Analytic Computations of One-Loop Massive Amplitudes

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26$^{\text{th}}$ April 2010

Loops and Legs 2010, Wörlitz

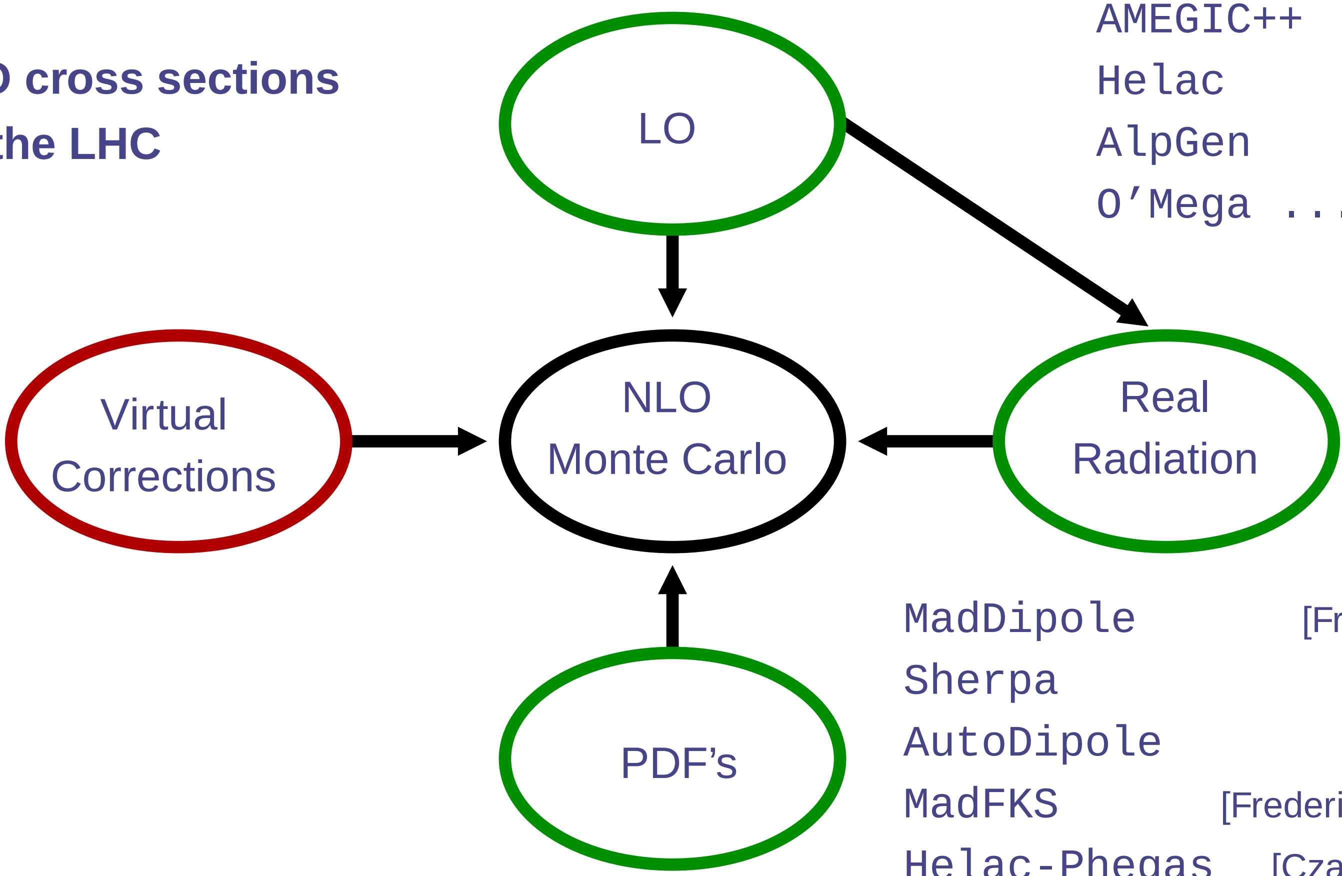
Outline

- Review of on-shell methods and amplitude structures
  - Complex analysis and generalised unitarity
- Application to massive fermion amplitudes
  - Spinor/Helicity with massive fermions
  - Compact helicity amplitudes for $gg \to t\bar{t}$

QCD for the LHC

NLO cross sections
for the LHC

MadGraph
AMEGIC++
Helac
AlpGen
O'Mega ...



MadDipole [Frederix,Gehrmann,Greiner]
Sherpa [Gleisberg,Krauss]
AutoDipole [Hasegawa,Moch,Uwer]
MadFKS [Frederix,Frixione,Maltioni,Stelzer]
Helac-Phegas [Czakon,Papadopoulos,Worek]

POWHEG [Nason et al]
MC@NLO [Frixione,Webber]

Computations of Virtual Corrections

- A lot of recent progress in computational methods for virtual corrections:

  Bern,Dixon,Dunbar,Kosower,Britto,Cachazo,Feng,Mastrolia,
  Ossola,Papadopoulos,Pittau,Ellis,Giele,Kunszt,Melnikov,Forde,. . .

