATION TO QCD



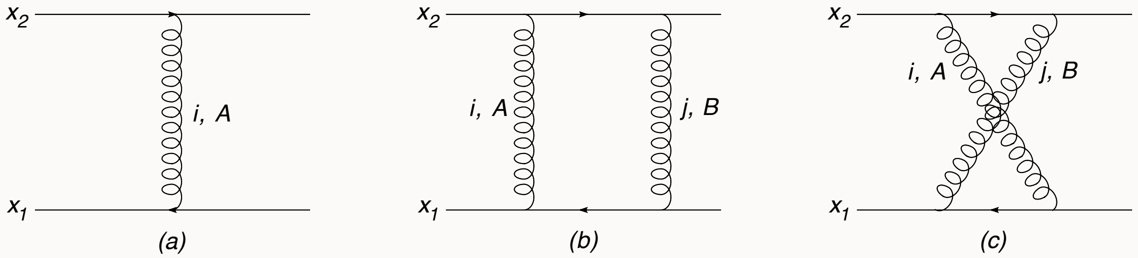
Amplitude for two colored lines

$$S(p_1,p_2) = H(p_1,p_2)\int \mathcal{D}A_s f(\infty) e^{iS[A_s]}$$

Replicate, and introduce ordering operator

$$f(\infty) = \mathcal{P}\exp\left[\int dx\cdot A(x)\right] \qquad \prod_{i=1}^{N}\mathcal{P}\exp\left[\int dx\cdot A_i(x)\right] = \mathcal{R}\mathcal{P}\exp\left[\sum_{i=1}^{N}\int dx\cdot A_i(x)\right]$$

Look for diagrams of replica order N. These will go into exponent



(a) (b) (c)

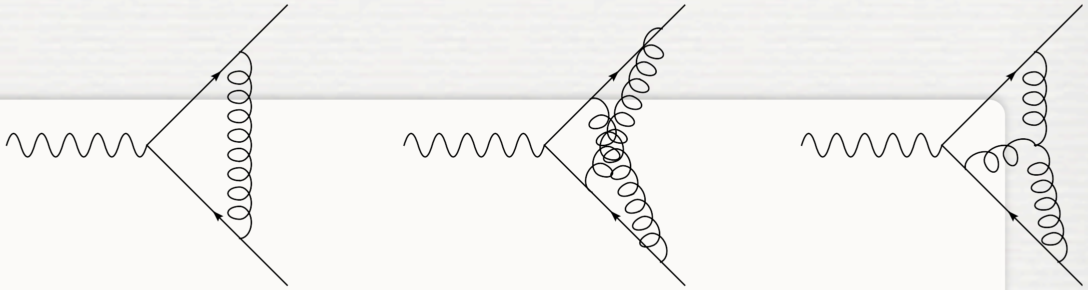
(a) is order N

(b) for equal replica number (i=j): C_F^2. For i≠j also C_F^2. Sum: $NC_F^2 + N(N-1)C_F^2 = N^2C_F^2$

(c) for equal replica number (i=j): $C_F^2 - C_F C_A/2$.

For i≠j C_F^2. Term linear in N: $N\left(C_F^2 - \frac{C_F C_A}{2}\right) + (-N)C_F^2 = N\left(-\frac{C_F C_A}{2}\right)$

