Dimensional recurrence relations: an easy way to evaluate higher orders of expansion in ϵ

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and

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in collaboration with R.N. Lee and A.V. Smirnov

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- Non-planar massless propagator diagram (as a by-product)
- Conclusion

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Theorem [A. Smirnov & A. Petukhov'10]

The number of master integrals is finite

Solving IBP relations by Laporta's algorithm

[S. Laporta & E. Remiddi'96; S. Laporta'00; T. Gehrmann & E. Remiddi'01]

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Solving IBP relations in other ways

[Baikov, Tarasov, Lee]

A new method of evaluating master integrals is based on the use of dimensional recurrence relations [O. Tarasov'96] and analytic properties of Feynman integrals as functions of d.

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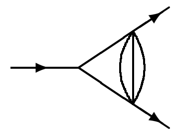
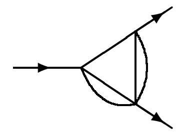
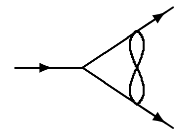
The missing finite parts of $A_{9,2}$ and $A_{9,4}$ were recently analytically evaluated [R. Lee, A. and V. Smirnovs'10]



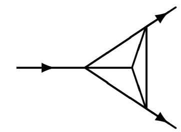
$A_{5,1}$



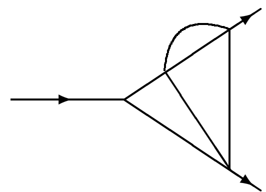
$A_{5,2}$



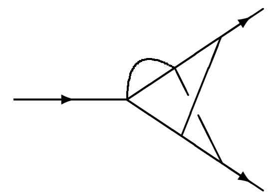
$A_{6,1}$



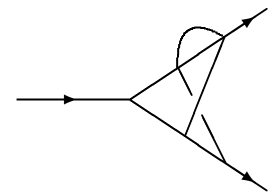
$A_{6,2}$



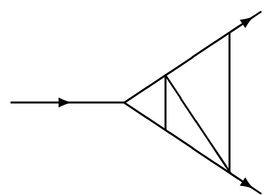
$A_{6,3}$



$A_{7,1}$



$A_{7,2}$



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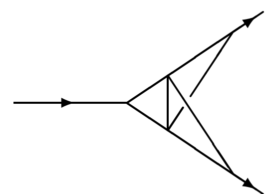
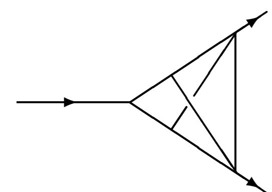
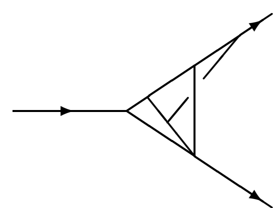


