Towards a Loop–Tree Duality at Two Loops

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- Motivation
- Review of the duality–relation at one–loop order
- Towards a duality–relation at two–loop order
- Conclusions and outlook

Motivation

The goal of this method is:

Find a “duality–relation” between

one–/two–loop integrals (scattering amplitudes) with arbitrary number of external legs (momenta)	$\Leftrightarrow$	single–/double–cut Bremsstrahlung integrals

For one–loop case, see: [Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09(2008)064]

Review of the duality–relation at one–loop

Duality–theorem relies on the pole structure of a amplitude.

Consider different propagators:

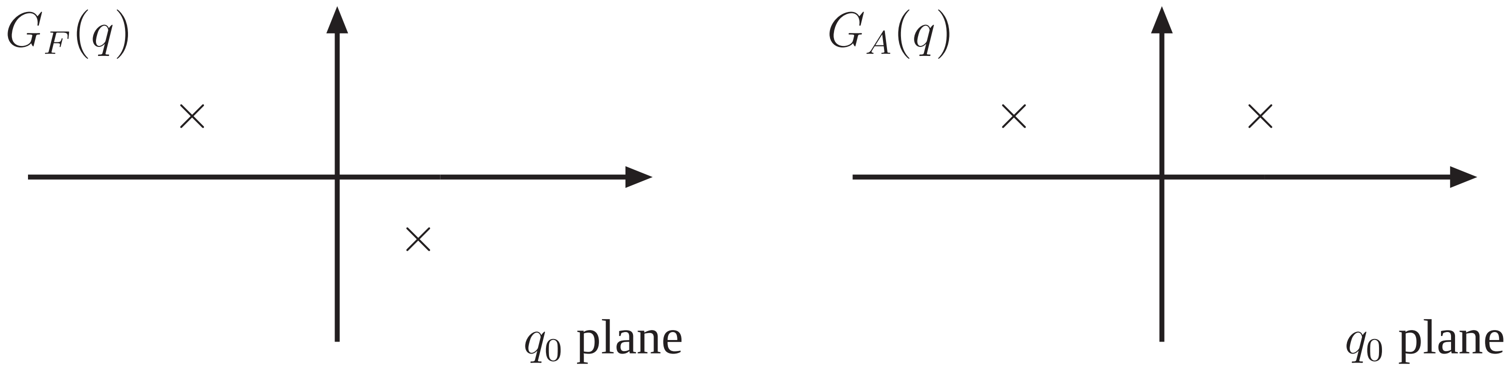
$$G_F(q) \equiv \frac{1}{q^2 + i0} \quad , \qquad G_A(q) \equiv \frac{1}{q^2 - i0\, q_0} \quad , \qquad G_R(q) \equiv \frac{1}{q^2 + i0\, q_0} \quad ,$$

$$[G_F(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 = \pm\sqrt{\mathbf{q}^2 - i0} \qquad \text{and} \qquad [G_A(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 \simeq \pm\sqrt{\mathbf{q}^2} + i0$$

$\Rightarrow$ in the complex plane of the variable q_0:

- Feynman propagator: the pole with positive/negative energy is slightly displaced below/above the real axis,
- both poles (independently of the sign of the energy) of the advanced/retarded propagator are slightly displaced above/below the real axis

Review of the duality–relation at one–loop



Using the identity:

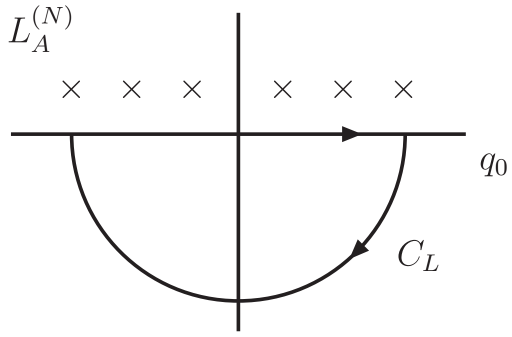
$$\frac{1}{x \pm i0} = \mathrm{PV}\left(\frac{1}{x}\right) \mp i\pi\,\delta(x) \quad ,$$

where PV is the principal value, we can define transformations from one description of propagators into the other:

$$G_A(q) \equiv G_F(q) + \widetilde{\delta}\,(q) \;\;, \quad G_R(q) \equiv G_F(q) + \widetilde{\delta}\,(-q) \;\;, \quad G_A(-q) \equiv G_R(q) \;, \quad G_F(-q) \equiv G_F(q)$$

$$\widetilde{\delta}\,(q) \equiv 2\pi\, i\, \theta(q_0)\; \delta(q^2) = 2\pi\, i\; \delta_+(q^2)$$

Feynman tree theorem:



$$0 \;=\; L_A^{(N)}(p_1,p_2,\dots,p_N) = \int_q \prod_{i=1}^{N} G_A(q_i) \;=\; \int_q \prod_{i=1}^{N} \left[G_F(q_i) + \widetilde{\delta}\,(q_i) \right]$$

