The two-loop hexagon Wilson loop in N=4 SYM

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- The BDS ansatz
- The Wilson loop-scattering amplitude duality
- The two-loop six-point remainder function

The BDS ansatz

- Bern, Dixon and Smirnov (BDS) conjectured that MHV amplitudes N=4 SYM can be written as:

$$M_n(\epsilon) = 1 + \sum_{l=1}^{\infty} a^l \, M_n^{(l)}(\epsilon) = \exp \sum_{l=0}^{\infty} a^l \left[f^{(l)}(\epsilon) \, M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right],$$

- The BDS ansatz reproduces not only correctly the infrared poles of the amplitude to all orders in perturbation theory, but it is supposed to also provide the finite part of the amplitude.

The BDS ansatz

- In practice, the BDS ansatz implies a tower of iteration formulæ in the number of loops, e.g. for two loops

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon),$$

$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2 \qquad\qquad C^{(2)} = -\zeta_2^2/2$$

The BDS ansatz

	n=4	n=5	n=6
l=2			
l=3			

The BDS ansatz

	n=4	n=5	n=6
l=2	✓		
l=3	✓		

The BDS ansatz

	n=4	n=5	n=6
l=2	✓	✓ (num.)	
l=3	✓		

The BDS ansatz

	n=4	n=5	n=6
l=2	✓	✓ (num.)	● (num.)
l=3	✓		

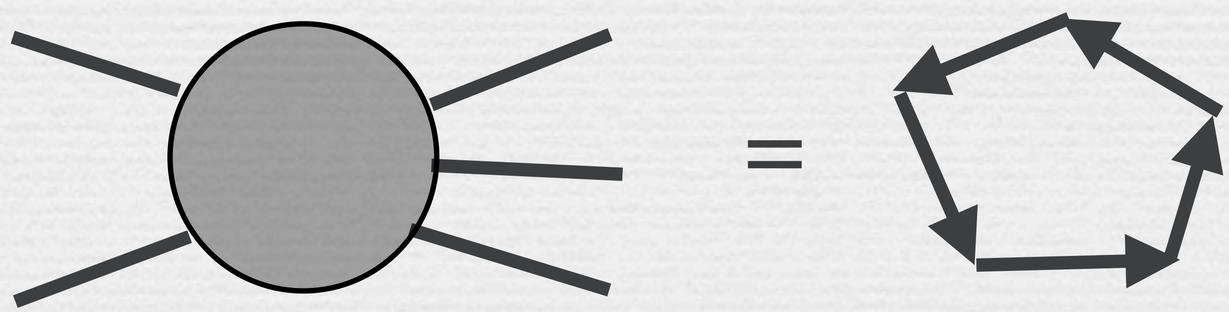
- What goes wrong for n = 6 ..?
- The answer comes from the Wilson loop!

Wilson loops in N=4 SYM

- Definition of a Wilson loop:

$$W[\mathcal{C}_n] = \mathrm{Tr}\ \mathcal{P}\ \exp\left[ig \oint \mathrm{d}\tau \dot{x}^\mu(\tau) A_\mu(x(\tau))\right]$$

- It is conjectured that Wilson loop along an *n*-edged polygon is equal to an *n*-point MHV scattering amplitude:



$$p_i = x_{i,i+1} = x_i - x_{i+1}$$

[Alday, Maldacena; Drummond, Korchemsky, Sokatchev]

- Proven analytically at one-loop for arbitrary *n*, and at two-loops for *n* = 4, 5, 6. [Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, Spence]

Wilson loops in N=4 SYM

- Wilson loops possess a conformal symmetry, and it was shown that a solution to the corresponding Ward identities is the BDS ansatz, e.g., at two-loops,

[Drummond, Henn, Korchemsky, Sokatchev]

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + \mathcal{O}(\epsilon)\,,$$

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$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} \; + R_n^{(2)}(u_{ij}) + \mathcal{O}(\epsilon)\,,$$

- ... but we can always add a arbitrary function of conformal invariants and we still obtain a solution to the Ward identities!

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

The breakdown of BDS

	n=4	n=5	n=6
l=2	✓	✓ (num.)	● (num.)
l=3	✓		

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No non trivial conformal cross-ratios,