APPLICATION TO QCD



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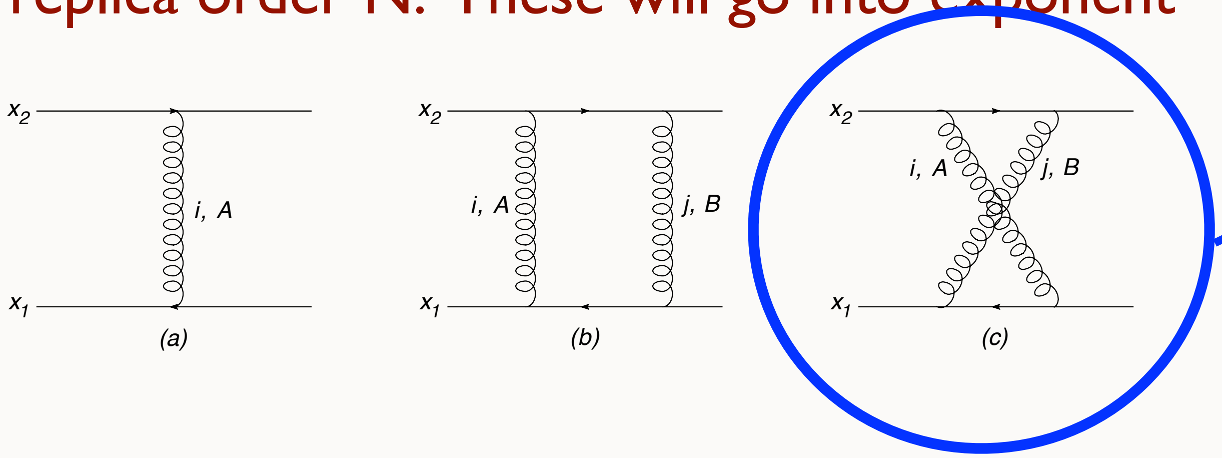


Web
Modified color factor

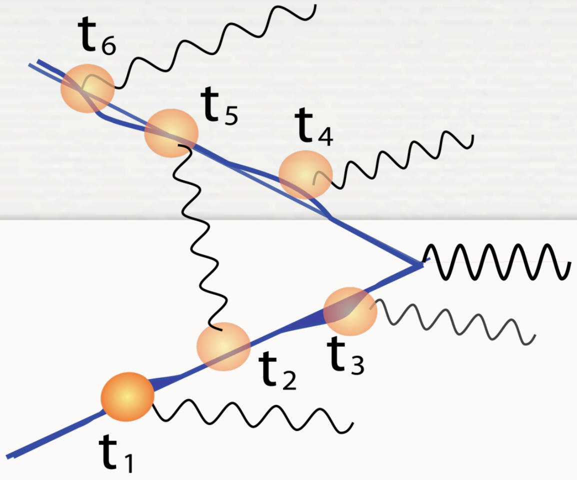
(a) is order N

(b) for equal replica number (i=j): C_F^2. For i≠j also C_F^2. Sum: $NC_F^2 + N(N-1)C_F^2 = N^2 C_F^2$

(c) for equal replica number (i=j): C_F^2-C_F C_A /2.
For i≠j C_F^2. Term linear in N:

$$N\left(C_F^2 - \frac{C_F C_A}{2}\right) + (-N)C_F^2 = N\left(-\frac{C_F C_A}{2}\right)$$

NEXT-TO-EIKONAL



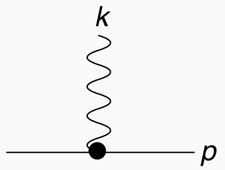
- Wilson lines are classical solutions of path integral
- Fluctuations around classical path at NE corrections
  - This class of NE corrections exponentiates
  - Keep track via scaling variable λ $p^\mu = \lambda n^\mu$

$$f(\infty) = \int_{x(0)=0} \mathcal{D}x \, \exp\left[i \int_0^\infty dt \left(\frac{\lambda}{2} \dot{x}^2 + (n + \dot{x}) \cdot A(x_i + nt + x) \right.\right.$$
$$\left.\left. + \frac{i}{2\lambda} \partial \cdot A(x_i + p_f t + x) \right) \right]$$