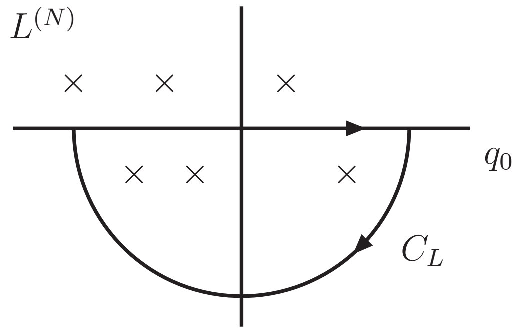
$$=\; L^{(N)}(p_1,p_2,\dots,p_N) + L^{(N)}_{1-\mathrm{cut}}(p_1,p_2,\dots,p_N) + \cdots + L^{(N)}_{\mathrm{N-cut}}(p_1,p_2,\dots,p_N)$$

with

$$L^{(N)}_{\mathrm{m-cut}}(p_1,p_2,\dots,p_N) = \int_q \left\{ \widetilde{\delta}\,(q_1)\dots\widetilde{\delta}\,(q_m)\;G_F(q_{m+1})\dots G_F(q_N) + \text{uneq. perms.} \right\}$$

$$L^{(N)}(p_1,p_2,\dots,p_N) = -\left[\, L^{(N)}_{1-\mathrm{cut}}(p_1,p_2,\dots,p_N) + \cdots + L^{(N)}_{\mathrm{N-cut}}(p_1,p_2,\dots,p_N) \,\right]$$

Duality theorem:



$$L^{(N)}(p_1, p_2, \ldots, p_N) = -2\pi i \int_{\mathbf{q}} \sum \left[\mathrm{Res}_{\{i\text{–th pole}\}} \, G_F(q_i)\right] \left[\prod_{\substack{j=1 \\ j\neq i}}^{N} G_F(q_j)\right]_{\{i\text{–th pole}\}}$$

$$\left[\mathrm{Res}_{\{i\text{–th pole}\}} \, G_F(q_i)\right] = \left[\mathrm{Res}_{\{i\text{–th pole}\}} \frac{1}{q_i^2 + i0}\right] = \int dq_0 \, \delta_+(q_i^2)$$

$$\left[\prod_{j\neq i} G_F(q_j)\right]_{\{i\text{–th pole}\}} = \left[\prod_{j\neq i} \frac{1}{q_j^2 + i0}\right]_{\{i\text{–th pole}\}} = \prod_{j\neq i} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$

where η is a *future-like* vector, $\eta_\mu = (\eta_0, \eta)$, $\eta_0 \geq 0, \; \eta^2 = \eta_\mu \eta^\mu \geq 0$

Review of the duality–relation at one–loop

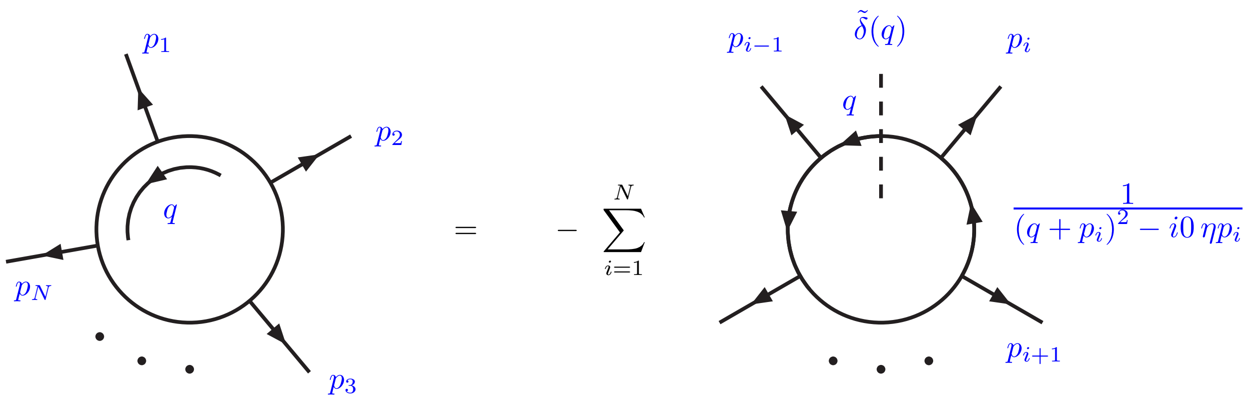
Define a dual propagator as: $$G_D(q_i, q_j) := \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$

which is related to the Feynman propagator via:

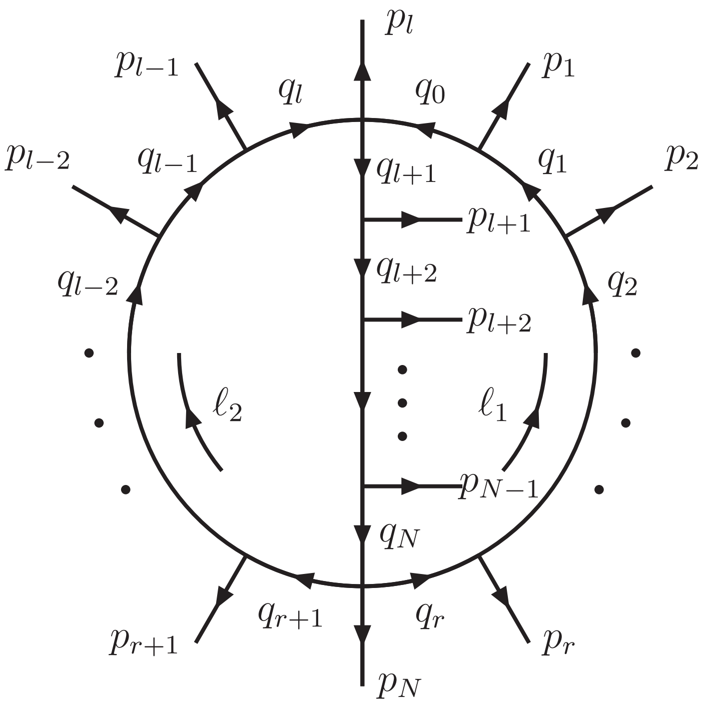
$$\tilde{\delta}(q_i)\; G_D(q_i, q_j) = \tilde{\delta}(q_i)\; \left[G_F(q_j) + \tilde{\theta}(q_j - q_i)\; \tilde{\delta}(q_j)\right], \qquad \tilde{\theta}(q) = \theta(\eta q)$$

Using this definition and putting everything together:

$$L^{(N)}(p_1, p_2, \ldots, p_N) \; = \; - \sum \int_q \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} G_D(q_i, q_j)$$



Towards a duality–relation at two–loop



$$q_i = \begin{cases} \ell_1 + p_{1,i} & 0 \le i \le r \ , \\ \ell_2 + p_{i,l-1} & r+1 \le i \le l \ , \\ \ell_{12} + p_{1,l-1} + p_{i,N} = \ell_{12} + p_{i,l-1} & l+1 \le i \le N \ , \end{cases}$$

where $\ell_{12} = \ell_1 + \ell_2$.