$R_4^{(l)} = R_5^{(l)} = 0.$

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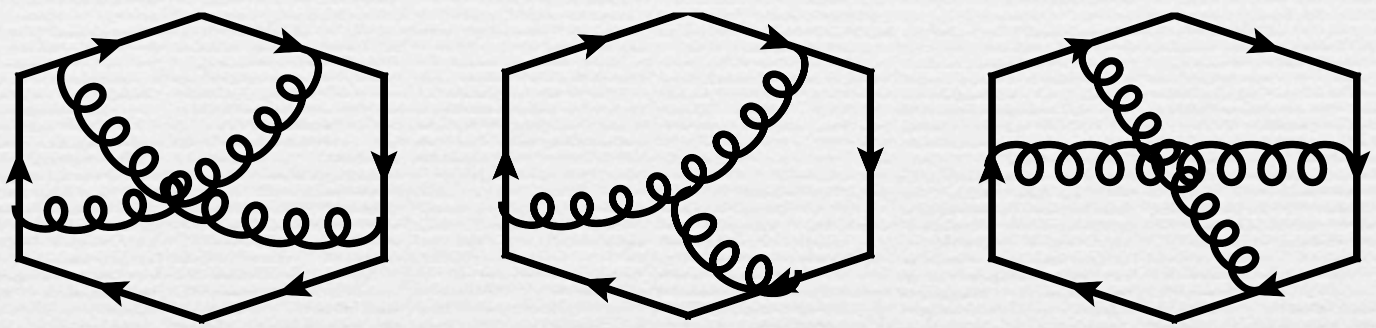
There are three non trivial cross ratios:

$$u_1 = \frac{s_{12}\, s_{45}}{s_{123}\, s_{345}}, \quad u_2 = \frac{s_{23}\, s_{56}}{s_{123}\, s_{234}}, u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}},$$

How can we compute this function?

- Anastasiou, Brandhuber, Heslop, Khoze, Spence and Travaglini worked out the two-loop Wilson loop diagrams:



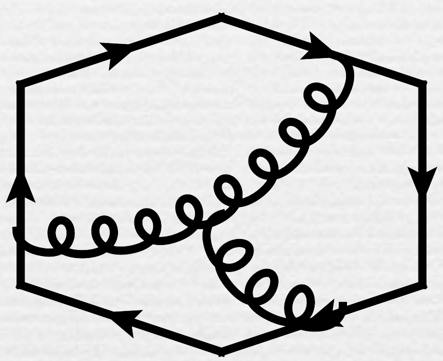
...

- Each of these diagrams is an integral, similar to a Feynman parameter integral.

- They studied these integrals extensively numerically, but no analytic solution was known.

How can we compute this function?

- For n = 6, many of the integrals can be computed explicitly, but one is particularly 'hard':



$$f_H(p_1, p_2, p_3; Q_1, Q_2, Q_3)$$
$$:= \frac{\Gamma(2-2\epsilon_{\mathrm{UV}})}{\Gamma(1-\epsilon_{\mathrm{UV}})^2} \int_0^1 \Big(\prod_{i=1}^3 d\tau_i\Big) \int_0^1 \Big(\prod_{i=1}^3 d\alpha_i\Big) \delta(1 - \sum_{i=1}^3 \alpha_i)\ (\alpha_1\alpha_2\alpha_3)^{-\epsilon_{\mathrm{UV}}} \frac{\mathcal{N}}{\mathcal{D}^{2-2\epsilon_{\mathrm{UV}}}}\ ,$$

$$\begin{aligned}\mathcal{N} &= 2(p_1p_2)(p_1p_3)\Big[\alpha_1\alpha_2(1-\tau_1) + \alpha_3\alpha_1\tau_1\Big] + 2(p_1p_3)(p_2p_3)\Big[\alpha_3\alpha_1(1-\tau_3) + \alpha_2\alpha_3\tau_3\Big] \\ &+ 2(p_1p_2)(p_2p_3)\Big[\alpha_2\alpha_3(1-\tau_2) + \alpha_1\alpha_2\tau_2\Big] + 2\alpha_1\alpha_2\Big[2(p_1p_2)(p_3Q_3) - (p_2p_3)(p_1Q_3) - (p_3p_1)(p_2Q_3)\Big] \\ &+ \ldots\end{aligned}$$

- The integrals do not explicitly depend on conformal ratios.
- But is all this complexity really needed..?
- Could we go to simplified kinematics?

Regge limits

- Multi-Regge kinematics

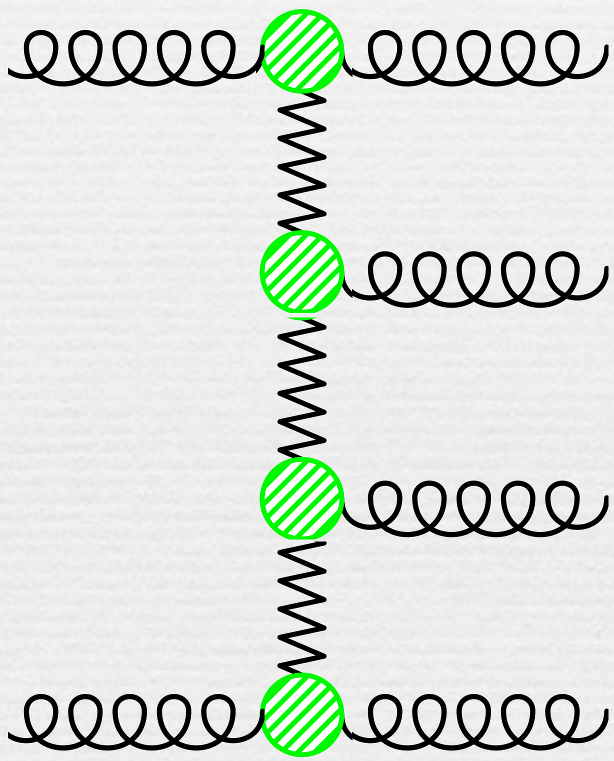
$$y_3 \gg y_4 \gg y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$

- s-type invariants are large.

  t-type invariants are small.