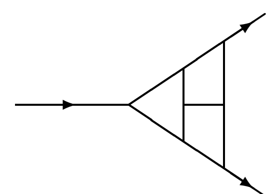
$A_{7,4}$



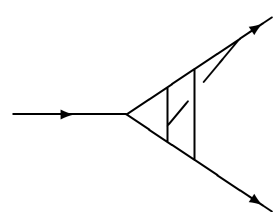
$A_{7,5}$



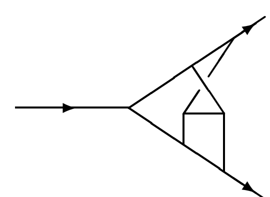
A_8



$A_{9,1}$

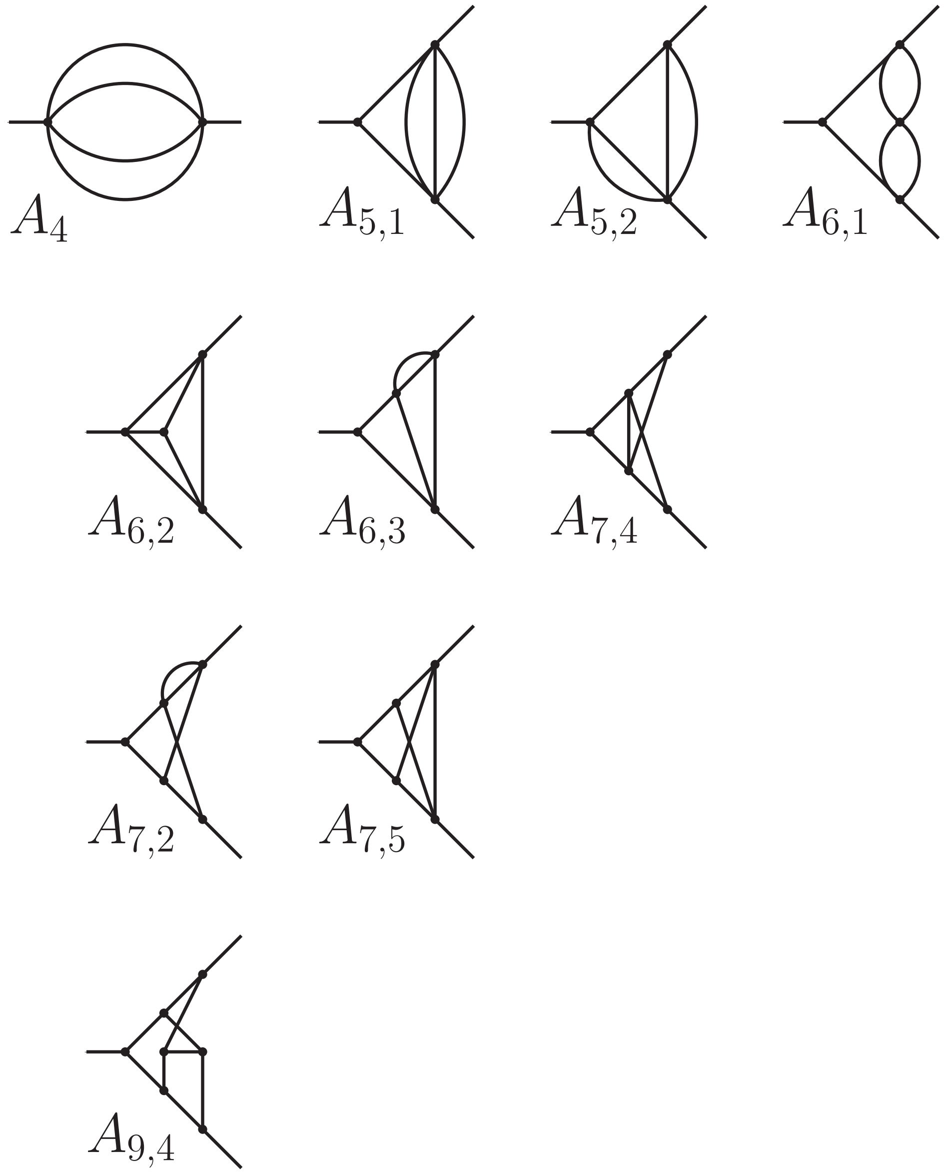


$A_{9,2}$



$A_{9,4}$

$A_{9,4}$ and lower master integrals



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Four lower master integrals, A_4, $A_{5,1}$, $A_{5,2}$, and $A_{6,3}$.

A_4, $A_{5,1}$, $A_{5,2}$ are of complexity level 0.

$A_{6,3}$ is of complexity level 1.
Solving dimensional recurrence relation $\to$

$$A_{6,3}(d) \;=\; A^{1,1}_{6,3}(d)\sum_{k=0}^{\infty} A^{1,2}_{6,3}(d+2k) + A^{2}_{6,3}(d)\,,$$

$$A^{1,1}_{6,3}(d) \;=\; -\sin(\pi d)A^{2}_{6,3}(d) = \frac{\pi^4 2^{11-3d}\csc\left(\frac{3\pi d}{2}\right)\csc\left(\frac{\pi d}{2}\right)}{(3d-10)\Gamma\left(d-\frac{5}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\,,$$

$$A^{1,2}_{6,3}(d) \;=\; \frac{(7d-18)\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^3}{3\pi^2(d-3)\Gamma\left(\frac{3d}{2}-3\right)}\,.$$

After $A_{6,3}$ turn to $A_{7,2}$.

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The dimensional recurrence relation gives

$$A_{7,2}(d+2) = c_{7,2}(d)A_{7,2}(d)$$
$$+c_{6,3}(d)A_{6,3}(d) + c_{5,2}(d)A_{5,2}(d) + c_{5,1}(d)A_{5,1}(d) + c_4(d)A_4(d)$$

where c_n are rational functions.