Momentum conservation: $$\sum_{i=1}^{N} p_i = 0 \ .$$

$$\alpha_1 \equiv [0, r] \ , \qquad \alpha_2 \equiv [r+1, l] \ , \qquad \alpha_3 \equiv [l+1, N] \ .$$

Towards a duality–relation at two–loop

I) Twice–dual:

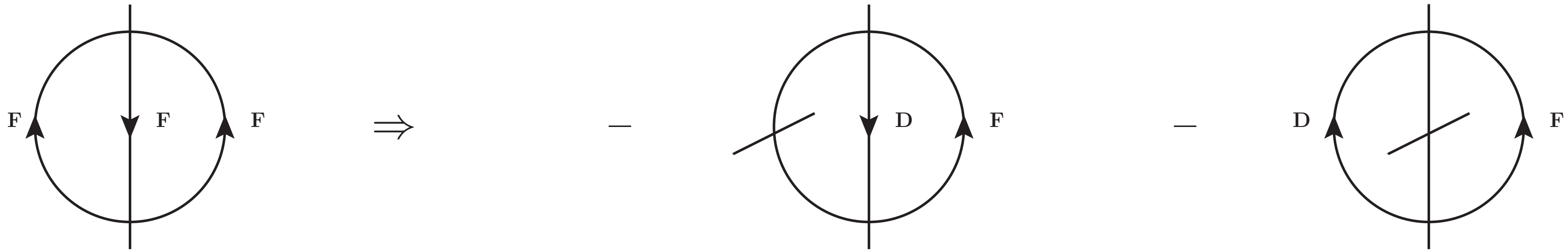
- Apply the Duality–relation to first loop momentum

- Re–express "appropriate" dual propagators via

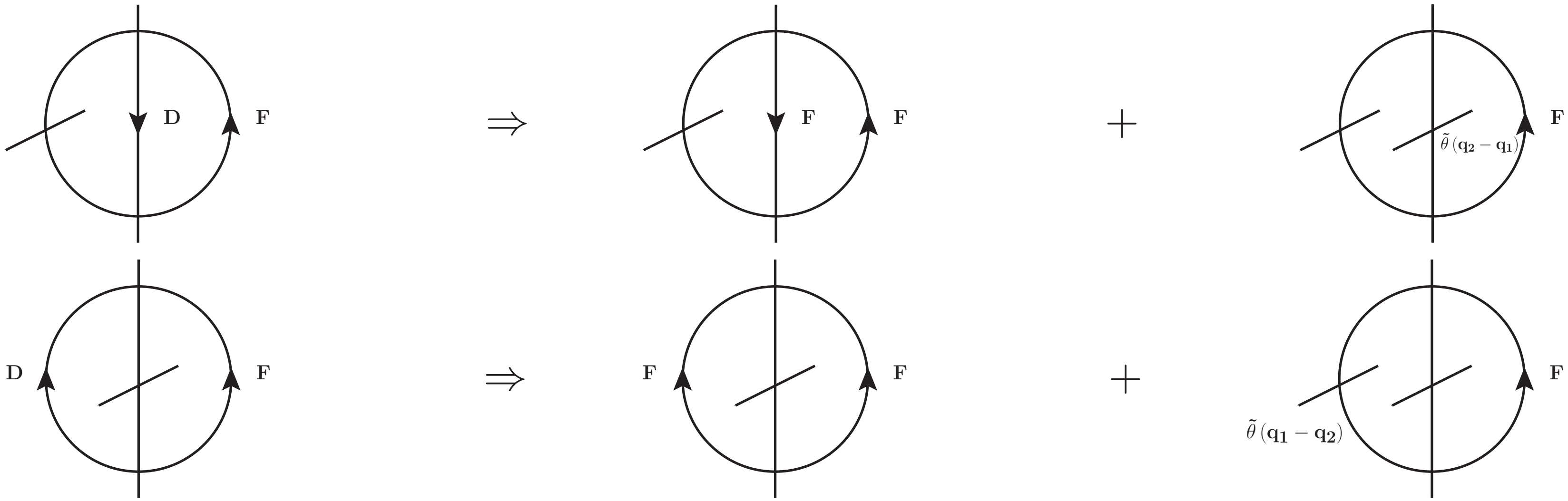
$$\widetilde{\delta}\,(q_i)\;G_D(q_i,q_j) = \widetilde{\delta}\,(q_i)\;\Big[G_F(q_j) + \tilde{\theta}\,(q_j - q_i)\;\widetilde{\delta}\,(q_j)\Big]$$