  Conformal cross ratios become trivial



[Del Duca, CD, Glover]

Regge limits

- Quasi-multi-Regge kinematics

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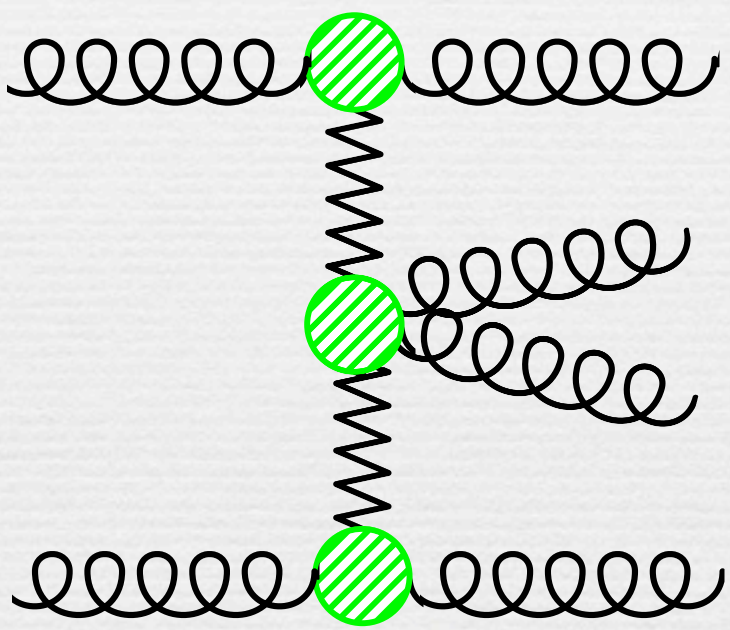


- Conformal cross ratios are no longer trivial

[Del Duca, CD, Glover]

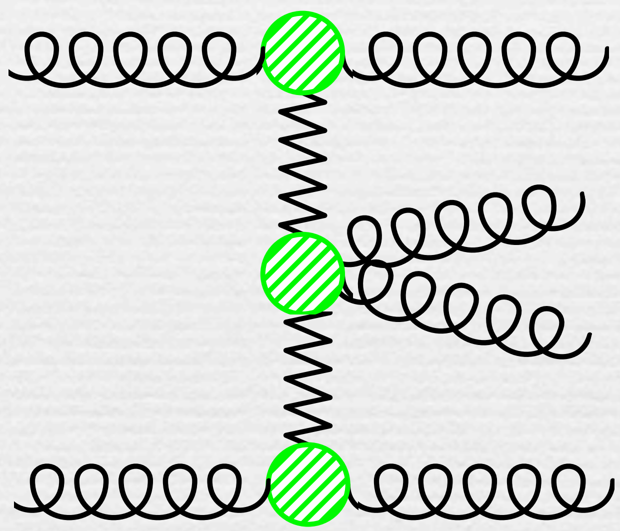
Regge-exactness of Wilson loops

- The result is in fact even stronger:

  The Wilson-loop is **Regge-exact** in this limit, i.e., it is the same in this special kinematics and in arbitrary kinematics

$$y_3 \gg y_4 \simeq y_5 \gg y_6$$

$$|p_{3\perp}|^2 \simeq |p_{4\perp}|^2 \simeq |p_{5\perp}|^2 \simeq |p_{6\perp}|^2$$



