Top-Quark Pair Production Beyond Next-To-Leading Order

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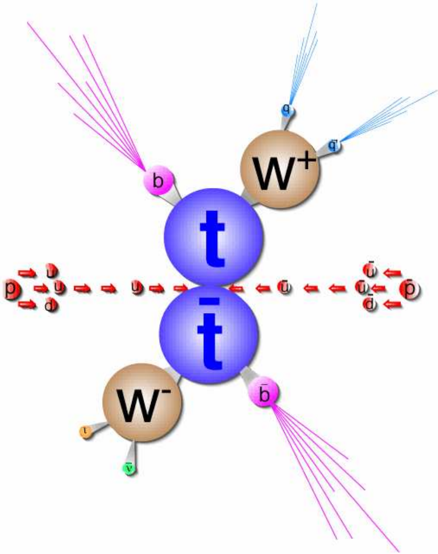
Loops & Legs, April 26, 2010

Outline

Top-Quark Pair Hadroproduction

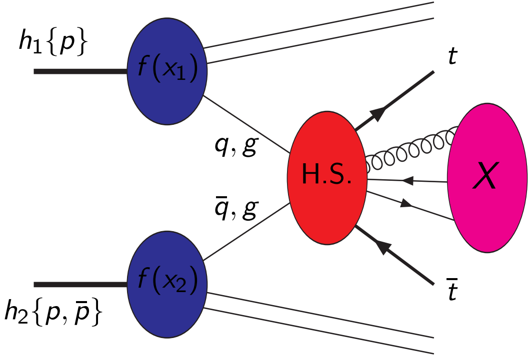


Several observables measured at Tevatron (few thousand observed events)

- Total Cross Section
- Invariant Mass Distribution
- Charge / Forward-Backward Asymmetry
- ...

- Next 2 years at the LHC $\sim$ few thousand observed events
- LHC ($\sqrt{s} = 14\,\text{TeV}$, $10\,\text{fb}^{-1}$/year)
  $\Longrightarrow$ millions of top-quarks / year

Top-quark pair production is a hard scattering process which can be computed in perturbative QCD

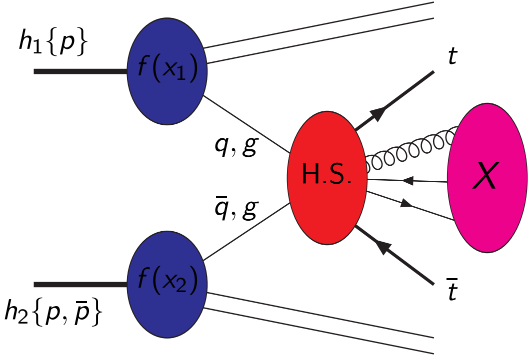


$$\sigma^{t\bar{t}}_{h_1,h_2} = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 f_i^{h_1}(x_1, \mu_F) f_j^{h_2}(x_2, \mu_F) \hat{\sigma}_{ij}(s, m_t, \alpha_s(\mu_R), \mu_F, \mu_R)$$

$$s_{\text{had}} = (p_{h_1} + p_{h_2})^2 \;, \; s = x_1 x_2 s_{\text{had}}$$

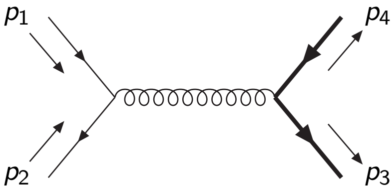
Top Quark Pair Hadroproduction & QCD

Top-quark pair production is a hard scattering process which can be computed in perturbative QCD

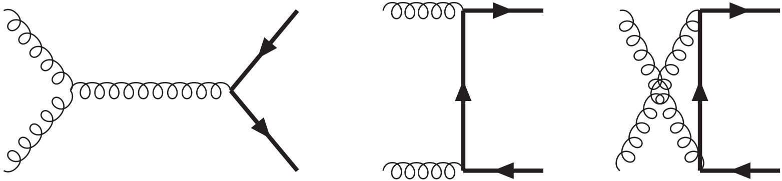


$$\sigma_{h_1,h_2}^{t\bar{t}}(s_{\text{had}}, m_t^2) = \sum_{ij} \int_{4m_t^2}^{s_{\text{had}}} ds \; \underbrace{L_{ij}\left(s, s_{\text{had}}, \mu_f^2\right)}_{\text{partonic luminosity}} \; \overbrace{\hat{\sigma}_{ij}(s, m_t^2, \mu_f^2, \mu_r^2)}^{\text{partonic cross section}}$$