- Automated numerical approaches:

  [`BlackHat`, `Rocket`, `CutTools/Helac-1loop`, `GOLEM`, Denner et al., . . .]
  - Large number of phenomenological studies
- Efficiency:
  - Numerical stablility
  - Fast numerical evaluation
  - Complexity of processes with additional jets
- Analytical computations good for numerical stability and speed
- Amplitude structures can give insight into new techniques

$t\bar{t}$ production at the LHC

- LHC is a top factory! ($> 10^6$ events/year)
- $pp \to t\bar{t}j$ and $pp \to t\bar{t}b\bar{b}$: backgrounds for $t\bar{t}H$ production channel

[Dittmaier,Uwer,Weinzierl]

[Melnikov,Schulze]

[Bredenstein,Denner,Dittmaier,Pozzorini; Bevilaqua et al.`Helac-1loop`]

- Recent computation of $pp \to t\bar{t} + 2j$ [Bevilaqua et al. `Helac-1loop`]
- $t\bar{t} + n(j)$ backgrounds for SUSY and LED searches
- Define heavy quark helicity states w.r.t massless direction η [Kleiss,Stirling]

$$u_\pm(Q;\eta) = \frac{(\not{Q} + m)|\eta\mp\rangle}{\langle Q^\flat \pm |\eta\mp\rangle}$$

where

$$Q^{\flat,\mu} = Q^\mu - \frac{m^2}{2Q\cdot\eta}\eta^\mu$$

Spinor-Helicity Formalism

- All amplitudes described by two-component Weyl spinors

$$\langle pq \rangle = e^{i\theta_{pq}} \sqrt{p \cdot q} \qquad [pq] = e^{-i\theta_{pq}} \sqrt{p \cdot q}$$

- Momenta and polarisation vectors

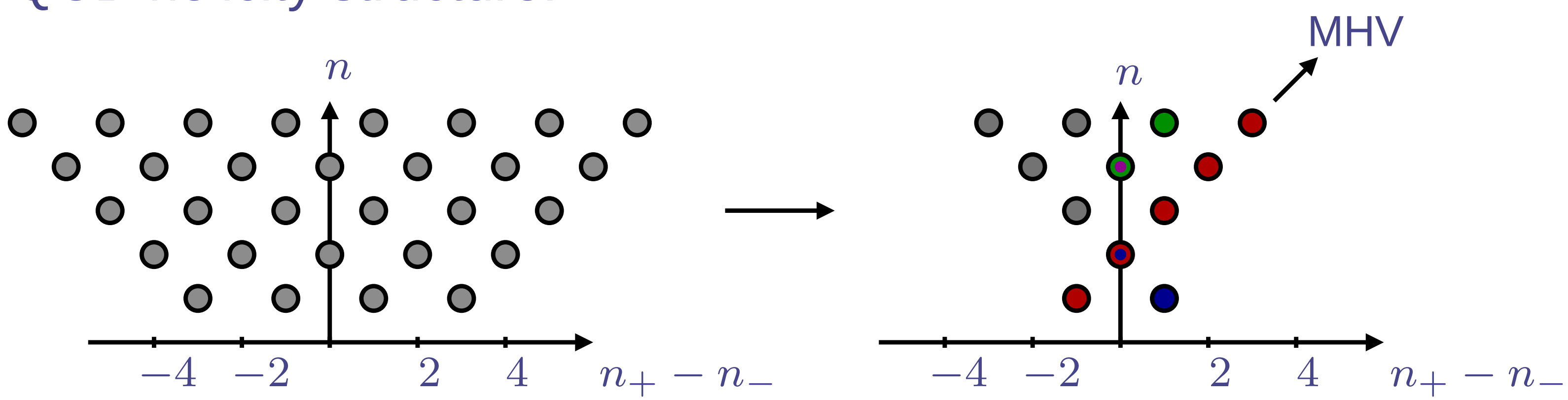
$$p^\mu = \frac{1}{2} \langle p | \sigma^\mu | p] \qquad \varepsilon_+^\mu(p,\xi) = \frac{\langle \xi | \sigma^\mu | p]}{\sqrt{2} \langle \xi p \rangle} \qquad \varepsilon_-^\mu(p,\xi) = \frac{\langle p | \sigma^\mu | \xi]}{\sqrt{2} [p\xi]}$$

- Massive momenta $\Rightarrow$ longer “spinor strings"

$$\langle p | P | q] \qquad \langle p | PQ | q \rangle \qquad \langle p | P(p+q | r \rangle$$