- Apply the Duality–relation to second loop momentum for terms with single cut

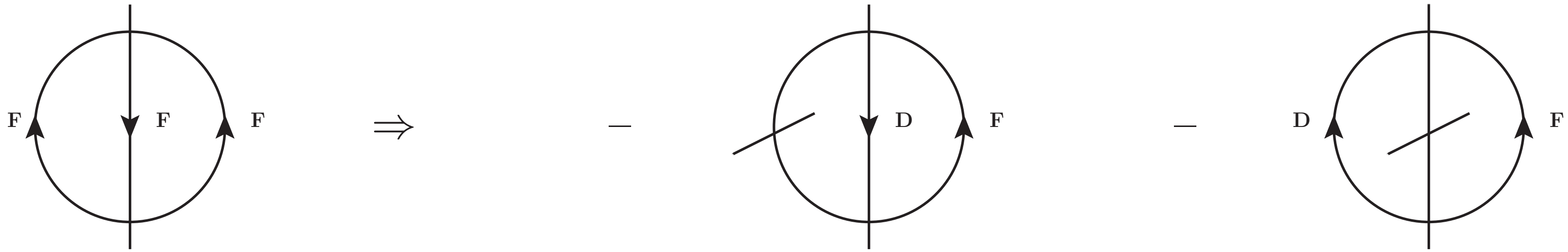
Towards a duality–relation at two–loop



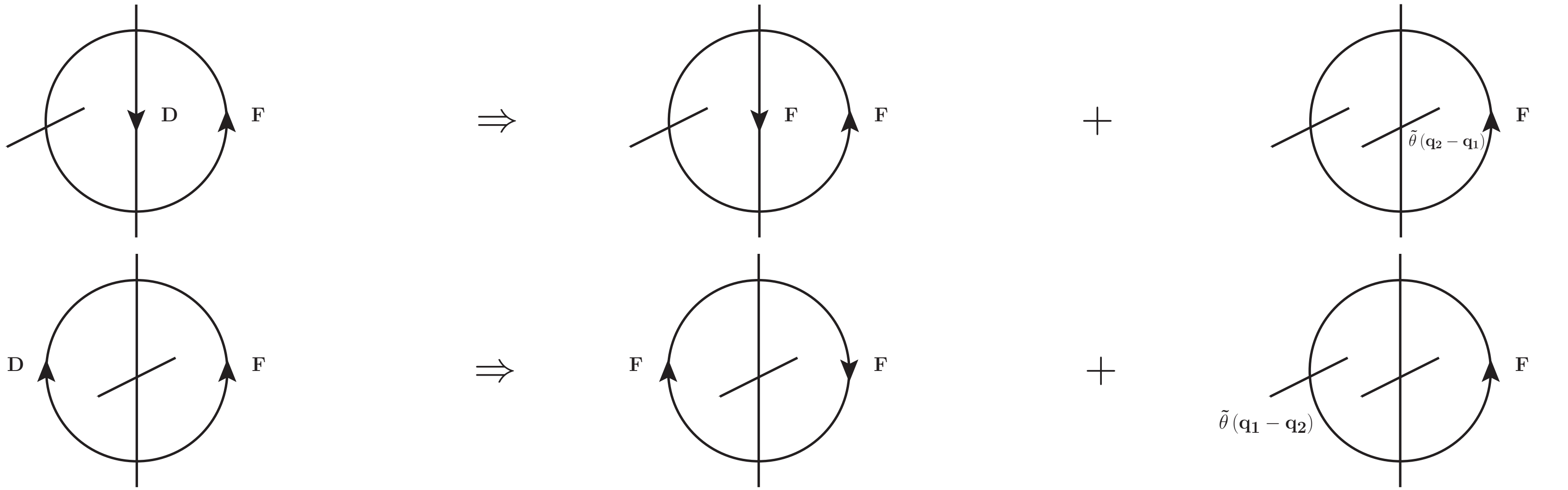
$$\widetilde{\delta}\,(q_i)\;\; G_D(q_i,q_j) = \widetilde{\delta}\,(q_i)\;\; \left[G_F(q_j) + \tilde{\theta}\,(q_j - q_i)\;\; \widetilde{\delta}\,(q_j)\right], \qquad G_D(q_i,q_j) = \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$



Towards a duality–relation at two–loop



$$\widetilde{\delta}\,(q_i)\;\; G_D(q_i,q_j) = \widetilde{\delta}\,(q_i)\;\left[G_F(q_j) + \tilde{\theta}\,(q_j - q_i)\;\; \widetilde{\delta}\,(q_j)\right], \quad G_D(q_i,q_j) = \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$



change: $q_0 \to -q_0$

Towards a duality–relation at two–loop

The final expression becomes

$$L^{(N)}_{2-\mathrm{loop}}(p_1,p_2,\ldots,p_N) = \int \frac{d^d\ell_1}{(2\pi)^d}\,\frac{d^d\ell_2}{(2\pi)^d}$$

$$\Bigg[\sum_{\substack{i\in\alpha_1\\ j\in\alpha_2,\alpha_3}} \widetilde{\delta}\,(q_i)\,\widetilde{\delta}\,(q_j) \prod_{\substack{k\in\alpha_1\\ k\neq i}} G_D(q_i;q_k) \prod_{\substack{l\in\alpha_2,\alpha_3\\ l\neq j}} G_D(q_j;q_l)$$

$$+ \sum_{\substack{i\in\alpha_1,\alpha_2\\ j\in\alpha_3}} \widetilde{\delta}\,(\beta_i q_i)\,\widetilde{\delta}\,(q_j) \prod_{\substack{k\in\alpha_1,\alpha_2\\ k\neq i}} G_D(\beta_i q_i;\beta_k q_k) \prod_{\substack{l\in\alpha_3\\ l\neq j}} G_D(q_j;q_l)$$

$$- \sum_{\substack{i\in\alpha_1\\ j\in\alpha_3}} \widetilde{\delta}\,(q_i)\,\widetilde{\delta}\,(q_j) \Bigg(\tilde{\theta}\,(q_j-q_i) \prod_{\substack{k\in\alpha_1,k\neq i\\ k\in\alpha_3,k>j}} G_D(q_i;q_k) \prod_{\substack{l\in\alpha_2\\ l\in\alpha_3,l<j}} G_F(q_l)$$

$$+\tilde{\theta}\,(q_i-q_j) \prod_{\substack{k\in\alpha_1,k>i\\ k\in\alpha_3,k\neq j}} G_D(q_j;q_k) \prod_{\substack{l\in\alpha_1,l<i\\ l\in\alpha_2}} G_F(q_l) \Bigg) \Bigg] \,.$$