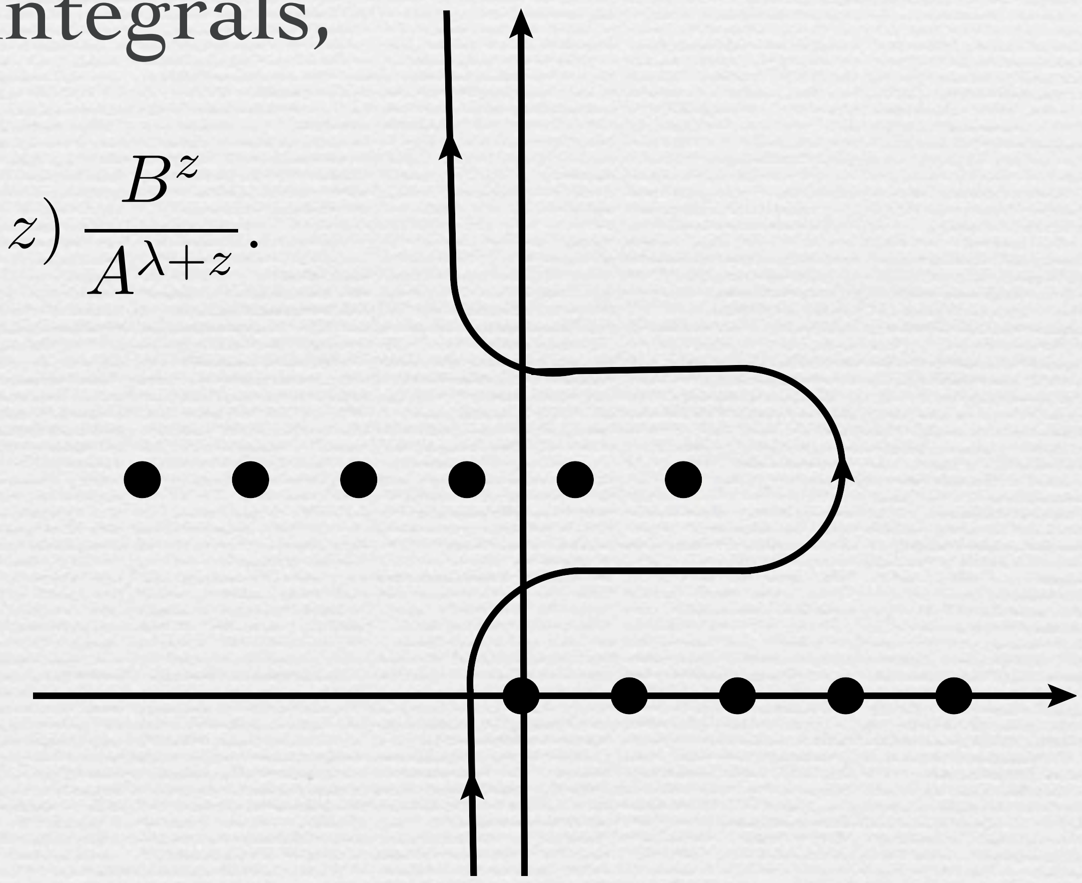
- This result is in fact true for Wilson loops with an arbitrary number of edges and loops! [Del Duca, CD, Smirnov]
- Bottomline: it is enough to perform the computation in these **simplified** kinematics to obtain the two-loop six-point Wilson loop in **arbitrary** kinematics!

A Recipe to compute Wilson loops

- Step 1:

  We write down a Mellin-Barnes representation for each diagram, i.e., we replace denominators in the Feynman parameter integrals by contour integrals,

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z\,\Gamma(-z)\,\Gamma(\lambda+z)\,\frac{B^z}{A^{\lambda+z}}.$$



- This turns the Feynman parameter integral into residue calculus:

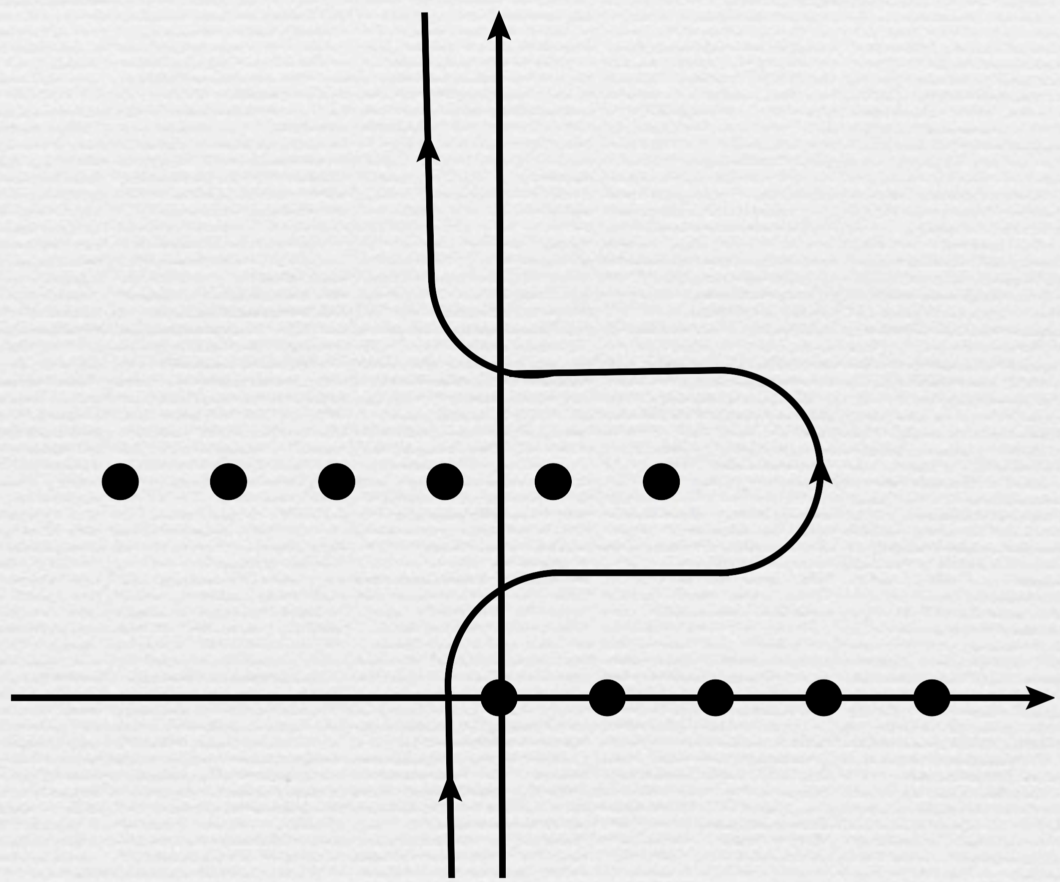
$$\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$$