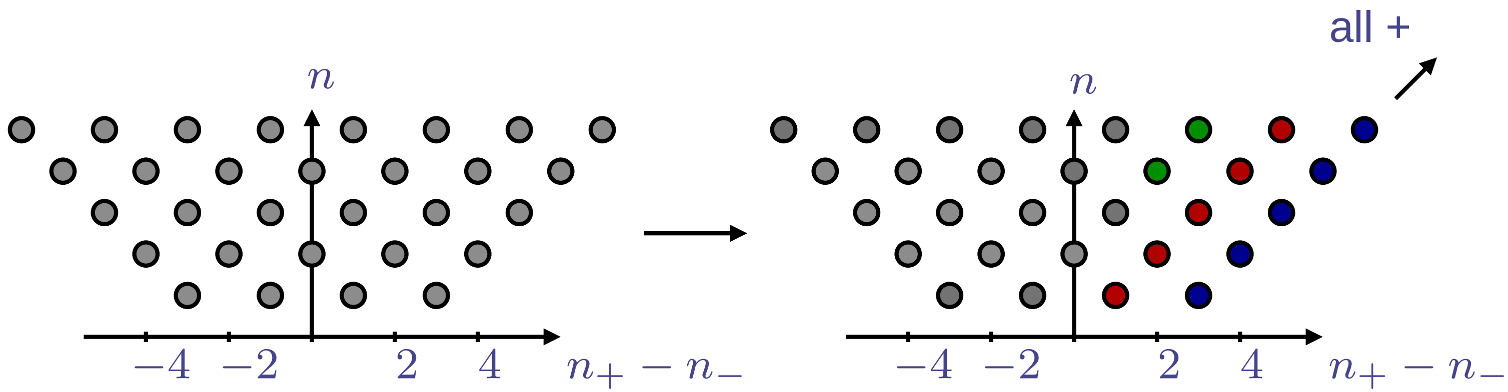
Massive Helicity Amplitudes

Massless QCD helicity structure:



Symmetry relates massive quark helicity states:

$$u_-(p, m; p^\flat, \eta) = \frac{\langle p^\flat \eta \rangle}{m} u_+(p, m; \eta, p^\flat)$$



Tree Level Amplitudes

- Compact tree-level amplitudes from BCFW recursion

$$A_4(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) = -im^3 \frac{[23]\langle\eta_1\eta_4\rangle}{\langle 23\rangle\langle 2|1|2]\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle}$$

$$A_4(1_t^+, 2^+, 3^-, 4_{\bar{t}}^+) = -im \frac{\langle 3|1|2]\left(\langle\eta_1\eta_4\rangle\langle 3|1|2] + [23]\langle\eta_1 3\rangle\langle\eta_4 3\rangle\right)}{s_{23}\langle 2|1|2]\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle}$$

- Easy to automate analytically: All helicity amplitudes for $n \leq 6$
- Turn trees into loops via generalised unitarity
  - Compact expressions at one-loop?
- Study $gg \to t\bar{t}$ in detail...

$t\bar{t}gg$ Colour Ordering

- Decompose into primitive amplitudes [Bern,Dixon,Kosower (1994)]

$$\mathcal{A}_4^{(0)}(1_t,2,3,4_{\bar{t}}) = \sum_{P(2,3)} (T^{a_2}T^{a_3})_{i_1 i_4} A_4^{(0)}(1_t,2,3,4_{\bar{t}})$$