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Turn to $\tilde{A}_{7,2}(d) = \Sigma(d)A_{7,2}(d)$
where the summing factor $\Sigma(d)$ is chosen as

$$\Sigma\left(d\right) = \frac{(d-3)\cos\left(\frac{\pi d}{2}\right)\cos\left(\frac{\pi}{6}-\frac{\pi d}{2}\right)\cos\left(\frac{\pi d}{2}+\frac{\pi}{6}\right)\Gamma\left(\frac{5d}{2}-9\right)}{\Gamma\left(\frac{d}{2}-2\right)^2}\,.$$

For $\tilde{A}_{7,2}(d)$, the recurrence relation is simpler:

$$\tilde{A}_{7,2}(d+2) \;\; = \;\; \tilde{A}_{7,2}(d) + \tilde{A}_{6,3}(d) + \tilde{A}_{5,2}(d) + \tilde{A}_{5,1}(d) + \tilde{A}_{4}(d)\,,$$

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where $\tilde{A}_n(d) = \Sigma(d+2)c_n(d)A_n(d)$.

The general solution:

$$\tilde{A}_{7,2}(d) = \sum_{l=0}^{\infty} \left[\tilde{A}_{5,2}(d-2-2l) + \tilde{A}_{5,1}(d-2-2l) + \tilde{A}^{2}_{6,3}(d-2-2l) \right]$$

$$-\sum_{l=0}^{\infty} \tilde{A}^{1,1}_{6,3}(d+2l) \sum_{k=0}^{\infty} A^{1,2}_{6,3}(d+2l+2k) - \sum_{l=0}^{\infty} \tilde{A}_4(d+2l) + \omega(z)\,,$$

where $z = \exp[i\pi d]$.

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where `U` and `F` are the basic functions, h is the number of loops, `degrees` are the indices, and `dmin` and `dmax` are values of the real part of d that determine the basic stripe.

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$\omega(z)$ is fixed up to the function $a_1 + a_2 \cot\left(\frac{\pi}{2}(d-6)\right)$

To fix the two constants, an MB representation can be used

$$A_{7,2}(d) = \frac{\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}-1\right)^2\Gamma(d-3)}{\Gamma(d-2)\Gamma\left(\frac{3d}{2}-5\right)\Gamma(2d-7)} \frac{1}{(2\pi)^2} \int\int \frac{\Gamma(-z_1)\Gamma(-z_2)}{\Gamma(d-z_1-4)}$$

$$\times \frac{\Gamma\left(\frac{d}{2}-z_1-2\right)}{\Gamma\left(\frac{3d}{2}-z_1-5\right)} \Gamma\left(\frac{3d}{2}-z_2-6\right)\Gamma(z_1+z_2+1)\Gamma(d-z_1-z_2-5)$$

$$\times\Gamma(z_2+1)^2\Gamma\left(\frac{3d}{2}-z_1-z_2-6\right)\Gamma\left(-\frac{3d}{2}+z_1+z_2+7\right) \mathrm{d}z_1\,\mathrm{d}z_2\,.$$

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MB tools at http://projects.hepforge.org/mbtools/

MB.m [M. Czakon'05]
MBresolve.m [A. Smirnov'09]
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$$\times\Gamma(z_2+1)^2\Gamma\left(\frac{3d}{2}-z_1-z_2-6\right)\Gamma\left(-\frac{3d}{2}+z_1+z_2+7\right) \mathrm{d}z_1\, \mathrm{d}z_2 \,.$$

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$$A_{7,2}(6-2\epsilon) = -\frac{41}{15552\epsilon} + O(\epsilon^0), \quad A_{7,2}(5-2\epsilon) = -\frac{\pi^{5/2}}{24\epsilon} + O(\epsilon^0)$$

$$
\begin{aligned}
\omega(z) \quad = \quad & \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi}{10} - \frac{\pi d}{2}\right) - \frac{\pi^3}{36} \tan\left(\frac{\pi}{6} - \frac{\pi d}{2}\right) \\
& - \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi d}{2} + \frac{\pi}{10}\right) + \frac{\pi^3}{36} \tan\left(\frac{\pi d}{2} + \frac{\pi}{6}\right) \\
& + \frac{\pi^3}{60} \cot^3\left(\frac{\pi d}{2}\right) + \frac{13\pi^3}{180} \cot\left(\frac{\pi d}{2}\right) \\
& + \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi}{5} - \frac{\pi d}{2}\right) - \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi d}{2} + \frac{\pi}{5}\right) \; .
\end{aligned}
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& -\frac{\pi^3}{20\sqrt{5}}\tan\left(\frac{\pi d}{2}+\frac{\pi}{10}\right)+\frac{\pi^3}{36}\tan\left(\frac{\pi d}{2}+\frac{\pi}{6}\right) \\
& +\frac{\pi^3}{60}\cot^3\left(\frac{\pi d}{2}\right)+\frac{13\pi^3}{180}\cot\left(\frac{\pi d}{2}\right) \\
& +\frac{\pi^3}{20\sqrt{5}}\cot\left(\frac{\pi}{5}-\frac{\pi d}{2}\right)-\frac{\pi^3}{20\sqrt{5}}\cot\left(\frac{\pi d}{2}+\frac{\pi}{5}\right) .
\end{aligned}
$$