A Recipe to compute Wilson loops

- Step 2:

  We exploit Regge exactness and we only compute the leading behavior of each integral in the quasi-multi-Regge limit

- The Mellin-Barnes approach is very suitable for this!



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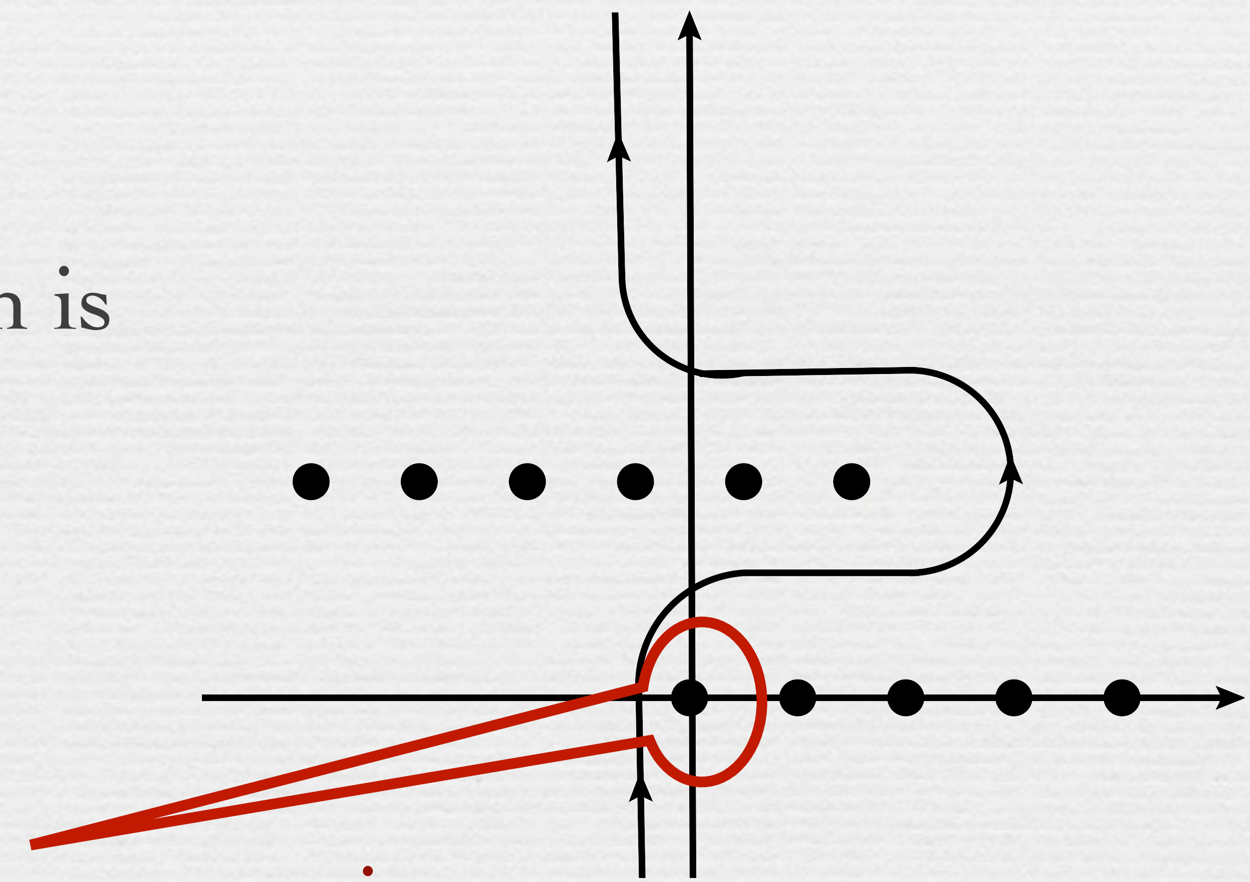