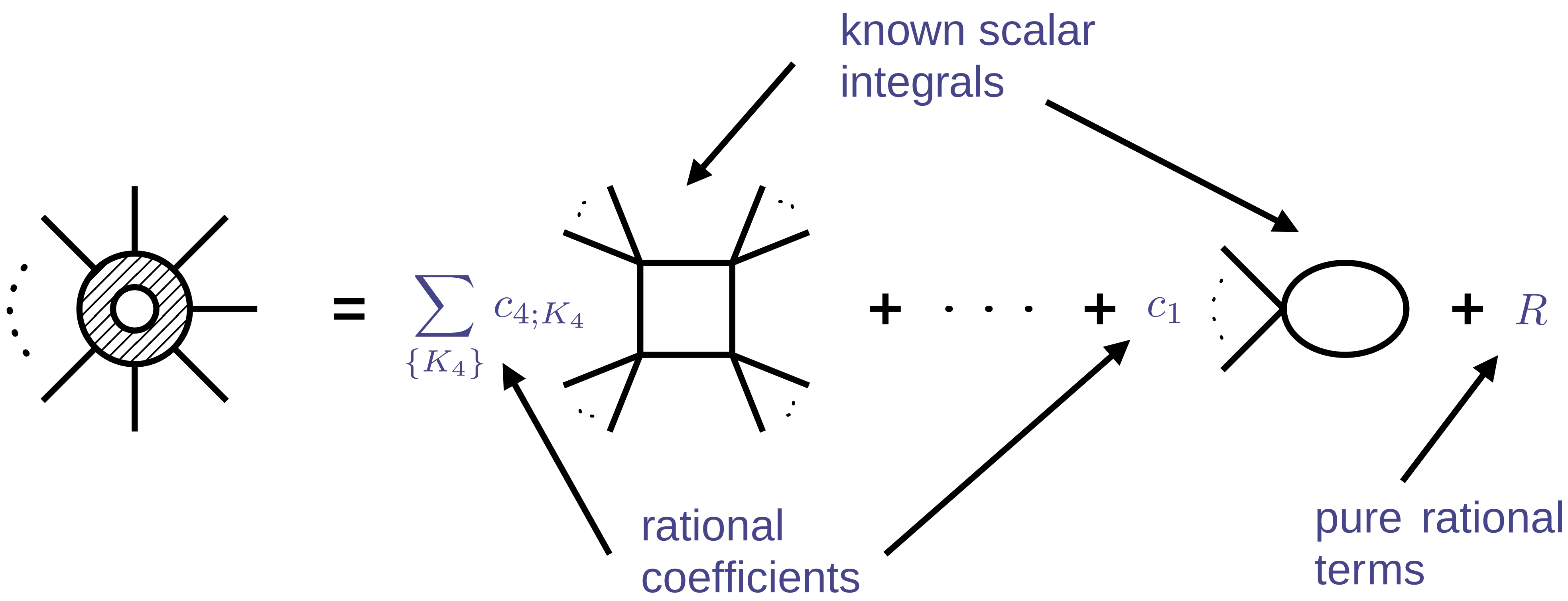
$$\mathcal{A}_4^{(1)}(1_t,2,3,4_{\bar{t}}) = \sum_{P(2,3)} N(T^{a_2}T^{a_3})_{i_1 i_4} A_{4;1}^{(1)}(1_t,2,3,4_{\bar{t}}) + \delta^{a_2 a_3}\delta_{i_1 i_4} A_{4;3}^{(1)}(1_t,4_{\bar{t}};2,3)$$

where

$$A_{4;1}^{(1)}(1_t,2,3,4_{\bar{t}}) = A^{[L]}(1_t,2,3,4_{\bar{t}}) - \frac{1}{N^2} A^{[R]}(1_t,2,3,4_{\bar{t}}) + \frac{N_f}{N} A^{[f]}(1_t,2,3,4_{\bar{t}}) + \frac{N_H}{N} A^{[H]}(1_t,2,3,4_{\bar{t}})$$

$$A_{4;3}^{(1)}(1_t,4_{\bar{t}};2,3) = \sum_{P(2,3)} \left\{ A^{[L]}(1_t,2,3,4_{\bar{t}}) + A^{[L]}(1_t,2,4_{\bar{t}},3) + A^{[R]}(1_t,2,3,4_{\bar{t}}) \right\}.$$

Structure of One-Loop Amplitudes



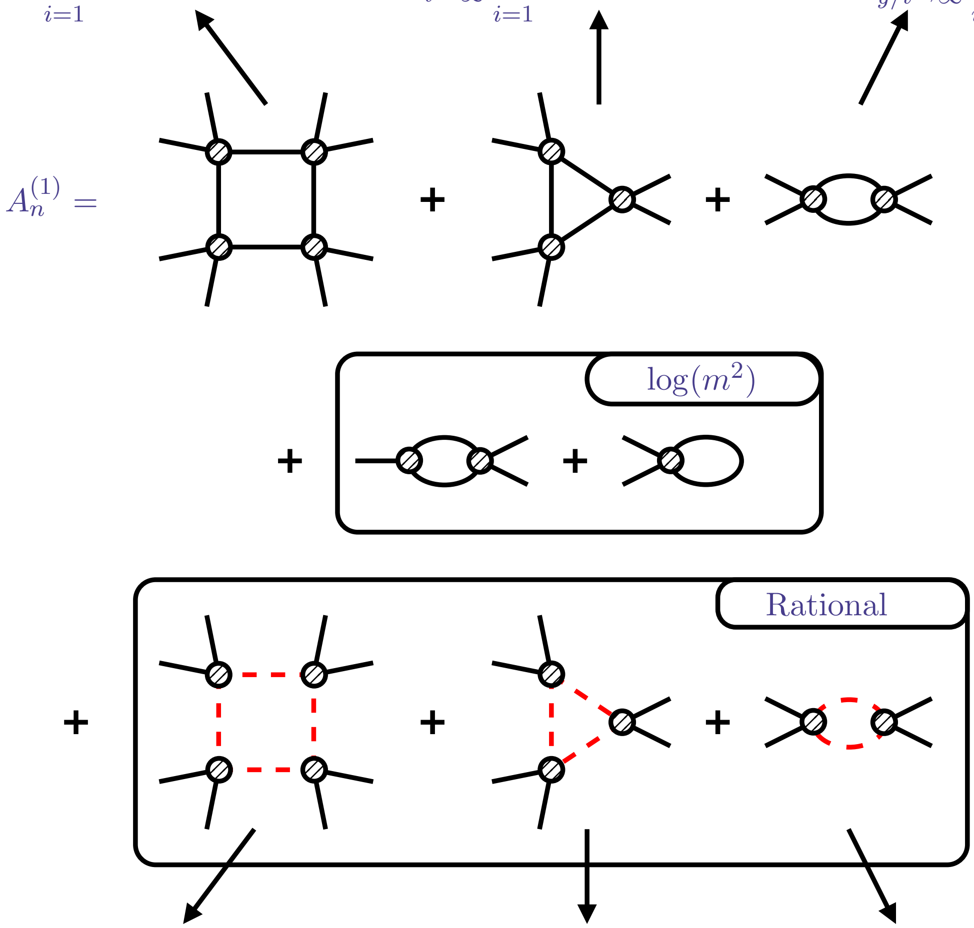
- General gauge theory amplitudes reduced to box topologies or simpler