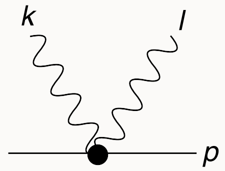
Use 1-D field theory propagator

$$\langle x(t) x(t') \rangle = G(t, t') = \frac{i}{\lambda} \min(t, t')$$

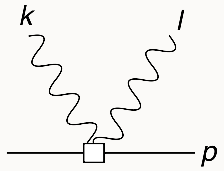
NE Feynman rules



$$\frac{k^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2}$$



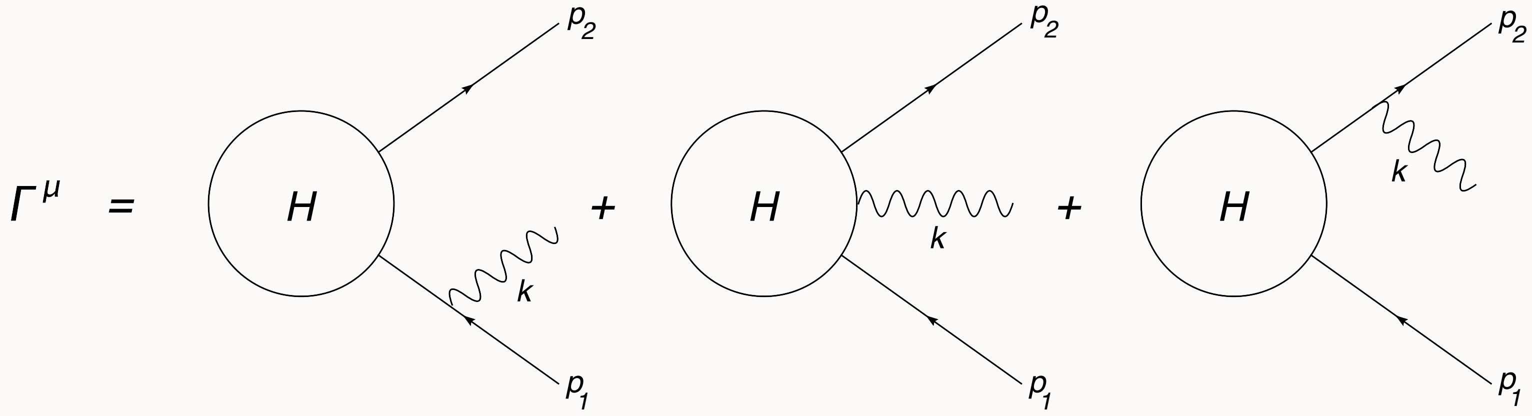
$$+ \frac{\eta^{\mu\nu}}{p \cdot (k + l)}$$



$$- \frac{l^\mu p^\nu p \cdot k + k^\nu p^\mu p \cdot l}{p \cdot (k + l) p \cdot k p \cdot l}$$

LOW-BURNETT-KROLL

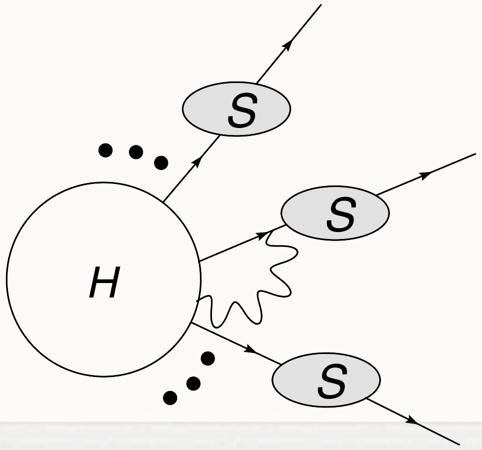
One soft emission determined by elastic amplitude to eikonal and next-to-eikonal order



$$\Gamma^\mu = \left[\frac{(2p_1-k)^\mu}{-2p_1\cdot k}+\frac{(2p_2+k)^\mu}{2p_2\cdot k}\right]\Gamma + \left[\frac{p_1^\mu(k\cdot p_2 - k\cdot p_1)}{p_1\cdot k}+\frac{p_2^\mu(k\cdot p_1-k\cdot p_2)}{p_2\cdot k}\right]\frac{\partial\Gamma}{\partial p_1\cdot p_2}$$

Analyzed in context of jet-soft factorization by Del Duca

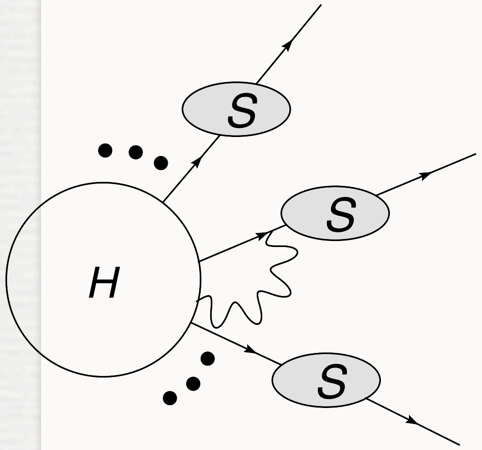
One emission from H still missing in our approach