Where $\beta_i = +1$ if $i\in\alpha_2$, and $\beta_i=-1$ if $i\in\alpha_1$.

Several of the $G_D(q_j,q_l)$ depend on ℓ_1 or ℓ_2 in the $i\eta$–description !

II) Advanced–dual: (Check)

- Express the first loop in terms of advanced propagators

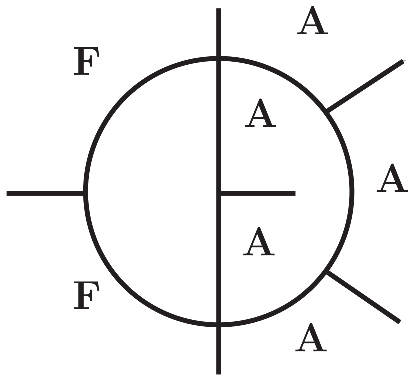
$$0 = \int_{\ell_1} \int_{\ell_2} \prod_{\substack{i \in \alpha_1 \vee \alpha_3 \\ j \in \alpha_2}} G_A(q_i) G_F(q_j)$$

- Re–express the advanced propagators via

$$G_A(q) \equiv G_F(q) + \widetilde{\delta}\,(q)$$

- Apply the Duality–relation to second loop momentum to generate the second cut

Towards a duality–relation at two–loop

$0 =$ 

$$= \int \frac{d^d\ell_1}{(2\pi)^d} \frac{d^d\ell_2}{(2\pi)^d} \left[G_A(q_0)G_A(q_1)G_A(q_2)\right] \left[G_F(q_3)G_F(q_4)\right] \left[G_A(q_5)G_A(q_6)\right]$$

For line α_1, we find:

$$G_A(q_0)G_A(q_1)G_A(q_2) = \left[G_F(q_0) + \widetilde{\delta}\,(q_0)\right] \left[G_F(q_1) + \widetilde{\delta}\,(q_1)\right] \left[G_F(q_2) + \widetilde{\delta}\,(q_2)\right]$$

$$= G_F(q_0)G_F(q_1)G_F(q_2) + \left[\widetilde{\delta}\,(q_0)\, G_F(q_1)G_F(q_2) + G_F(q_0)\widetilde{\delta}\,(q_1)\, G_F(q_2) + G_F(q_0)G_F(q_1)\widetilde{\delta}\,(q_2)\right]$$

$$+ \left[\widetilde{\delta}\,(q_0)\,\widetilde{\delta}\,(q_1)\, G_F(q_2) + G_F(q_0)\widetilde{\delta}\,(q_1)\,\widetilde{\delta}\,(q_2) + \widetilde{\delta}\,(q_0)\, G_F(q_1)\widetilde{\delta}\,(q_2)\right] + \widetilde{\delta}\,(q_0)\,\widetilde{\delta}\,(q_1)\,\widetilde{\delta}\,(q_2)$$

$$= G_F(q_0)G_F(q_1)G_F(q_2)$$

$$+ \left[\widetilde{\delta}\,(q_0)\, G_D(q_0,q_1)G_D(q_0,q_2) + \widetilde{\delta}\,(q_1)\, G_D(q_1,q_0)G_D(q_1,q_2) + \widetilde{\delta}\,(q_2)\, G_D(q_2,q_0)G_D(q_2,q_1)\right]$$

Which is generalizable to n propagators!

We can express the multiple–cut–contributions in α_1, α_3 respectively, in terms of dual propagators and single cuts only (proof by induction):

$$\sum_{\substack{unequal\\ perm(\alpha_k)}} \prod_{\substack{i_1,i_2\in\alpha_1\\ i_1+i_2=|\alpha_k|}} G_F(q_{i_1})\,\widetilde{\delta}(q_{i_2}) + \prod_{i=1}^{|\alpha_k|} \widetilde{\delta}(q_i) = \sum_{i\in\alpha_k} \widetilde{\delta}(q_i) \prod_{\substack{j\in\alpha_k\\ j\neq i}} G_D(q_i,q_j) =: G_D(\alpha_k)$$

with $G_D(q_i) := \widetilde{\delta}(q_i)$

$$G_F(\alpha_k) := \prod_{i\in\alpha_k} G_F(q_i)\;, \quad G_A(\alpha_k) := \prod_{i\in\alpha_k} G_A(q_i)\;, \quad G_R(\alpha_k) := \prod_{i\in\alpha_k} G_R(q_i)$$

Hence:

$$G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k)\;, \quad G_R(\alpha_k) = G_F(\alpha_k) + G_D(-\alpha_k)$$

Towards a duality–relation at two–loop

Going back to the example:

$$0 = \int_{\ell_1} \int_{\ell_2} G_A(\alpha_1) G_F(\alpha_2) G_A(\alpha_3) \quad = \quad \int_{\ell_1} \int_{\ell_2} \left[G_F(\alpha_1) + G_D(\alpha_1)\right] G_F(\alpha_2) \left[G_F(\alpha_3) + G_D(\alpha_3)\right]$$