$$q(p_1) + \bar{q}(p_2) \longrightarrow t(p_3) + \bar{t}(p_4)$$

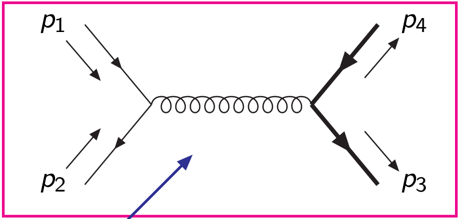


$$g(p_1) + g(p_2) \longrightarrow t(p_3) + \bar{t}(p_4)$$



Tree Level QCD Partonic Processes

$$q(p_1) + \bar{q}(p_2) \longrightarrow t(p_3) + \bar{t}(p_4)$$



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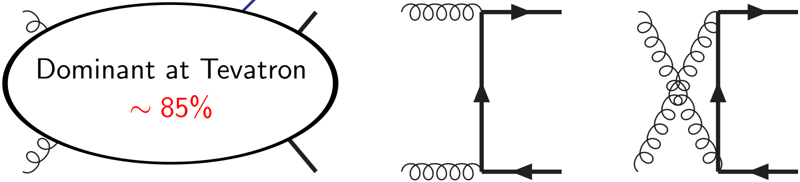

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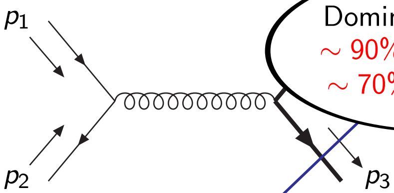
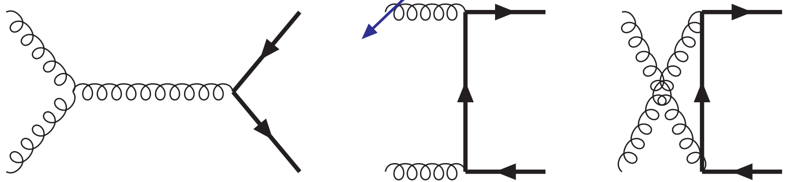


Dominant at LHC
$\sim$ 90% at 14 TeV
$\sim$ 70% at 7 TeV

$$g(p_1) + g(p_2) \longrightarrow t(p_3) + \bar{t}(p_4)$$



NLO corrections

The NLO corrections to top-quark pair production have been a subject of active research for more than 20 years
(too many authors to list them all here!)

- NLO QCD corrections to the total cross section
- NLO QCD corrections to the distributions (p_T, rapidity, invariant mass, ...)
- NLL resummation of threshold effects
- Mixed QCD-EW corrections
- NLO corrections keeping into account top spins and top decays
- NLO QCD corrections to $t\bar{t}+$ additional hard particles

(see talks by A. Denner and M. Czakon)

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At the LHC some observables (ex. total cross section) will be affected experimental errors which are smaller than the current NLO + NLL theoretical uncertainties

To take full advantage of the LHC potential, we need to go
beyond the current NLO+NLL calculation

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COMPLETE NNLO

- Goal: to calculate all the virtual and real corrections
- Requires to use and develop cutting edge calculational techniques
- Very time consuming
- The only ultimate solution to the problem

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APPROXIMATE NNLO

- Goal: to capture the numerically dominant NNLO corrections
- Can be done using Effective Field Theory methods
- One can obtain predictions relatively fast
- By construction, one introduces systematic uncertainties

APPROXIMATE NNLO CALCULATIONS:
Partonic Threshold Expansion (and Resummation) for the Invariant Mass Distribution