  [Passarino,Veltman;Melrose]

- Isolate logarithms with cuts $\rightarrow$ exploit on-shell simplifications
- General cutting principle:
  - apply δ-functions to L and R sides
  - generate and solve the linear system for the coefficients

Generalised Unitarity for One-Loop Amplitudes

$$C_4 = \prod_{i=1}^{4} A_i(l) \qquad C_3 = \lim_{t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0} \qquad C_2 \sim \lim_{y/t\to\infty} \prod_{i=1}^{2} A_i(l(y,t))$$



$$C_4^{[4]} = \lim_{\mu^2\to\infty} \prod_{i=1}^{4} A_i(l)|_{\mu^4} \qquad C_3^{[2]} = \lim_{\mu^2,t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0,\mu^2} \qquad C_2^{[2]} \sim \lim_{\mu^2,y/t\to\infty} \prod_{i=1}^{2} A_i(l(y,t))$$

Generalised cuts
$\Rightarrow$ loops from trees

Bern,Dixon,Dunbar,Kosower;

Britto,Cachazo,Feng,Mastrolia,Yang;

Ossola,Papadopoulos,Pittau;

Forde

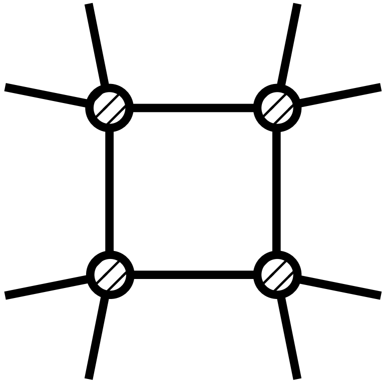
D-dimensional cuts:

[Ossola,Papadopoulos,Pittau;

Giele,Kunszt,Melnikov;

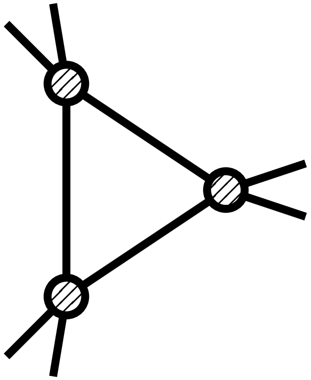
Britto,Feng,Mastrolia;SB]

Multiple Cuts and Integrand Reduction



- Quadruple cut$\rightarrow$ 4 on-shell δ-functions
- $l_k^2 = m_k^2 \rightarrow$ fixed loop momentum
- $C_4 = \frac{1}{2}\sum_{\sigma=\pm} A_1 A_2 A_3 A_4(l_1^\sigma)$ [BCF]

- Triple cut$\rightarrow$ 3 on-shell δ-functions
- Parametrise free integration [OPP,Forde]



$$\oint J_t dt A_1 A_2 A_3 = \oint J_t dt \mathrm{Inf}_t[A_1 A_2 A_3(t)] + \sum_k \frac{\mathrm{Res}_{t=t_k}(A_1 A_2 A_3)}{\xi_k (t - t_k)}$$

- $C_3 = \frac{1}{2}\sum_{\sigma=\pm} \mathrm{Inf}_t[A_1 A_2 A_3(l_1^\sigma(t))]|_{t^0}$

Bubble coefficients follow from a similar analysis:

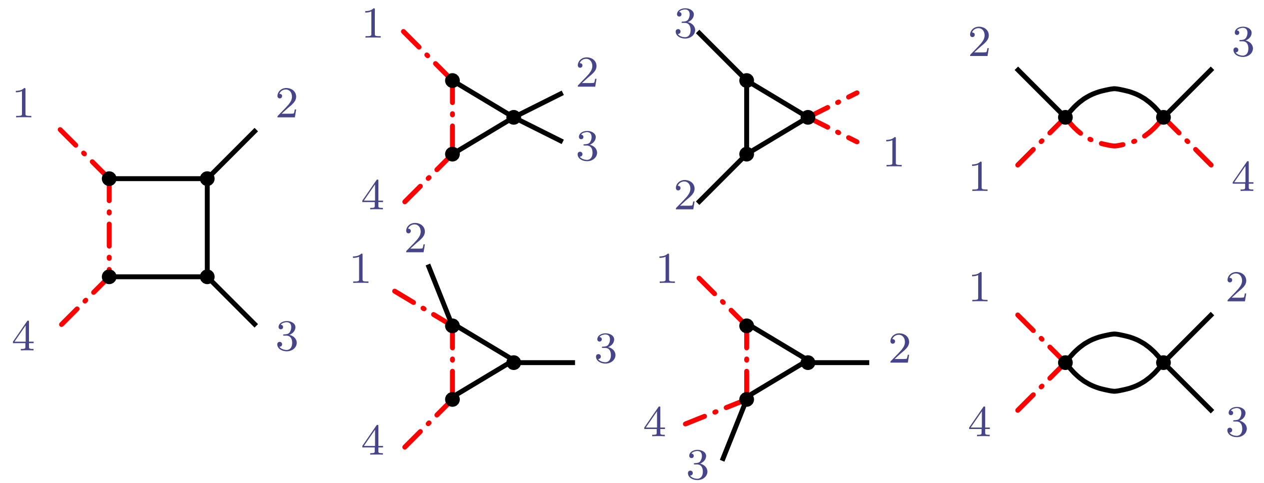
$$C_2 = \mathrm{Inf}_t \mathrm{Inf}_y [A_1 A_2(t,y)] - \frac{1}{2}\sum_{\sigma=\pm} \mathrm{Inf}_t[A_1 A_2' A_3'(t, y_\pm)]$$

3-cut: Cauchy's Theorem [Dunbar,Perkins,Warwick]

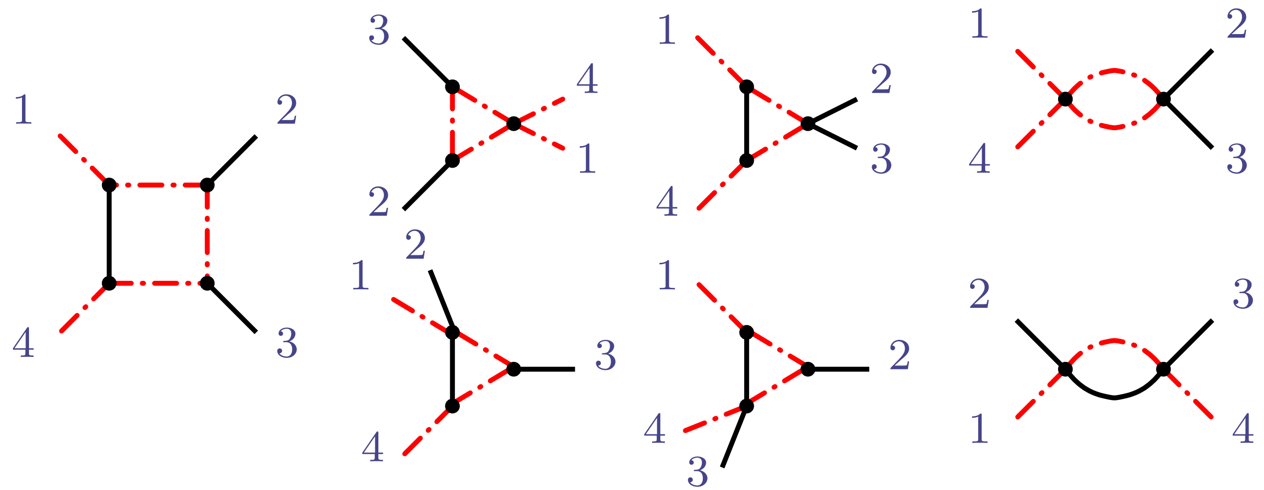
2-cut: Stokes' Theorem [Mastrolia]

The Integral Basis

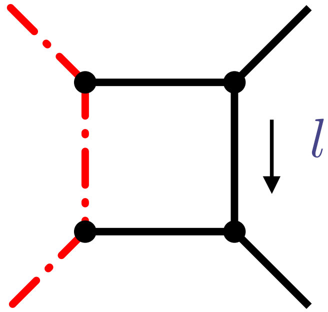
$$A^{[L]}(1_t, 2, 3, 4_{\bar{t}})$$



$$A^{[R]}(1_t, 2, 3, 4_{\bar{t}})$$



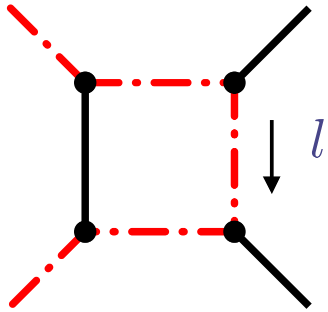
Extracting the Coefficients



$$l^{(1),\mu} = -\frac{\langle 2|1|2]}{2\langle 2|1|3]}\langle 2|\gamma^\mu|3]$$

$$l^{(2),\mu} = -\frac{\langle 2|1|2]}{2\langle 3|1|2]}\langle 3|\gamma^\mu|2]$$

“Left-Moving" box coefficient has simple solution



$$l^{\pm\mu} = \frac{c_\pm}{2}\langle 2|\gamma^\mu|3] - \frac{m^2}{2c_\pm}\langle 3|\gamma^\mu|2]$$