Solving for the propagators with G_F only:

$$\begin{aligned} & \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1)\, G_F(\alpha_2)\, G_F(\alpha_3) \\ = \quad & -\int_{\ell_1} \int_{\ell_2} \left\{ G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_F(\alpha_2)\, G_F(\alpha_3) + G_F(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) \right\} \end{aligned}$$

- The first term is in the correct form
- Terms 2 and 3 need one further cut: apply dual method

Towards a duality–relation at two–loop

$$\int_{\ell_1}\int_{\ell_2} G_F(\alpha_1)\, G_F(\alpha_2)\, G_F(\alpha_3) = -\int_{\ell_1}\int_{\ell_2} G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3)$$

$$+\int_{\ell_1}\int_{\ell_2} G_D(\alpha_1) \left\{ \sum_{i\in\alpha_2\vee\alpha_3} \widetilde{\delta}\,(q_i) \prod_{\substack{j\in\alpha_2\vee\alpha_3 \\ j\neq i}} G_D(q_i,q_j) \right\}$$

$$+\int_{\ell_1}\int_{\ell_2} G_D(\alpha_3) \left\{ \sum_{i\in\alpha_1\vee\alpha_2} \widetilde{\delta}\,(\beta q_i) \prod_{\substack{j\in\alpha_1\vee\alpha_2 \\ j\neq i}} G_D(\beta q_i,\beta q_j) \right\}$$

Where $\beta_i = +1$ if $i \in \alpha_2$, and $\beta_i = -1$ if $i \in \alpha_1$.

Some of the $G_D(q_j, q_l)$ depend on ℓ_1 or ℓ_2 in the $i\eta$–description !

Towards a duality–relation at two–loop

Express all $\widetilde{\delta}\,(q_i)\;G_D(q_i,q_j)\;\rightarrow\;\widetilde{\delta}\,(q_i)\,G_F(q_j)+\tilde{\theta}\,(q_j-q_i)\;\widetilde{\delta}\,(q_i)\;\widetilde{\delta}\,(q_j)$

$\Rightarrow$ both final expressions are equal to the formula:

$$
\begin{aligned}
L^{(N)}_{2-\mathrm{loop}}(p_1,p_2,\ldots,p_N) &= \int_{\ell_1}\int_{\ell_2} \\
&\left[\sum_{\substack{n,m=1\\ i_r\in\alpha_1,\alpha_3\\ j_r\in\alpha_2}}^{n+m=Z}\left\{\widetilde{\delta}\,(q_{i_1})\cdots\widetilde{\delta}\,(q_{i_n})\;\widetilde{\delta}\,(q_{j_1})\cdots\widetilde{\delta}\,(q_{j_m})\;G_F(q_{k_1})\cdots G_F(q_{k_s})+\text{uneq. perm.}\right\}\right. \\
&\left.+\sum_{\substack{n,m=1\\ i_r\in\alpha_1\\ j_1\in\alpha_3\\ j_r\in\alpha_2,\alpha_3,r>1}}^{n+m=Z}\left\{\widetilde{\delta}\,(-q_{i_1})\cdots\widetilde{\delta}\,(-q_{i_n})\;\widetilde{\delta}\,(q_{j_1})\cdots\widetilde{\delta}\,(q_{j_m})\;G_F(q_{k_1})\cdots G_F(q_{k_s})+\text{uneq. perm.}\right\}\right]
\end{aligned}
$$

where Z is the number of loop internal lines.

This shows that both formulae are independent of the "artificial" vector η, as expected!

Towards a duality–relation at two–loop

We can find a description for the integral as:

$$L^{(N)}_{2-\mathrm{loop}}(p_1, \cdots, p_n) =$$
$$\int_{\ell_1} \int_{\ell_2} \left\{ G_D(\alpha_1)\, G_D(\alpha_2)\, G_F(\alpha_3) + G_D(-\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G^*(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3) \right\}$$

where: $$G^*(\alpha_k) \quad := \quad G_A(\alpha_k) + G_R(\alpha_k) - G_F(\alpha_k)$$

- This relation can be achieved using both descriptions
- It is not unique, but there are several possible representations, depending on the choice of propagators

Towards a duality–relation at two–loop

Using $$G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k)\ , \quad G_R(\alpha_k) = G_F(\alpha_k) + G_D(-\alpha_k)$$

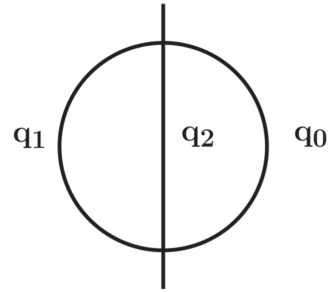
$$\begin{aligned} G^*(\alpha_k) &= G_A(\alpha_k) + G_R(\alpha_k) - G_F(\alpha_k) \\ &= G_F(\alpha_k) + G_D(\alpha_k) + G_D(-\alpha_k) \end{aligned}$$

Hence, if we allow for triple–cuts, we obtain:

$$L^{(N)}_{2-\mathrm{loop}}(p_1, \cdots, p_n) =$$
$$\int_{\ell_1} \int_{\ell_2} \Big\{ G_D(\alpha_1)\, G_D(\alpha_2)\, G_F(\alpha_3) + G_D(-\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G_F(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3)$$
$$+ G_D(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3) + G_D(-\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3) \Big\}$$

$\Rightarrow$ This double–integral is given in terms of only G_F and G_D !!