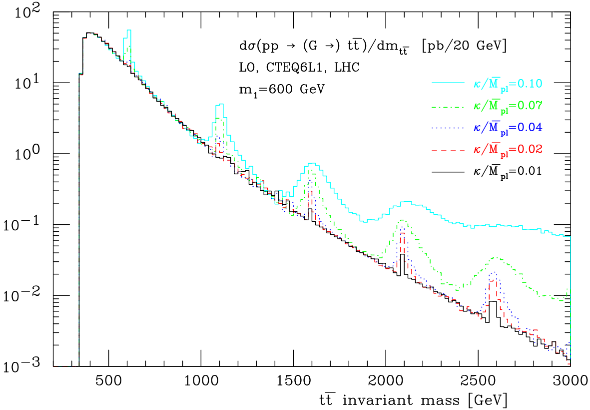
(WARNING: I will not discuss the recent results concerning the total partonic cross section in the limit $\beta = \sqrt{1 - 4m_t^2/s} \to 0$)

Langenfeld *et al.* ('09), Beneke *et al.* ('09)

Approximate NNLO Formulas for the Invariant Mass Distribution

Ahrens, AF, Neubert, Pecjak, and Yang ('09)

The distribution in the invariant mass $M^2 = (p_t + p_{\bar{t}})^2$ can be used to measure m_t, and to search for s-channel heavy resonances



Frederix and Maltoni ('07)

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$$\frac{d\sigma}{dM} = \frac{8\pi\beta}{3M} \int_\tau^1 \frac{dz}{z} \sum_{ij=(q\bar{q},gg,\bar{q}q)} L_{ij}\left(\frac{\tau}{z}, \mu\right) C_{ij}\left(z, \ldots, \mu\right)$$

We focus on the partonic threshold region $s \sim M^2$; $z \to 1$

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In this limit, there is little phase-space available for real gluon emission: this is a virtual-soft approximation

Approximate NNLO Formulas for the Invariant Mass Distribution

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The distribution in the invariant mass $M^2 = (p_t + p_{\bar{t}})^2$ can be used to measure

Two qu

We can calculate the C_{ij} up to terms of $\mathcal{O}(1-z)$

It will be possible to predict accurately $d\sigma/dM$ if:

a) $\tau \sim 1$; ...but the interesting region is $\tau < 0.3$

b) $L_{ij} \to 0$ for $z \to \tau$; Dynamical Threshold Enhancement

$$\frac{d\sigma}{dM} = \frac{8\pi\beta}{3M} \int_\tau^1 \frac{dz}{z} \sum_{ij=(q\bar{q},gg,\bar{q}q)} L_{ij}\left(\frac{\tau}{z}, \mu\right) C_{ij}(z, \ldots, \mu)$$

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Hard Scattering Kernels C_{ij}

In the limit $z \to 1$ we can distinguish three different scales

$$s, M^2, m_t^2 \gg s(1-z) \gg \Lambda_{\mathrm{QCD}}$$

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In the threshold region the hard scattering kernels factor into hard functions and soft functions (matrices in color space)

Kidonakis, Sterman ('97)

$$C(z, M, m_t, \mu) = \int_{-1}^{1} d\cos\theta \, \text{Tr}\left[\mathbf{H}(M, m_t, \cos\theta, \mu)\mathbf{S}(\sqrt{s}(1-z), m_t, \cos\theta, \mu)\right]$$

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The same factorization formula can be re-obtained using the language of Soft-Collinear Effective Theory

Hard Scattering Kernels C_{ij}-II

The soft functions include plus distributions of the form

$$\alpha_s^n \left[\frac{\ln^m(1-z)}{1-z} \right]_+ \qquad m = 0, \cdots, 2n-1$$

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In particular at NNLO

$$C_{ij}^{\rm NNLO}(z, M, m_t, \mu) = \int_{-1}^{1} d\cos\theta \Bigg\{ D_3 \left[\frac{\ln^3(1-z)}{1-z}\right]_+ + D_2 \left[\frac{\ln^2(1-z)}{1-z}\right]_+ + D_1 \left[\frac{\ln(1-z)}{1-z}\right]_+ + D_0 \left[\frac{1}{1-z}\right]_+ + C_0 \delta(1-z) + R(z) \Bigg\}$$

D_i, C_0 are functions of $M, m_t, \cos\theta, \mu$;

(D_3, D_2, D_1 first obtained by Kidonakis and Vogt ('03))

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$C_{ij}^{\text{NNLO}}(z, M$ ... $\left[\frac{\ln^2(1-z)}{1-z}\right]_+$