LOW-BURNETT-KROLL

Path integral method provides elegant way to derive Low's theorem

$$S(p_1,\dots,p_n) = \int \mathcal{D}A_s H(x_1,\dots,x_n;A_s)e^{-ip_1x_1}f(x_1,p_1;A_s)\dots e^{-ip_nx_n}f(x_1,p_1;A_s)e^{iS[A_s]}$$



Gauge transformation must cancel between f's and H

$$f(x_i,p_f;A) \to f(x_i,p_f;A+\partial\Lambda) = e^{-iq\Lambda(x_i)}f(x_i,p_f;A)$$

Opposite transformation in H, expand to first order in A and Λ

Low contribution is then:

$$S(p_1,\dots,p_n) = \int \mathcal{D}A\Bigg[\int \frac{d^dk}{(2\pi)^d}\sum_j^n q_j\Big(\frac{n_j^\mu}{n_j\cdot k}k_\nu\frac{\partial}{\partial p_{j_\nu}} - \frac{\partial}{\partial p_{j_\mu}}\Big)H(p_1,\dots,p_n)A_\mu(k)\Bigg]$$
$$\times f(0,p_1;A)\dots f(0,p_n;A)$$

First term is due to displacement of f(x,p,A)

Analagous result in non-abelian case, for n=2

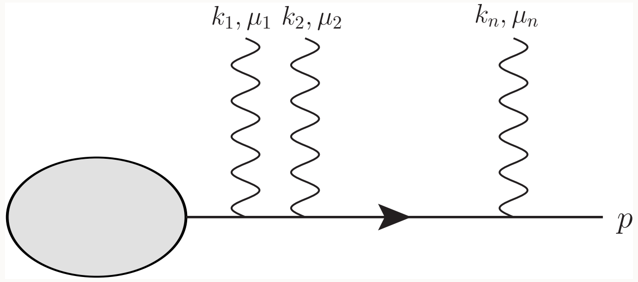
UPSHOT

- Exponentiation of soft emissions for matrix elements as “connectedness”
  - For both eikonal and next-to-eikonal contributions from external lines
  - Replica trick both for exponentiation, and for explicit expression for webs. New NE Webs.
  - 1 emission from hard part (1/N) also included
  - QCD: 2 lines. 3 lines also easy. 4, not so much.
- Can we arrive at the same results using diagrams, and inductive reasoning?
  - Combinatorics challenging..

DIAGRAMMATIC APPROACH

EL, Magnea, Stavenga, White

Recall: Abelian case, multiple emission, and sum over permutations



Eikonal identity:

$$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

For many emissions

$$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

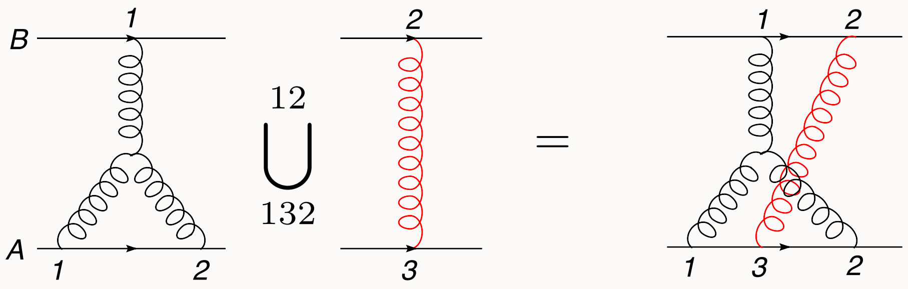
Non-abelian case requires

- web: two-eikonal irreducible graph
- "group" : projection of web on external line
- analogue of eikonal identity for permutations that leaves ordering in group invariant

$$\sum_{\tilde{\pi}}\frac{1}{2p\cdot k_{\tilde{\pi}_1}}\frac{1}{2p\cdot(k_{\tilde{\pi}_1}+k_{\tilde{\pi}_2})}\cdots\frac{1}{p\cdot(k_{\tilde{\pi}_1}+\ldots+k_{\tilde{\pi}_n})}=\prod_g\frac{1}{2p\cdot k_{g_1}}\frac{1}{2p\cdot(k_{g_1}+k_{g_2})}$$
$$\cdots\frac{1}{2p\cdot(k_{g_1}+\ldots+k_{g_m})}$$