A result is then expressed in terms of a double series with factorized expressions. Calculating it with very high precision and using PSLQ [H.R.P. Ferguson & D.H. Bailey'91]

$$A_{9,4}(4-2\epsilon) = e^{-3\gamma_E\epsilon}\left\{-\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \left[1+\frac{43\pi^2}{108}\right]\frac{1}{\epsilon^4}\right.$$

$$+\left[\frac{109\zeta(3)}{9} + \frac{14}{9} + \frac{53\pi^2}{27}\right]\frac{1}{\epsilon^3}$$

$$+\left[\frac{608\zeta(3)}{9} - 17 - \frac{311\pi^2}{108} - \frac{481\pi^4}{12960}\right]\frac{1}{\epsilon^2}$$

$$+\left[-\frac{949\zeta(3)}{9} - \frac{2975\pi^2\zeta(3)}{108} + \frac{3463\zeta(5)}{45} + 84 + \frac{11\pi^2}{18} + \frac{85\pi^4}{108}\right]\frac{1}{\epsilon}$$

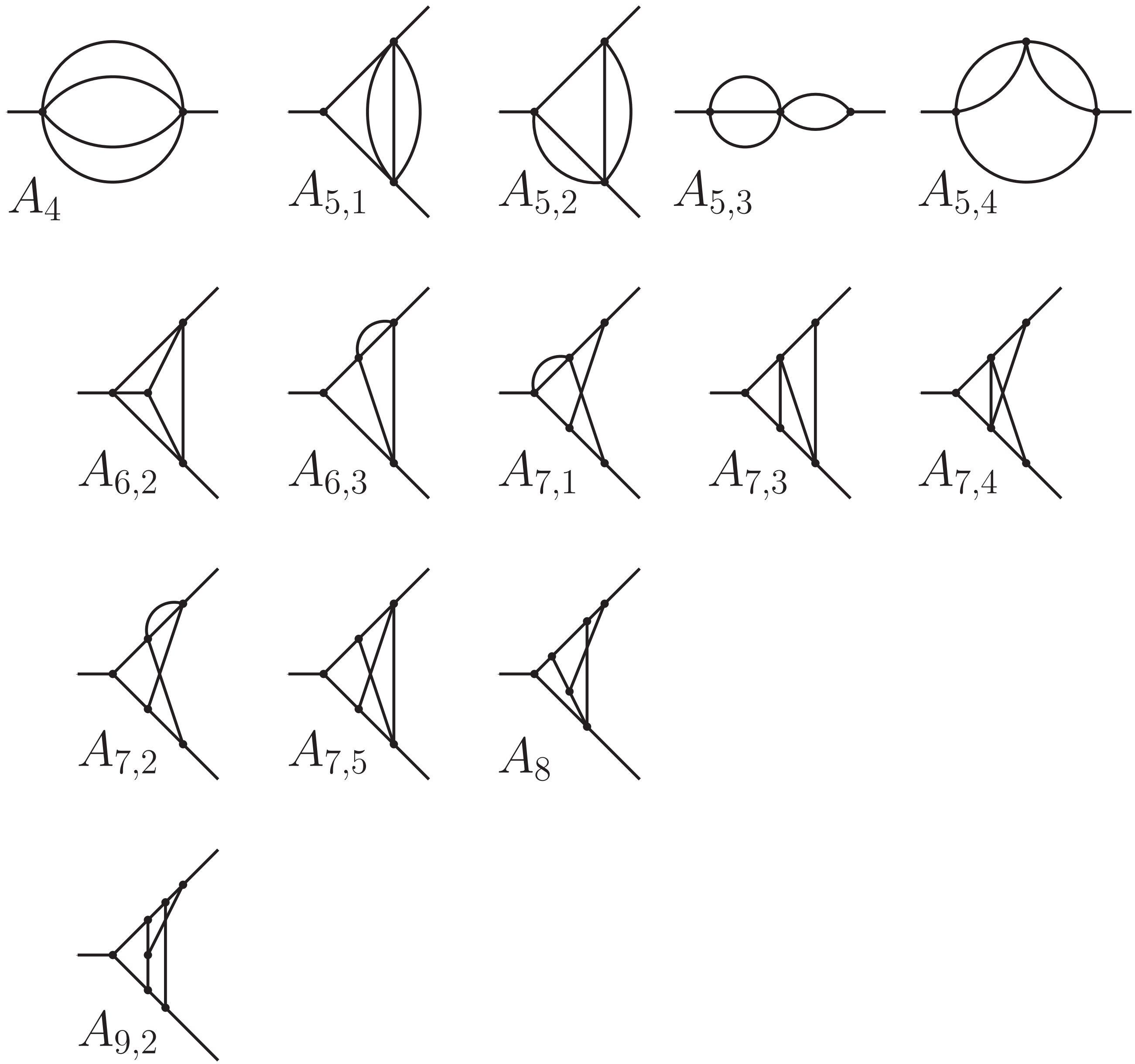
[P. Baikov, K. Chetyrkin, A. and V. Smirnovs, & M. Steinhauser'09]