H and S obey RGE of the form

$$\frac{d}{d\ln\mu}\mathbf{H} = \mathbf{\Gamma}_H \mathbf{H} + \mathbf{H}\mathbf{\Gamma}_H^\dagger$$

where $\mathbf{\Gamma}$ is known up to NNLO

(AF, Neubert, Pecjak, and Yang ('09))

$$+C_0\delta(1-z) + R(z)\Big\}$$

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Hard Scattering Kernels C_{ij}-II

By exploiting the information encoded in $\mathbf{\Gamma}$ and RGEs

it was possible to calculate D_3, D_2, D_1, D_0 and

the scale dependence of C_0

1

In particular at NNLO

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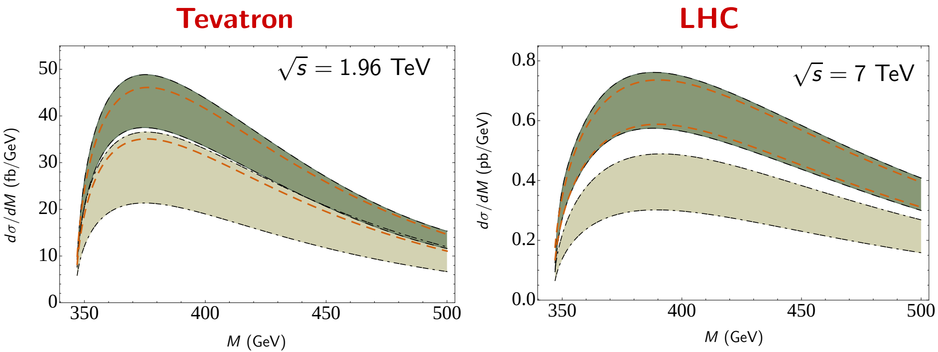
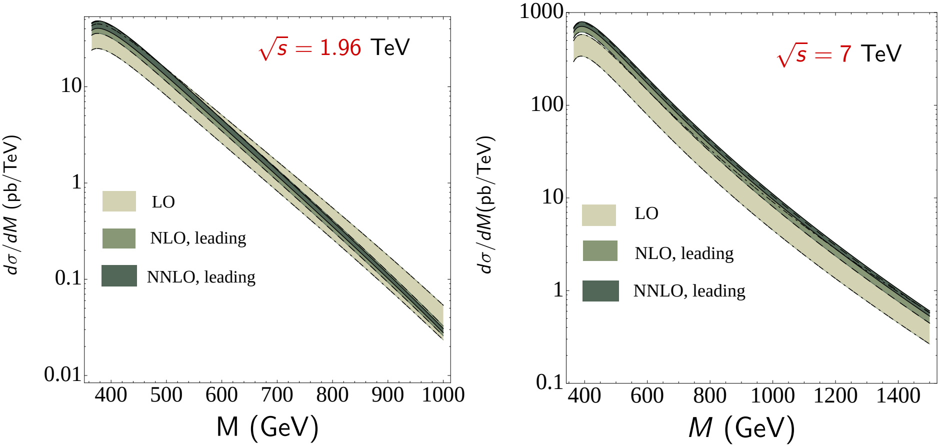
Threshold Expansion vs Exact NLO



- Exact NLO result (dark grey band) obtained with MCFM (Campbell, Ellis)
- The NLO threshold expansion $\longrightarrow$ band between the dashed lines ($200\,\text{GeV} \leq \mu \leq 800\,\text{GeV}$; close to $M/2 \leq \mu \leq 2M$)
- The threshold expansion agrees quite well with the exact result, even in the low invariant mass region