$$c\pm = -\frac{1}{2\langle 2|1|3]}\Bigg(s_{12} + m^2 \pm \sqrt{(s_{12}+m^2)^2 + 4m^2\langle 2|1|3]\langle 3|1|2]/s_{23}}\Bigg)$$

“Right-Moving" box coefficient contains square roots. . .

Using the universal IR constraints

- Taylor expansions can result in “relatively" long expressions
  $\Rightarrow$ Further simplifications needed
- Coefficients satisfy set of constraints imposed from universal pole structure:

$$\frac{1}{\epsilon}\log\left(\frac{-m^2}{s}\right) \qquad \frac{1}{\epsilon}\log\left(1-\frac{t}{m^2}\right) \qquad \frac{1}{\epsilon}\log\left(\frac{\beta+1}{\beta-1}\right)$$

- Three IR consistency equations:

$$C^{[L]}_{1|2|3|4} + \langle 2|1|2] C^{[L]}_{3;23|4|1} = s_{23}\langle 2|1|2] A^{(0)}(1_t, 2, 3, 4_{\bar{t}})$$
$$C^{[L]}_{1|2|3|4} + s_{23} C^{[L]}_{3;12|3|4} + s_{23} C^{[L]}_{3;1|2|34} = s_{23}\langle 2|1|2] A^{(0)}(1_t, 2, 3, 4_{\bar{t}})$$
$$C^{[R]}_{1|2|3|4} + \langle 2|1|2] C^{[R]}_{3;2|3|41} = (s_{23} - 2m^2)\langle 2|1|2] A^{(0)}(1_t, 2, 3, 4_{\bar{t}})$$

- Eliminate three-mass triangle from “Right-Moving” primitive

Cut-Constructible Contributions: $A^{[L]}$

$$
\begin{aligned}
-iA^{[L]}(1_t^+,2^+,3^+,4_t^+) = &-\frac{\langle\eta_1\eta_4\rangle[32]^2 m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle} I_4\left(s_{23},s_{12},0,0,m^2,m^2,0,0,m^2,0\right)\\
&-\frac{\left(2s_{12}\langle\eta_1\eta_4\rangle-\langle\eta_1|K_{12}K_{23}|\eta_4\rangle\right)[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2}\widehat{I}_2\left(s_{12},0,m^2\right)\\
&+\frac{c_\Gamma}{2\epsilon}\left(\frac{\mu_R^2}{m^2}\right)^\epsilon A^{(0)}(1_t^+,2^+,3^+,4_t^+)\\
&+R_4^{[L]},
\end{aligned}
$$

- Wave-function cuts not well defined by unitarity cuts [Ellis,Giele,Kunszt,Melnikov]
- $\log(m^2)$ coefficient from universal IR behaviour [Catani,Dittmaier,Trocsanyi] [Moch,Mitov]
- Correct IR pole structure (FDH scheme)
- Checked against known results [Körner,Merebashvili] [Anastasiou,Aybat]

Cut-Constructible Contributions: $A^{[R]}$

$$
\begin{aligned}
&-iA^{[R]}(1_t^+,2^+,3^+,4_{\bar t}^+) = F_4\left(s_{23},s_{12},0,0,m^2,m^2,m^2,m^2,0,m^2\right)\Bigg(\\
&-\frac{\langle 2\eta_1\rangle\langle 2\eta_4\rangle\left(2m^2+\langle 2|1|2]\right)[32]^2m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]} + \frac{\langle 3\eta_1\rangle\langle 3\eta_4\rangle\left(2m^2+\langle 2|1|2]\right)[32]^2m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 3|1|2]}\\
&-\frac{\left(2\left(2m^2+s_{23}\right)\langle\eta_1\eta_4\rangle+2\langle\eta_1|K_{12}K_{23}|\eta_4\rangle\right)[32]m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle}\Bigg)\\
&+I_3\left(s_{23},m^2,m^2,m^2,0,m^2\right)\frac{\left(2m^2+s_{23}\right)\langle\eta_1\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]} + I_3\left(s_{12},0,m^2,0,m^2,m^2\right)\Bigg(\\
&\frac{(2\langle\eta_1\eta_4\rangle\langle 23\rangle+4\langle 2\eta_4\rangle\langle 3\eta_1\rangle)[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2} + \frac{\langle 3\eta_1\rangle\langle 3\eta_4\rangle\langle 2|1|3][32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|2]} - \frac{\langle 2\eta_1\rangle\langle 2\eta_4\rangle\langle 3|1|2][32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|2]}\\
&+\frac{\langle 2\eta_1\rangle\langle 2\eta_4\rangle\langle 2|1|2][32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 2|1|3]} - \frac{\langle 3\eta_1\rangle\langle 3\eta_4\rangle\langle 2|1|2][32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2\langle 3|1|2]}\Bigg) + I_3\left(0,0,s_{23},m^2,m^2,m^2\right)\Bigg(\\
&\frac{\langle 2\eta_1\rangle\langle 2\eta_4\rangle[32]^2m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|3]} - \frac{\langle 3\eta_1\rangle\langle 3\eta_4\rangle[32]^2m^3}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 3|1|2]} - \frac{\langle\eta_1\eta_4\rangle[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle}\Bigg)\\
&-\widehat{I}_2\left(s_{12},0,m^2\right)\frac{\left(2s_{12}\langle\eta_1\eta_4\rangle-\langle\eta_1|K_{12}K_{23}|\eta_4\rangle\right)[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2} + \frac{c_\Gamma}{2\epsilon}\left(\frac{\mu_R^2}{m^2}\right)^\epsilon A^{(0)}(1_t^+,2^+,3^+,4_{\bar t}^+) + R_4^{[R]}
\end{aligned}
$$