[G. Heinrich, T. Huber, D. Kosower and V. Smirnov'09]

$$+\left[\frac{434\zeta(3)}{9} - \frac{299\pi^2\zeta(3)}{3} - \frac{3115\zeta(3)^2}{6} + \frac{7868\zeta(5)}{15} - 339\right.$$

$$\left.\left.+\frac{77\pi^2}{4} - \frac{2539\pi^4}{2592} - \frac{247613\pi^6}{466560}\right] + O(\epsilon)\right\}$$ [R. Lee, A. and V. Smirnovs'10]

$A_{9,2}$ and lower master integrals



$$A_{9,2}(4-2\epsilon) = e^{-3\gamma_E\epsilon}\left\{ -\frac{2}{9\epsilon^6} - \frac{5}{6\epsilon^5} + \left[\frac{20}{9} + \frac{17\pi^2}{54}\right]\frac{1}{\epsilon^4} \right.$$

$$+ \left[\frac{31\zeta(3)}{3} - \frac{50}{9} + \frac{181\pi^2}{216}\right]\frac{1}{\epsilon^3}$$

$$+ \left[\frac{347\zeta(3)}{18} + \frac{110}{9} - \frac{17\pi^2}{9} + \frac{119\pi^4}{432}\right]\frac{1}{\epsilon^2}$$

$$+ \left[-\frac{514\zeta(3)}{9} - \frac{341\pi^2\zeta(3)}{36} + \frac{2507\zeta(5)}{15} - \frac{170}{9} + \frac{19\pi^2}{6} + \frac{163\pi^4}{960}\right]\frac{1}{\epsilon}$$

[P. Baikov, K. Chetyrkin, A. and V. Smirnovs, & M. Steinhauser'09]

[G. Heinrich, T. Huber, D. Kosower and V. Smirnov'09]

$$+ \left[\frac{1516\zeta(3)}{9} - \frac{737\pi^2\zeta(3)}{24} - 29\zeta(3)^2 + \frac{2783\zeta(5)}{6} - \frac{130}{9}\right.$$

$$\left. \left. + \frac{\pi^2}{2} - \frac{943\pi^4}{1080} + \frac{195551\pi^6}{544320}\right] + O(\epsilon) \right\}$$ [R. Lee, A. and V. Smirnovs'10]

Evaluating three-loop quark and gluon form factors

[Baikov, Chetyrkin, A. and V. Smirnovs, and Steinhauser’09]

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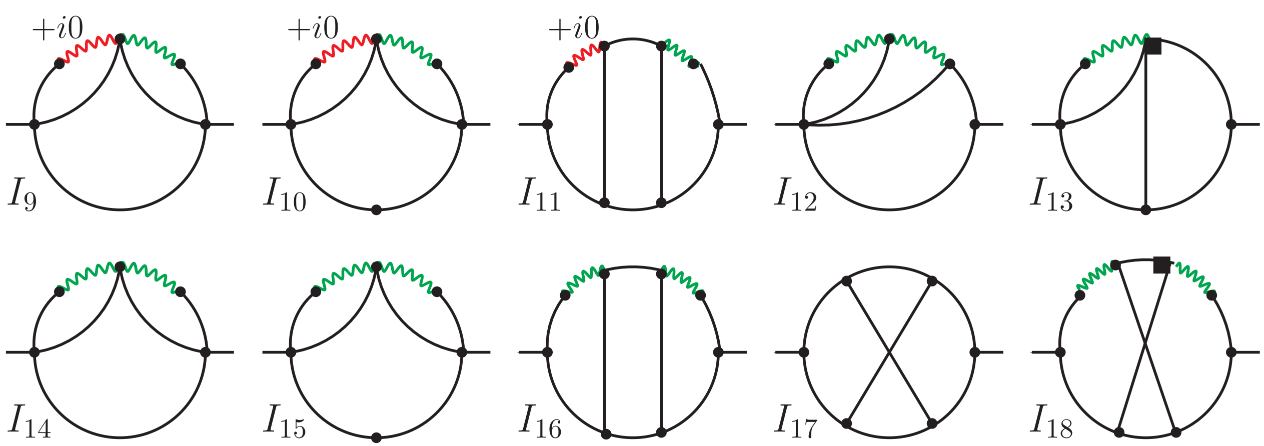
Also the subleading $O(\epsilon)$ terms for the fermion-loop type contributions were calculated.