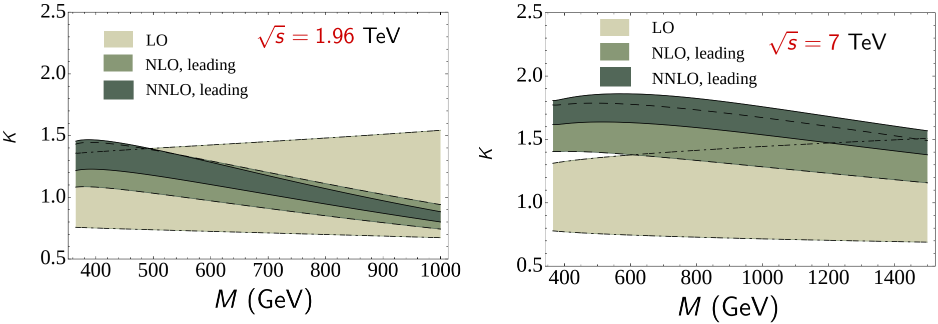
Invariant Mass Distribution

$$\left(\frac{M}{2} \leq \mu \leq 2M\right)$$



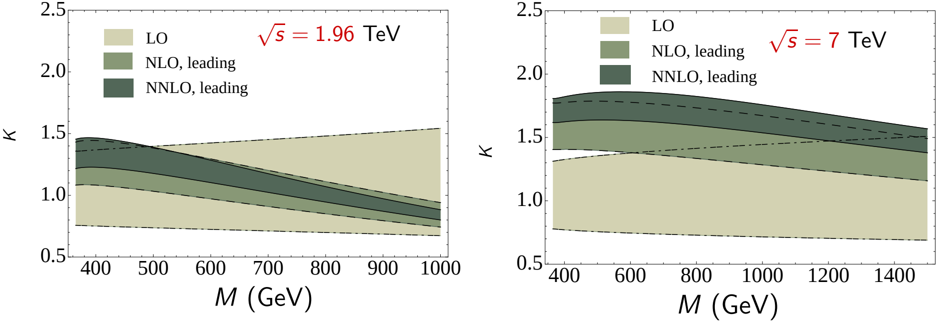
Threshold Expansion at NNLO

$$K = \frac{d\sigma(\mu)/dM}{d\sigma^{\mathrm{LO}}(\mu = M)/dM}$$



Using MSTW2008 pdfs

$$K = \frac{d\sigma(\mu)/dM}{d\sigma^{\mathrm{LO}}(\mu = M)/dM}$$



- Scale dependence still sizable
- Resummation and/or complete NNLO needed!

By solving the RGE satisfied by **H** and **S** it is possible to rewrite the scattering kernel as

$$C(z, M, m_t, \cos\theta, \mu_f) = \exp\left[4a_{\gamma^\phi}(\mu_s, \mu_f)\right]$$

$$\times \mathrm{Tr}\left[\mathbf{U}(M, m_t, \cos\theta, \mu_h, \mu_s)\,\mathbf{H}(M, m_t, \cos\theta, \mu_h)\,\mathbf{U}^\dagger(M, m_t, \cos\theta, \mu_h, \mu_s)\right.$$

$$\left.\times\tilde{\mathbf{s}}\left(\ln\frac{M^2}{\mu_s^2} + \partial_\eta, M, m_t, \cos\theta, \mu_s\right)\right]\frac{e^{-2\gamma_E\eta}}{\Gamma(2\eta)}\frac{z^{-\eta}}{(1-z)^{1-2\eta}}$$

The $\ln(\mu_s/\mu_h) \sim \ln(1-z)$ are exponentiated in **U**

$$\mathbf{U}(M, m_t, \cos\theta, \mu_h, \mu) = \mathcal{P}\exp\int_{\mu_h}^{\mu}\frac{d\mu'}{\mu'}\,\mathbf{\Gamma}_H(M, m_t, \cos\theta, \mu')$$

Resummation and Matching

All the scales can be fixed (and varied) separately:

- $\mu_h \sim \mu_f \sim M$
- μ_s is chosen to minimize the corrections coming from the soft function $\left(\frac{M}{4} \leq \mu_s \leq \frac{M}{10}\right)$

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RG-impr. PT	log accuracy	Γ_{cusp}	γ^h, γ^ϕ	$\mathbf{H}$, $\tilde{\mathbf{s}}$
LO	NLL	2-loop	1-loop	tree-level
NLO	NNLL	3-loop	2-loop	1-loop

All the pieces for the first NNLL calculation are now available

Ahrens, AF, Neubert, Pecjak, and Yang ('10)