Towards a duality–relation at two–loop



Examples:

The massless sunrise two–loop integral is given by

$$L^{(2)}_{2-\text{loop}}(p_1, p_2) = \int_{q_1} \int_{q_2} G_F(\ell_1) G_F(\ell_2) G_F(\ell_1 + \ell_2 + p_1) \, ,$$

with $q_0 = \ell_1$, $q_1 = \ell_2$, and $q_2 = \ell_1 + \ell_2 + p_1$.

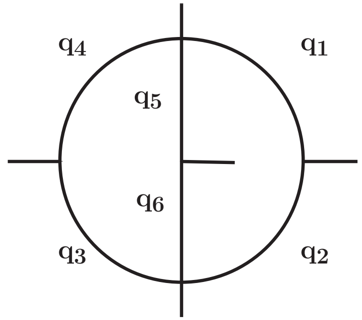
$$L^{(N)}_{2-\text{loop}}(p_1) \quad = \quad \int_{\ell_1} \int_{\ell_2} \left\{ \widetilde{\delta}\,(q_0)\, \widetilde{\delta}\,(q_1)\, G_F(q_2) + \widetilde{\delta}\,(-q_0)\, G_F(q_1) \widetilde{\delta}\,(q_2) + G^*(q_0) \widetilde{\delta}\,(q_1)\, \widetilde{\delta}\,(q_2) \right\} .$$

Where the last term is given by

$$\begin{aligned} G^*(q_0) \quad &\equiv \quad \underbrace{G_F(q_0) + \widetilde{\delta}\,(q_0)}_{G_A(q_0)} + \underbrace{G_F(q_0) + \widetilde{\delta}\,(-q_0)}_{G_R(q_0)} - G_F(q_0) \\ &= \quad G_F(q_0) + \widetilde{\delta}\,(q_0) + \widetilde{\delta}\,(-q_0) \; = \; \frac{1}{q_0^2 - i0} \, . \end{aligned}$$

Towards a duality–relation at two–loop

Consider the case: $\{q_1, q_2\} \in \alpha_1$, $\{q_3, q_4\} \in \alpha_2$, and $\{q_5, q_6\} \in \alpha_3$.



$$L^{(N)}_{2-\text{loop}}(p_1,p_2,p_3,p_4) = \int_{\ell_1}\int_{\ell_2}$$

$$\times \Big\{ \left[\widetilde{\delta}\,(q_1)\; G_D(q_1,q_2) + \widetilde{\delta}\,(q_2)\; G_D(q_2,q_1)\right] \left[\widetilde{\delta}\,(q_3)\; G_D(q_3,q_4) + \widetilde{\delta}\,(q_4)\; G_D(q_4,q_3)\right] G_F(q_5)\, G_F(q_6)$$

$$+ \left[\widetilde{\delta}\,(-q_1)\; G_D(-q_1,-q_2) + \widetilde{\delta}\,(-q_2)\; G_D(-q_2,-q_1)\right] G_F(q_3)\, G_F(q_4)$$

$$\times \left[\widetilde{\delta}\,(q_5)\; G_D(q_5,q_6) + \widetilde{\delta}\,(q_6)\; G_D(q_6,q_5)\right]$$

$$+ \Big[G_A(q_1)\, G_A(q_2) + G_R(q_1)\, G_R(q_2) \;-\; G_F(q_1)\, G_F(q_2)\Big] \left[\widetilde{\delta}\,(q_3)\; G_D(q_3,q_4) + \widetilde{\delta}\,(q_4)\; G_D(q_4,q_3)\right]$$

$$\times \left[\widetilde{\delta}\,(q_5)\; G_D(q_5,q_6) + \widetilde{\delta}\,(q_6)\; G_D(q_6,q_5)\right] \Big\}\,.$$

Or with triple cuts:

$$L^{(N)}_{2-\text{loop}}(p_1,p_2,p_3,p_4) = \int_{\ell_1}\int_{\ell_2}$$

$$\times\Big\{ \left[\widetilde{\delta}\,(q_1)\; G_D(q_1,q_2) + \widetilde{\delta}\,(q_2)\; G_D(q_2,q_1)\right] \left[\widetilde{\delta}\,(q_3)\; G_D(q_3,q_4) + \widetilde{\delta}\,(q_4)\; G_D(q_4,q_3)\right] G_F(q_5)\, G_F(q_6)$$

$$+ \left[\widetilde{\delta}\,(-q_1)\; G_D(-q_1,-q_2) + \widetilde{\delta}\,(-q_2)\; G_D(-q_2,-q_1)\right] G_F(q_3)\, G_F(q_4) \left[\widetilde{\delta}\,(q_5)\; G_D(q_5,q_6) + \widetilde{\delta}\,(q_6)\; G_D(q_6,q_5)\right]$$

$$+ G_F(q_3)\, G_F(q_4) \left[\widetilde{\delta}\,(q_3)\; G_D(q_3,q_4) + \widetilde{\delta}\,(q_4)\; G_D(q_4,q_3)\right] \left[\widetilde{\delta}\,(q_5)\; G_D(q_5,q_6) + \widetilde{\delta}\,(q_6)\; G_D(q_6,q_5)\right]$$

$$+ \Bigg[\left[\widetilde{\delta}\,(q_1)\; G_D(q_1,q_2) + \widetilde{\delta}\,(q_2)\; G_D(q_2,q_1)\right] + \left[\widetilde{\delta}\,(-q_1)\; G_D(-q_1,-q_2) + \widetilde{\delta}\,(-q_2)\; G_D(-q_2,-q_1)\right] \Bigg]$$

$$\times \left[\widetilde{\delta}\,(q_3)\; G_D(q_3,q_4) + \widetilde{\delta}\,(q_4)\; G_D(q_4,q_3)\right] \left[\widetilde{\delta}\,(q_5)\; G_D(q_5,q_6) + \widetilde{\delta}\,(q_6)\; G_D(q_6,q_5)\right] \Big\}\,.$$

Conclusions:

- Duality at one-loop is under testing for numerical efficiency (ask Petros)
- We found a formula which expresses a general two–loop diagram with n propagators in terms of double–cuts only, where the propagators are dual–propagators, with most of them, however not all, independent of integration momenta in their $i\eta$–description.
- If we allow for triple cuts, we can define a formula which expresses a general two–loop diagram with n propagators in terms of double– and triple–cuts, where all propagators are dual– or Feynman–propagators, meaning that the expression is completely independent of integration momenta in its $i\eta$–description.