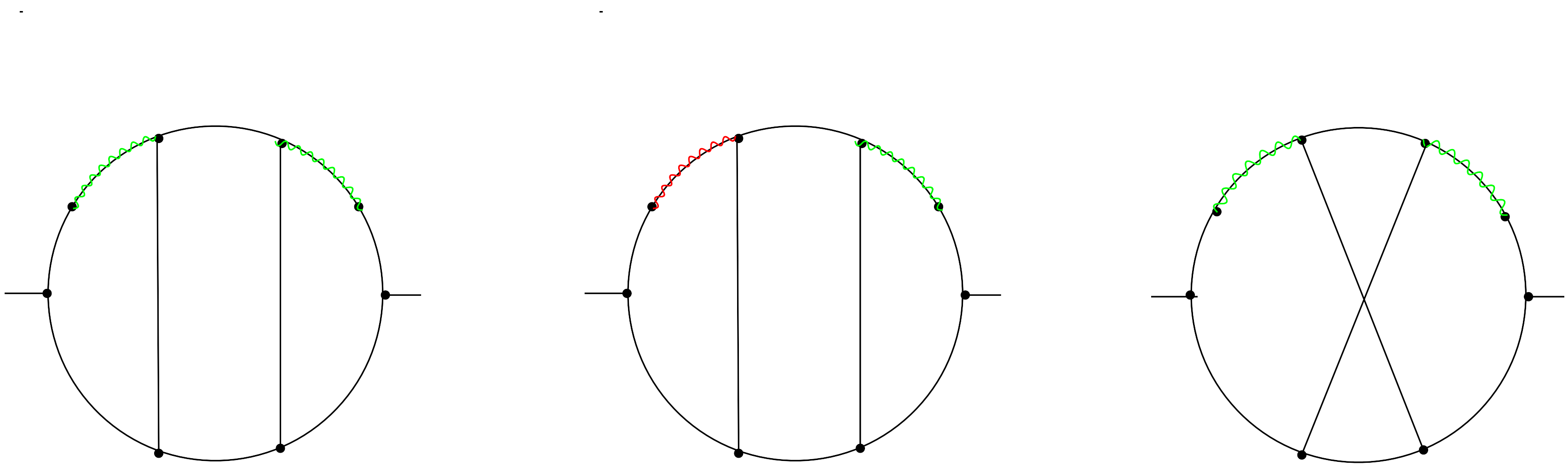
It is possible to evaluate the whole $O(\epsilon)$ part of the form factors :)