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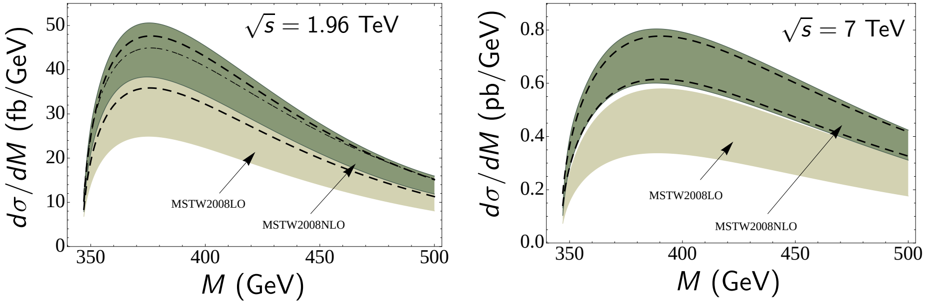
It is possible to match fixed order NLO and NNLL

$$d\sigma^{\text{NLO+NNLL}} = d\sigma^{\text{NNLL}}\Big|_{\mu_h,\mu_s,\mu_f} + \\ + \left(d\sigma^{\text{NLO}}\Big|_{\mu_f} - d\sigma^{\text{NNLL}}\Big|_{\mu_s=\mu_h=\mu_f} \right)$$

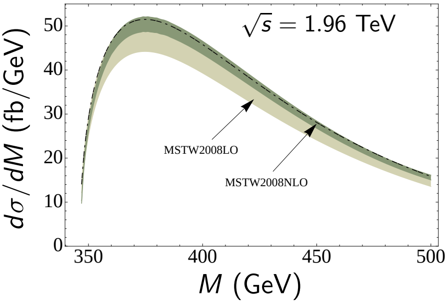
Ahrens, AF, Neubert, Pecjak, and Yang ('10)

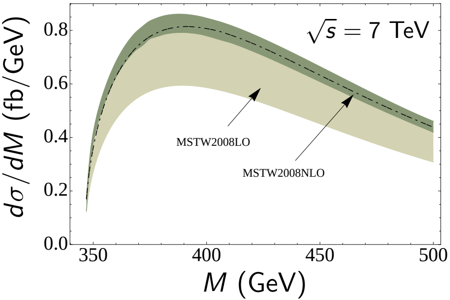
Fixed Order (LO and NLO)



NLO + NNLL Spectrum

Resummation (NLL and NLO+NNLL)





- $\mu_f = 400$ GeV ($\mu_f \sim M$ in the plotted range)
- reduced scale uncertainty

The Total Cross Section

The total cross section is obtained by integrating the distribution over M

$$\sigma\left(s_{\text{had}}, m_t\right) = \int_{2m_t}^{\sqrt{s_{\text{had}}}} dM \frac{d\sigma}{dM}$$

$\mu_f = 173$ GeV	Tevatron	LHC (7 TeV)
$\sigma_{\text{NLO,lead}}$	$6.20^{+0.39\,+0.31}_{-0.71\,-0.23}$	$144^{+5\,+7}_{-13\,-8}$
σ_{NLO}	$6.49^{+0.33\,+0.33}_{-0.70\,-0.24}$	$150^{+18\,+8}_{-19\,-9}$
$\sigma_{\text{NNLL+NLO}}$	$6.48^{+0.17\,+0.32}_{-0.21\,-0.25}$	$146^{+7\,+8}_{-7\,-8}$

- The NLO total cross section is known analytically

Czakon and Mitov ('08)

- First error from scale variations, second from PDFs (MSTW2008NNLO at 90%)

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$\mu_f = 400$ GeV	Tevatron	LHC (7 TeV)
$\sigma_{\text{NLO,lead}}$	$5.34^{+0.73\,+0.28}_{-0.73\,-0.21}$	$127^{+14\,+6}_{-15\,-7}$
σ_{NLO}	$5.64^{+0.73\,+0.30}_{-0.75\,-0.22}$	$126^{+19\,+7}_{-18\,-7}$
$\sigma_{\text{NNLL+NLO}}$	$6.30^{+0.19\,+0.31}_{-0.19\,-0.23}$	$149^{+7\,+8}_{-7\,-8}$

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Czakon and Mitov ('08)