Calculating the Rational Terms

- Remaining terms include tadpole coefficients

$$R^{[X]} = R^{[X],DD} + c_1 \left(I_1 - m^2 I_2(0; m^2, m^2) \right)$$

- Fix **all** from universal factorisation properties [Bern,Morgan]
- D-dimensional cuts efficient for numerical purposes [Ellis,Giele,Kunszt,Melnikov] [Melnikov,Schulze]
- Extract compact analytic forms from Feynman diagram computation [Körner,Merebashvili] [SB,Sattler (in progress)]
  - Diagram generation with Passarino-Veltman reduction
  - Spinor-Helicity representation
  - Removal of spurious poles

Rational Contributions

$$-iR^{[L]}(1_t^+,2^+,3^+,4_{\bar{t}}^+) = \frac{2\left(2m^2\langle\eta_1\eta_4\rangle - \langle\eta_1|(1+2)(2+3)|\eta_4\rangle\right)[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2}$$

$$-\frac{(\langle\eta_1|(1+2)(2+3)|\eta_4\rangle + \langle\eta_1\eta_4\rangle\langle 2|1|2])[32]m}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]} - \frac{(\langle\eta_1\eta_4\rangle\langle 2|1|2] + \langle\eta_1 2\rangle\langle\eta_4 3\rangle[32])m}{3\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle^2}$$

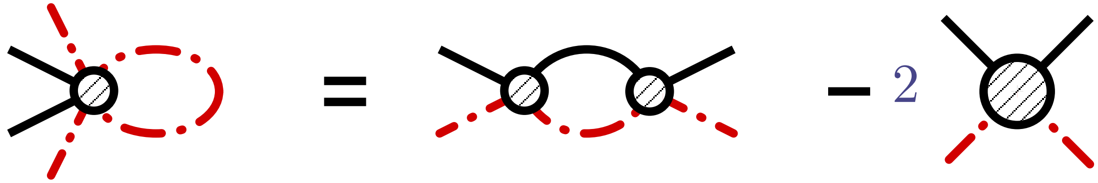
$$-iR^{[R]}(1_t^+,2^+,3^+,4_{\bar{t}}^+) = \frac{2\left(2m^2\langle\eta_1\eta_4\rangle - \langle\eta_1|(1+2)(2+3)|\eta_4\rangle\right)[32]m^3}{\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]^2}$$

$$+\frac{(\langle\eta_1|(1+2)(2+3)|\eta_4\rangle + \langle\eta_1\eta_4\rangle\langle 2|1|2])[32]m}{2\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle\langle 23\rangle\langle 2|1|2]}$$

- Full amplitude numerically verified [Ellis,Giele,Kunszt,Melnikov]
  - c++ implementation of spinor products
  - Scalar integrals with QCDloop [Ellis,Zanderighi]

Rational Contributions: Tadpole Coefficients

- Tadpole coefficients can be determined from cut-construcible terms:



$$C_1^{[L/R]}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) = C_{2;12}^{[L/R]}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) - 2A^{(0)}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+)$$

- Universal structure for both L and R amplitudes
- Verified by comparisons with D-dimensional cutting methods

[Ellis,Giele,Kunszt,Melnikov]

- Tadpole from small mass limit? [Moch,Mitov]

Outlook

- On-shell simplifications also persist in massive amplitudes
- Automated generalised unitarity extraction for arbitary masses
- IR and massless limits : compact expressions for $gg \to t\bar{t}$
- Extraction of tadpole coefficients without single cuts
- Automated approach : Extension for $t\bar{t} + j$ in progress