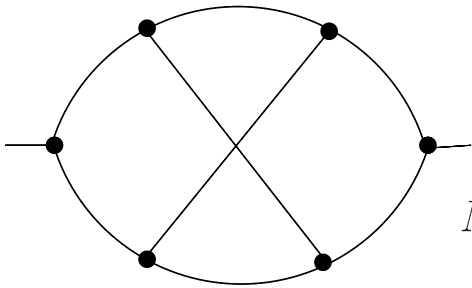
$$
\begin{aligned}
A_{9,1}(4-2\epsilon) = e^{-3\gamma_E\epsilon}\Bigg\{ & \frac{1}{18\epsilon^5} - \frac{1}{2\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{53}{18} + \frac{29\pi^2}{216}\right) \\
& + \frac{1}{\epsilon^2}\left(\frac{35\zeta(3)}{18} - \frac{29}{2} - \frac{149\pi^2}{216}\right) + \frac{1}{\epsilon}\left(-\frac{307\zeta(3)}{18} + \frac{129}{2} + \frac{139\pi^2}{72} + \frac{5473\pi^4}{25920}\right) \\
& + \left(\frac{793\zeta(5)}{10} + \frac{871\pi^2\zeta(3)}{216} + \frac{1153\zeta(3)}{18} - \frac{3125\pi^4}{5184} - \frac{19\pi^2}{8} - \frac{537}{2}\right) \\
& + \epsilon\left(-\frac{287\zeta(3)}{2} + \frac{2969\pi^2\zeta(3)}{216} + \frac{5521\zeta(3)^2}{36} - \frac{8251\zeta(5)}{30}\right. \\
& \left. + \frac{2133}{2} - \frac{97\pi^2}{8} + \frac{4717\pi^4}{28115} + \frac{761151\pi^6}{186624}\right) \\
& + \epsilon^2\left(\frac{195\zeta(3)}{2} - \frac{5887\pi^2\zeta(3)}{72} + \frac{138403\pi^4\zeta(3)}{25920} + \frac{799\zeta(3)^2}{4} + \frac{22487\zeta(5)}{30}\right. \\
& \left. - \frac{11987\pi^2\zeta(5)}{10115} + \frac{228799\zeta(7)}{126} - \frac{8181}{2} + \frac{969\pi^2}{8} - \frac{1333\pi^4}{320} - \frac{4286603\pi^6}{6531840}\right) + \ldots
\end{aligned}
$$

Most complicated master integrals for the three-loop static quark potential

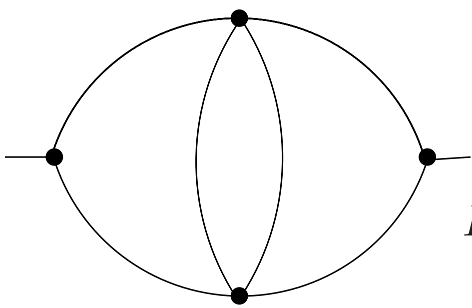
[A. and V. Smirnovs & M. Steinhauser'09]



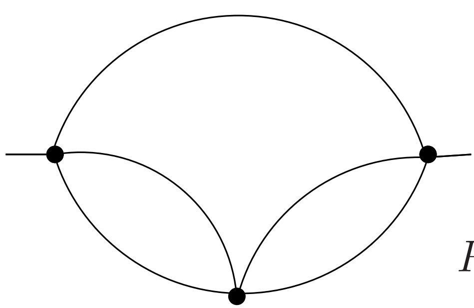




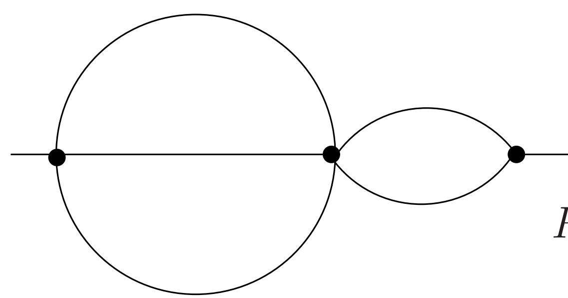
$N_{8,3}$



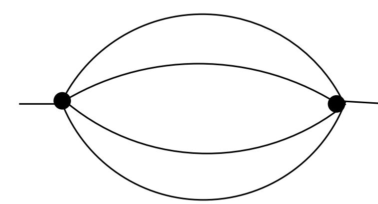
$P_{6,5}$



$P_{5,3}$



$P_{5,4}$



P_4

$$
\begin{aligned}
N_{8,3}(d+2) = {} & \frac{(d-4)}{8(d-2)(d-1)(2d-7)(2d-5)} N_{8,3} \\
& + \frac{4\left(5d^2-28d+38\right)}{(d-4)^2(d-2)(d-1)(2d-5)} P_{5,3} \\
& [4(d-4)(d-2)(d-1)(2d-7)(2d-5)(3d-8)]^{-1} \\
& \times \left(37d^3-313d^2+858d-752\right) P_{6,5} \\
& - \left[2(d-4)^2(d-3)(d-2)(d-1)(2d-7)(2d-5)\right]^{-1} \\
& \times \left(43d^4-478d^3+1963d^2-3530d+2352\right) P_{5,4} \\
& - \left[(d-4)^3(d-3)^2(d-2)(d-1)(2d-7)(3d-8)\right]^{-1} \\
& \times \left(401d^6-7251d^5+54491d^4-217784d^3\right. \\
& \left. +489064d^2-581248d+287232\right) P_4
\end{aligned}
$$