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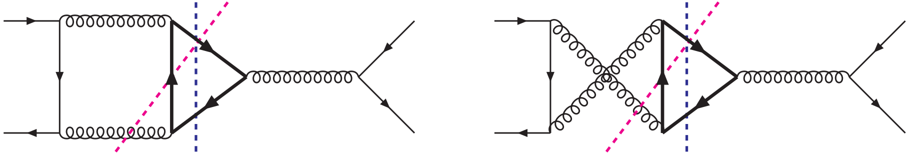
Charge Asymmetry

The charge asymmetry is the difference in production rate for top and antitop at fixed angle or rapidity

$$\underbrace{A(y) = \frac{N_t(y) - N_{\bar{t}}(y)}{N_t(y) + N_{\bar{t}}(y)}}_{\text{differential CA}} \qquad \underbrace{A = \frac{N_t(y \geq 0) - N_{\bar{t}}(y \geq 0)}{N_t(y \geq 0) + N_{\bar{t}}(y \geq 0)}}_{\text{integrated CA}} \qquad \left(N_i \equiv \frac{d\sigma^{t\bar{t}}}{dy_i} \right)$$

Arising at order α_s^3 in the channel $q\bar{q} \to t\bar{t}$



The Forward-Backward Asymmetry at Tevatron

Because of QCD charge conjugation invariance, $N_{\bar{t}}(y) = N_t(-y)$, and therefore A is equal to the forward-backward asymmetry

$$A_{\text{FB}} \equiv \frac{1}{\sigma} \int_{2m_t}^{\sqrt{s}} dM \left(\int_0^1 d\cos\theta \frac{d^2\sigma^{N_1 N_2 \to t\bar{t}X}}{dMd\cos\theta} - \int_{-1}^0 d\cos\theta \frac{d^2\sigma^{N_1 N_2 \to t\bar{t}X}}{dMd\cos\theta} \right)$$

- The measured asymmetry in the lab frame

$$A_{\text{FB}}^{p\bar{p}} = 19.3 \pm 6.9\%$$

- The predicted LO asymmetry is $A_{\text{FB}}^{p\bar{p}} = 5.1^{+0.7}_{-0.3}\%$

Kühn and Rodrigo ('08), Bernreuther and Si ('10)

- In the $t\bar{t}$-frame the asymmetry is $\sim 30\%$ larger

$$A_{\text{FB}}^{t\bar{t}} = 24 \pm 13\%$$

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$$A_{\text{FB}} \equiv \frac{1}{\ldots} \int^{\sqrt{s}} \left(\int^{1} \frac{d^2\sigma^{N_1 N_2 \to t\bar{t}X}}{\ldots} - \int^{0} \frac{d^2\sigma^{N_1 N_2 \to t\bar{t}X}}{\ldots \theta} \right)$$

- Th…
- Th… ('10)
- In …

AFNPY ('10) partonic frame	$A_{\text{FB}}(\%)$ $(\mu_f = m_t)$	$A_{\text{FB}}(\%)$ $(\mu_f = 400\,\text{GeV})$
"LO" QCD	$7.3^{+0.7}_{-0.6}$	$6.6^{+0.5}_{-0.5}$
"LO" + NNLL	$7.3^{+1.1}_{-0.7}$	$6.6^{+0.6}_{-0.5}$

$A^{t\bar{t}}_{\text{FB}} = 7.6^{+0.8}_{-0.5}$ LO QCD ; $A^{t\bar{t}}_{\text{FB}} = 8.0^{+0.7}_{-0.5}$ LO QCD + EW

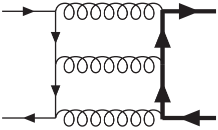
Bernreuther and Si ('10)

$$A^{t\bar{t}}_{\text{FB}} = 24 \pm 13\%$$

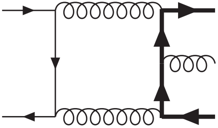
COMPLETE NNLO CALCULATIONS: STATUS

NNLO Laundry List

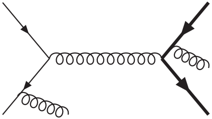
- Two-loop diagrams with a $t\bar{t}$ in the final state



- One-loop diagrams with a $t\bar{t}g(q,\bar{q})$ in the final state



- Tree-level diagrams with a $t\bar{t}gg(gq, g\bar{q}, q\bar{q})$ in the final state