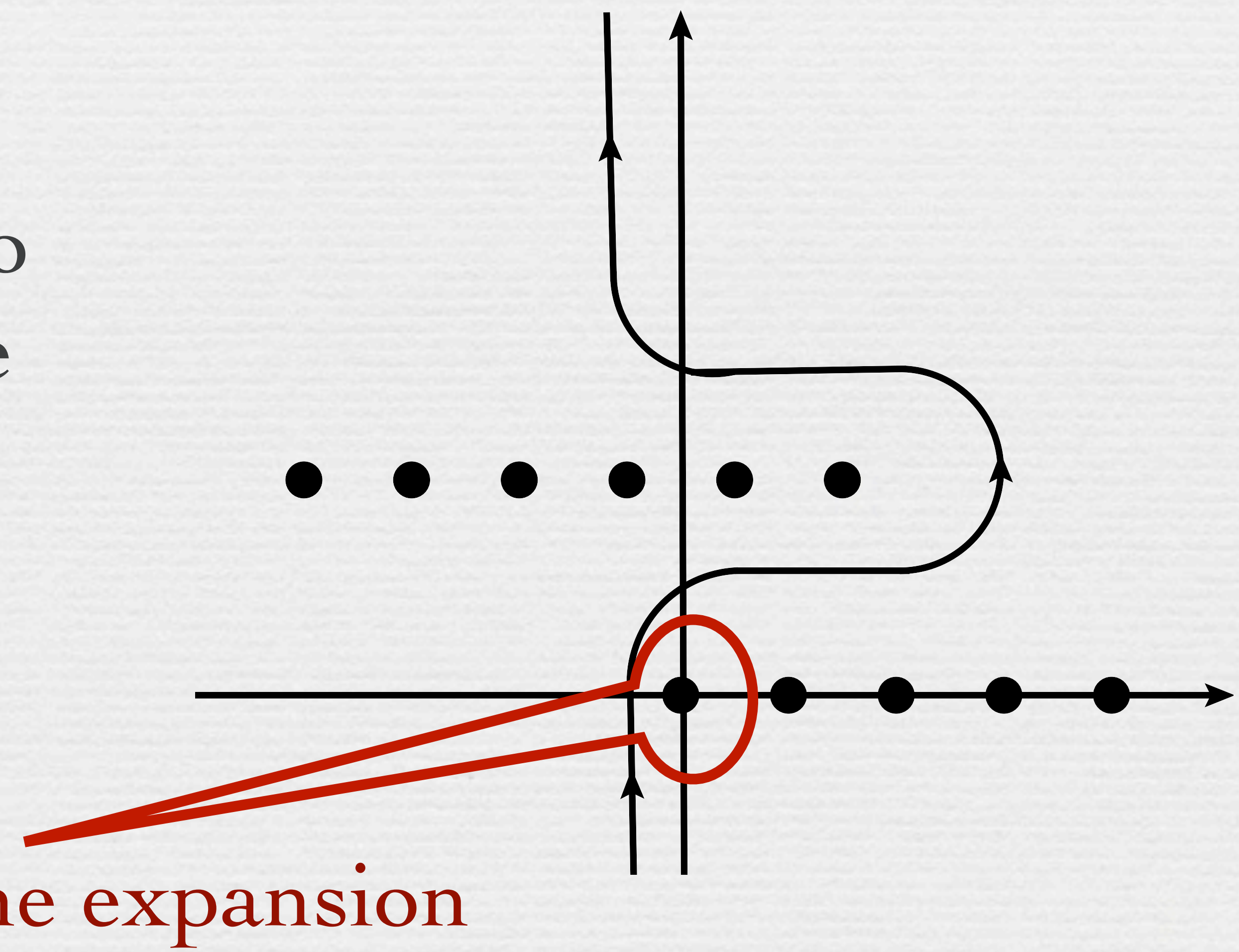
Leading term in the expansion

A Recipe to compute Wilson loops

- Step 3:

  Iterate the limits: There are six different ways to take the limits, corresponding to the six cyclic permutations of the external legs.

- Regge-exactness allows us to take all six limits at the same time!