MERGING

Can also use in reverse, as “merging”



$$\exp\left\{\sum_i \bar{c}_H E(H)\right\} = \prod_H \left(\sum_n \frac{1}{n!}[\bar{c}_H E(H)]^n\right) = \sum_G c_G E(G)$$

$$E^\mu(k) = \frac{2p^\mu}{2p \cdot k}$$

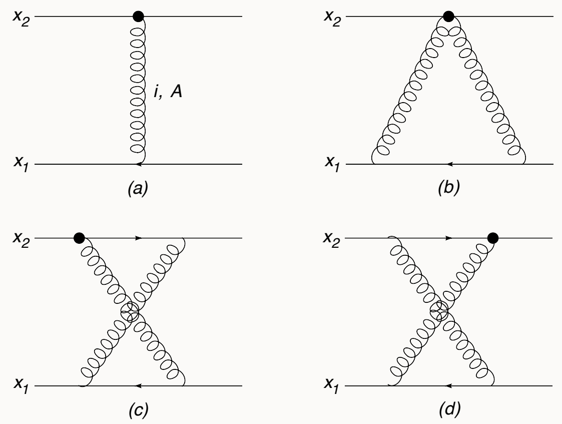
Prove that, for normal color factors on rhs, those on right side are those of webs

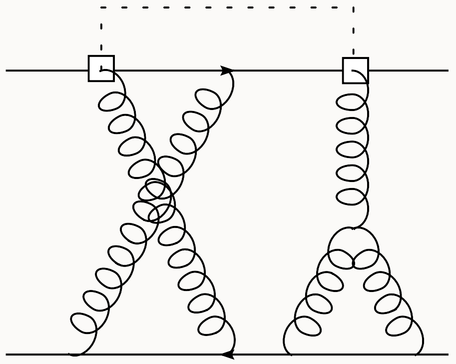
Proof uses

- induction
- combinatorics
- Schur’s lemma

NEXT-TO-EIKONAL CASE

- Identify next-to-eikonal vertices
  - show that they “decorrelate”, once summed over all perm’s. Use induction again
    - as eikonal webs, but now with a special vertex
    - for fermions: become spin-sensitive
  - new correlations between eikonal webs → NE webs
  - checked precise correspondence with path integral method





- Proof of exponentiation as for eikonal case

DRELL-YAN CHECK

- Check use of NE Feynman rules for Drell-Yan double real emission
  - Combine with exact phase space

$$K^{(2)NE} = \left(\frac{\alpha_S C_F}{4\pi}\right)^2 \left[\frac{1024\mathcal{D}_3}{3} - \frac{1024\log^3(1-z)}{3} + 640\log^2(1-z)\right.$$
$$+ \frac{512\mathcal{D}_2 - 512\log^2(1-z) + 640\log(1-z)}{\epsilon} + \frac{512\mathcal{D}_1 - 512\log(1-z)}{\epsilon^2}$$
$$\left. + \frac{256\mathcal{D}_0 - 256}{\epsilon^3}\right]$$

$$\mathcal{D}_i = \left[\frac{\log^i(1-z)}{1-z}\right]_+$$

  - Agrees with exact result, to similar accuracy

CONCLUSIONS

- Eikonal approximation important, yields simplification, symmetries, all-order results
- Next-to-eikonal contributions not negligible, but fairly little is known
- Found that certain next-to-eikonal contributions form new webs, and exponentiate
    - using path integrals, or diagrammatics
    - classified “Low’s theorem” contributions
- To do:
    - further test predictive power, application to cross sections
    - more legs,