$$\frac{e^{-3\gamma_E\epsilon}}{1-2\epsilon}\left\{20\zeta(5)+\epsilon\left(68\zeta(3)^2+\frac{10\pi^6}{189}\right)\right.$$ [Chetyrkin, Kataev & Tkachov'80, Kazakov'84]

$$+\epsilon^2\left(\frac{34\pi^4\zeta(3)}{15}-5\pi^2\zeta(5)+450\zeta(7)\right)$$ [Becavac'06]

$$+\epsilon^3\left(-\frac{9072}{5}\zeta(5,3)-2588\zeta(3)\zeta(5)-17\pi^2\zeta(3)^2+\frac{6487\pi^8}{10500}\right)$$

$$+\epsilon^4\left(-\frac{4897\pi^6\zeta(3)}{630}-\frac{6068\zeta(3)^3}{3}+\frac{13063\pi^4\zeta(5)}{120}-\frac{225\pi^2\zeta(7)}{2}+\frac{88036\zeta(9)}{9}\right)$$

$$+\epsilon^5\left(\frac{2268}{5}\pi^2\zeta(5,3)+42513\zeta(8,2)-145328\zeta(3)\zeta(7)\right.$$

$$\left.\left.-73394\zeta(5)^2+647\pi^2\zeta(3)\zeta(5)-\frac{11813\pi^4\zeta(3)^2}{120}+\frac{28138577\pi^{10}}{9355500}\right)+\ldots\right\}$$

Pull out another factor to kill terms with pure π^2

[Chetyrkin, Kataev & Tkachov'80, D. Broadhurst'99]

$$
(1-2\epsilon)^2\left(\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)}\right)^3\left\{20\zeta(5)+\epsilon\left(68\zeta(3)^2+\frac{10\pi^6}{189}\right)\right.
$$

$$
+\epsilon^2\left(\frac{34\pi^4\zeta(3)}{15}+450\zeta(7)\right)
$$

$$
+\epsilon^3\left(-\frac{12072}{5}\zeta(5,3)-2448\zeta(3)\zeta(5)+\frac{8519\pi^8}{13500}\right)
$$

$$
+\epsilon^4\left(-\frac{1292\pi^6\zeta(3)}{189}-\frac{4640\zeta(3)^3}{3}+\frac{1202\pi^4\zeta(5)}{5}+\frac{88036\zeta(9)}{9}\right)
$$

$$
+\epsilon^5\left(42513\zeta(8,2)-142178\zeta(3)\zeta(7)-73022\zeta(5)^2\right.
$$

$$
\left.\left.-\frac{232\pi^4\zeta(3)^2}{3}+\frac{593053\pi^{10}}{187120}\right)+\ldots\right\}
$$

Pull out another factor to kill terms with pure π^2 [Kazakov'84]

$$\frac{1}{(1-2\epsilon)\Gamma(1-\epsilon)^3}\left\{20\zeta(5)+\epsilon\left(68\zeta(3)^2+\frac{10\pi^6}{189}\right)\right.$$
$$+\epsilon^2\left(\frac{34\pi^4\zeta(3)}{15}+450\zeta(7)\right)$$
$$+\epsilon^3\left(-\frac{9072}{5}\zeta(5,3)-2568\zeta(3)\zeta(5)+\frac{8519\pi^8}{13500}\right)$$
$$+\epsilon^4\left(-\frac{1352\pi^6\zeta(3)}{189}-\frac{5864\zeta(3)^3}{3}+\frac{542\pi^4\zeta(5)}{5}+\frac{88036\zeta(9)}{9}\right)$$
$$+\epsilon^5\left(42513\zeta(8,2)-144878\zeta(3)\zeta(7)\right.$$
$$\left.\left.-73382\zeta(5)^2-\frac{1466\pi^4\zeta(3)^2}{15}+\frac{592063\pi^{10}}{187110}\right)+\ldots\right\}$$

The coefficients in the ϵ-expansion of planar massless propagator diagrams up to five loops should be expressed in terms of multiple zeta values, while the non-planar graphs may contain, in addition, multiple sums with 6th roots of unity.

[Brown'08]

The full color dependence of the 4-loop 4-gluon amplitude in N=4 SUSY YM in terms of 50 4-loop 4-point integrals.

[Bern, Carrasco, Johansson & Roiban'10]

The critical dimension at which the amplitude first diverge. For 4 loops, this is d=11/2.

The subleading-color parts of the divergence require the three-loop propagator non-planar master integral.

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We have

$$-6.19839922674949591682009254798193687634789879896791 52\ldots$$

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- to be further developed