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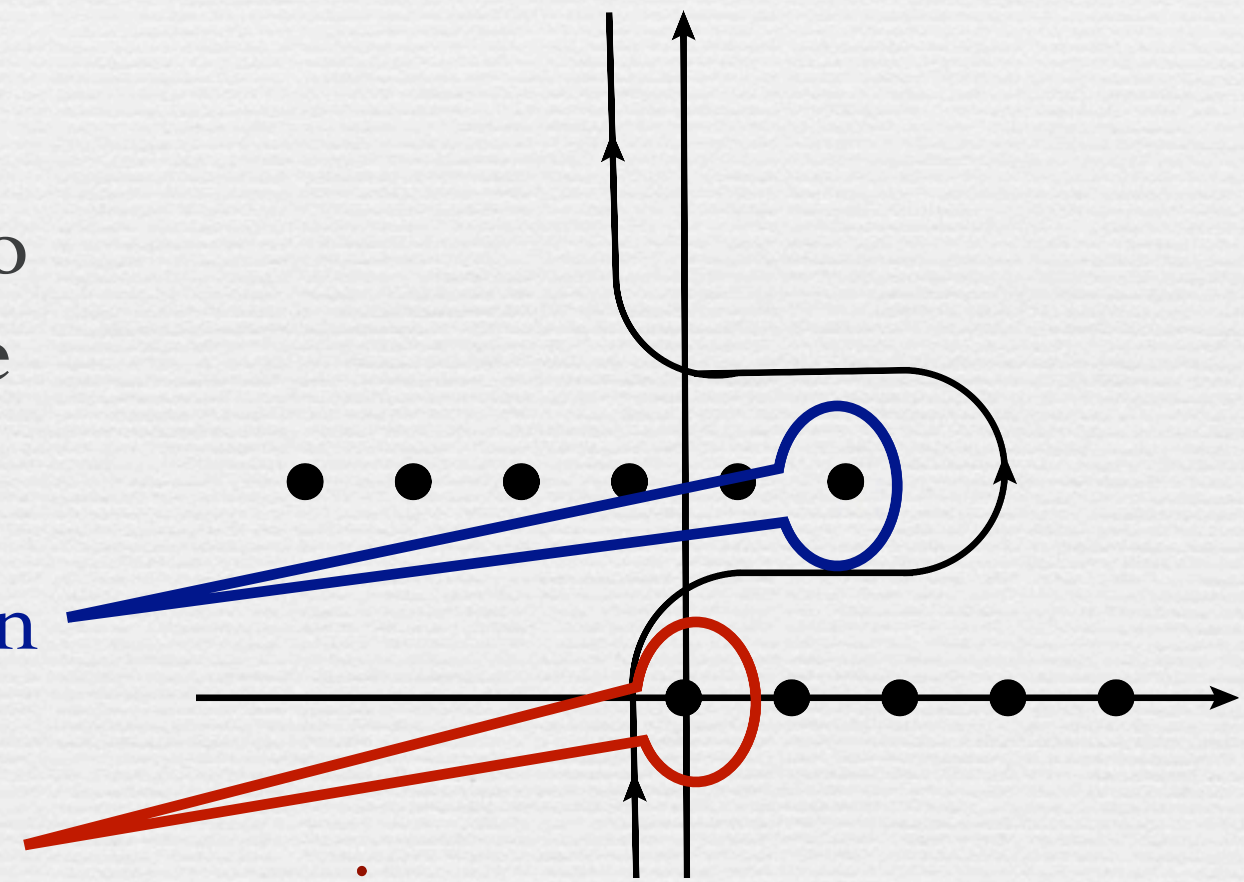