Outlook (of an impatient optimist):

- Duality can be extended to higher orders beyond two-loops (under investigation)

Thank you !

Proof of the relation:

$$\sum_{\substack{unequal\\ perm(\alpha_k)}} \prod_{\substack{i_1,i_2\in\alpha_1\\ i_1+i_2=|\alpha_k|}} G_F(q_{i_1})\,\widetilde{\delta}\,(q_{i_2}) + \prod_{i=1}^{|\alpha_k|}\widetilde{\delta}\,(q_i) = \sum_{i\in\alpha_k}\widetilde{\delta}\,(q_i)\prod_{\substack{j\in\alpha_k\\ j\neq i}} G_D(q_i,q_j)$$

Consider a set of n real variables λ_i, with $i = 1, 2, \ldots, n$, that fulfill the constraint

$$\sum_{i=1}^{n}\lambda_i = 0 \;\; .$$

which is basically the statement of momentum–conservation. We then have the following:

$$\theta(\lambda_1)\,\theta(\lambda_1+\lambda_2)\,\ldots\,\theta(\lambda_1+\lambda_2+\cdots+\lambda_{n-1}) + \text{cyclic perms.} = 1 \;\; ,$$

Setting $\lambda_i = q_i - q_{i+1}$ for $i \in \{1, ..., n\}$, with $(n+i) \equiv i \bmod n$, which automatically fulfills this requirement, this can also be written as:

$$\theta(q_1-q_2)\,\theta(q_1-q_3)\;\ldots\;\theta(q_1-q_n) + \text{cyclic perms.} = 1$$

$$\theta(q_2-q_1)\,\theta(q_3-q_1)\;\ldots\;\theta(q_n-q_1) + \text{cyclic perms.} = 1$$

Take the following example for $\alpha = \{q_1, q_2, q_3\}$, $|\alpha| = 3$:

$$
\begin{aligned}
&\sum_{i\in\alpha} \widetilde{\delta}\,(q_i) \prod_{\substack{j\in\alpha\\ j\neq i}} G_D(q_i, q_j) \\
= \; & \widetilde{\delta}\,(q_1)\, G_D(q_1, q_2) G_D(q_1, q_3) + \widetilde{\delta}\,(q_2)\, G_D(q_2, q_1) G_D(q_2, q_3) + \widetilde{\delta}\,(q_3)\, G_D(q_3, q_1) G_D(q_3, q_2) \\
= \; & \widetilde{\delta}\,(q_1) \left[G_F(q_2) + \tilde{\theta}\,(q_2 - q_1)\, \widetilde{\delta}\,(q_2) \right] \left[G_F(q_3) + \tilde{\theta}\,(q_3 - q_1)\, \widetilde{\delta}\,(q_3) \right] \\
& + \widetilde{\delta}\,(q_2) \left[G_F(q_1) + \tilde{\theta}\,(q_1 - q_2)\, \widetilde{\delta}\,(q_1) \right] \left[G_F(q_3) + \tilde{\theta}\,(q_3 - q_2)\, \widetilde{\delta}\,(q_3) \right] \\
& + \widetilde{\delta}\,(q_3) \left[G_F(q_1) + \tilde{\theta}\,(q_1 - q_3)\, \widetilde{\delta}\,(q_1) \right] \left[G_F(q_2) + \tilde{\theta}\,(q_2 - q_3)\, \widetilde{\delta}\,(q_2) \right] \\
= \; & \widetilde{\delta}\,(q_1)\, G_F(q_2) G_F(q_3) + \widetilde{\delta}\,(q_2)\, G_F(q_1) G_F(q_3) + \widetilde{\delta}\,(q_3)\, G_F(q_1) G_F(q_2) \\
& + G_F(q_2) \widetilde{\delta}\,(q_1)\, \widetilde{\delta}\,(q_3) \left[\tilde{\theta}\,(q_3 - q_1) + \tilde{\theta}\,(q_1 - q_3) \right] \\
& + G_F(q_3) \widetilde{\delta}\,(q_1)\, \widetilde{\delta}\,(q_2) \left[\tilde{\theta}\,(q_2 - q_1) + \tilde{\theta}\,(q_1 - q_2) \right] \\
& + G_F(q_1) \widetilde{\delta}\,(q_2)\, \widetilde{\delta}\,(q_3) \left[\tilde{\theta}\,(q_2 - q_3) + \tilde{\theta}\,(q_3 - q_2) \right] \\
& + \left[\tilde{\theta}\,(q_1 - q_2)\, \tilde{\theta}\,(q_3 - q_2) + \tilde{\theta}\,(q_2 - q_1)\, \tilde{\theta}\,(q_3 - q_1) + \tilde{\theta}\,(q_1 - q_3)\, \tilde{\theta}\,(q_2 - q_3) \right] \widetilde{\delta}\,(q_1)\, \widetilde{\delta}\,(q_2)\, \widetilde{\delta}\,(q_3)\,.
\end{aligned}
$$