Leading term in the expansion in limit 2

Leading term in the expansion in limit1

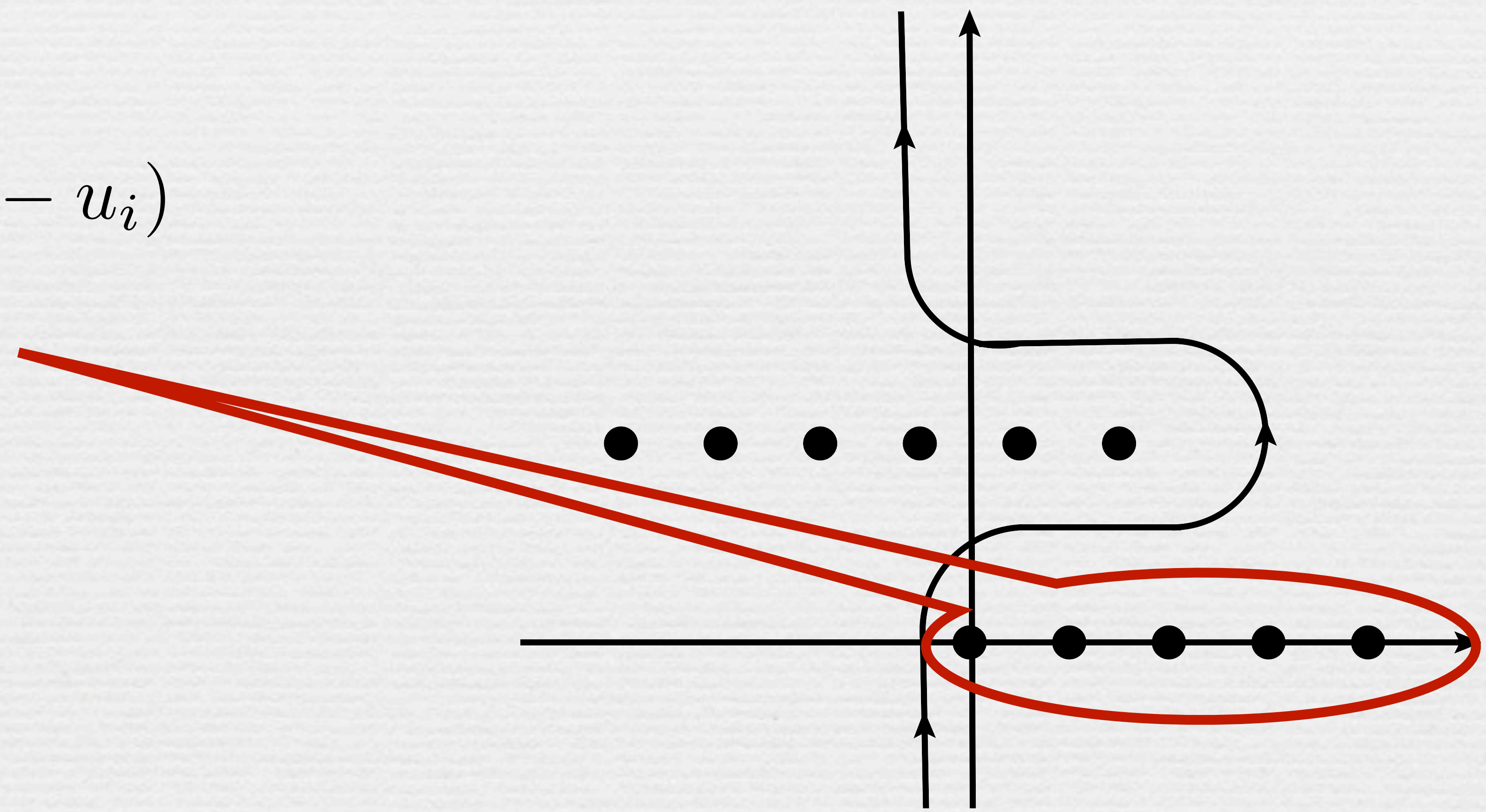
A Recipe to compute Wilson loops

- Step 4:

  Sum up the remaining towers of residues:

$$\sum_{n=1}^{\infty} \frac{u_i^n}{n} = -\ln(1-u_i)$$

$$\sum_{n=1}^{\infty} \frac{u_i^n}{n^k} = \mathrm{Li}_k(u_i)$$



The six-point remainder function

- We applied this recipe to the two-loop six-edged Wilson loop.
- In the limit, all integrals are
  - ➡ at most three-fold.
  - ➡ dependent on conformal cross ratios only.
- Hence, the resulting integrals are much simpler and can be solved in a closed form, and we can easily extract the two-loop six-point remainder function,

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

[Del Duca, CD, Smirnov]

The six-point remainder function

- The result is completely expressed in terms Goncharov's multiple polylogarithm,

$$G(\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a}\, G(\vec{w}';t) \qquad \Big| \qquad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t}\, \mathrm{Li}_{n-1}(t)$$

- Some of them depend on complicated arguments:

$$u_{jkl}^{(\pm)} = \frac{1-u_j-u_k+u_l \pm \sqrt{(u_j+u_k-u_l-1)^2 - 4\,(1-u_j)\,(1-u_k)\,u_l}}{2\,(1-u_j)\,u_l},$$

$$v_{jkl}^{(\pm)} = \frac{u_k-u_l \pm \sqrt{-4u_ju_ku_l+2u_ku_l+u_k^2+u_l^2}}{2\,(1-u_j)\,u_k}.$$

- For some values of the u's, the square roots can become complex.
- They however always come in pairs such that the full result is real.

The six-point remainder function

- We checked that our result has all the properties required for the remainder function:

  - ✓ the result is of uniform transcendental weight 4.
  - ✓ no new transcendental numbers appear (only $\zeta_2, \zeta_3, \zeta_4$).
  - ✓ explicitly dependent on conformal cross-ratios.
  - ✓ symmetric in all its arguments.
  - ✓ vanishes in all collinear and multi-Regge limits.
  - ✓ we checked numerically several points.

Conclusion

- Planar N = 4 SYM displays a lot of nice features related to scattering amplitudes and Wilson loops, both at strong and at weak coupling.
- Regge-exactness of the Wilson loops gives a powerful tool for analytic computations.
- We applied this tool and performed the first analytic calculation of the two-loop six-point BDS remainder function in terms of generalized polylogarithms.