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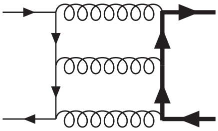
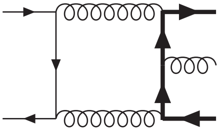


  - 200 diagrams in $q\bar{q} \to t\bar{t}$
  - 800 diagrams in $gg \to t\bar{t}$

- One-loop diagrams with a $t\bar{t}g(q,\bar{q})$ in the final state

Dittmaier, Uwer, Wenzierl ('07,'08)
Bevilacqua *et al.* ('10)
Melnikov, Schulze ('10)



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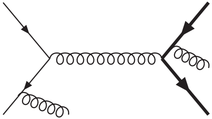

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

The two-loop corrections to $q\bar{q} \to t\bar{t}$ were first evaluated in the limit in which $s, |t|, |u| \gg m_t^2$

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Adding more terms in the expansion in powers of $m^2/s, m^2/|t|, m^2/|u|$ it is not sufficient for phenomenological studies (particularly near threshold)

An exact numerical evaluation of these correction is available

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M. Czakon ('08)

Analytic calculations:

- Diagrams with a closed (light or heavy) quark loop

  Bonciani, AF, Gehrmann, Maître, Studerus ('08)

- Leading color coefficient in the N_c expansion (planar diagrams only)

  Bonciani, AF, Gehrmann, Studerus ('09)

Some Technical Details

The leading color coefficient involves planar diagrams only

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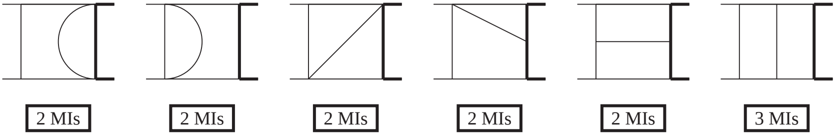
A new GiNaC/C++ implementation of the Laporta Algorithm was developed for this project: REDUZE (Studerus ('09))

It involves 6 new irreducible box topologies



C. Studerus ('09)

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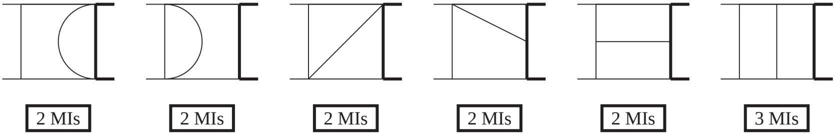


C. Studerus ('09)

The use new two-dim HPLs is unavoidable to obtain analytic expressions; they can be expanded analytically and evaluated numerically with a GiNaC package (Vollinga and Weinzierl ('04))

TWO-LOOP CORRECTIONS TO $gg \to t\bar{t}$

- The two-loop diagrams in the $gg \to t\bar{t}$ channel are available only in the $s \gg m_t^2$ limit

  Czakon, Mitov, and Moch ('08)

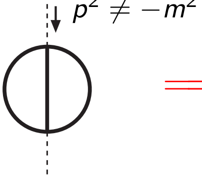
- The coefficients of all the IR poles are know analytically (see talk by L. Yang)

  AF, Neubert, Pecjak, and Yang ('09)

- The diagrams involving massless quark loops can be calculated analytically in the usual way

  Bonciani, AF, Gehrmann, Studerus (in progress)

- Part of the virtual corrections involve many MI which cannot be expressed in terms of HPLs only



$\Longrightarrow$

Elliptic Functions

$$K(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

Laporta Remiddi ('04)

Summary & Conclusions

- The top-quark pair production cross section will eventually be measurable with a $5 - 10\%$ uncertainty at the LHC; This requires the complete computation of the NNLO QCD corrections to the top-quark pair production cross section
- Approximate NNLO results are available for the total cross section and some distributions; they are a useful tool for phenomenology (estimate of the NNLO corrections, studies of the scale dependence, resummation of large logarithmic corrections, etc)
- Complete NNLO computations: The calculation of the two-loop corrections in both the $q\bar{q}$ and gg channels is major technical challenge; several results were obtained in the last two years, good perspective for the future

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The moral of the story is always the same:
there is a lot of work to do ...

Backup Slides

When will you finish?

"Obviously some team of theorists must have computed this at NLO a decade ago, and probably is close to having it at NNLO? Unfortunately not

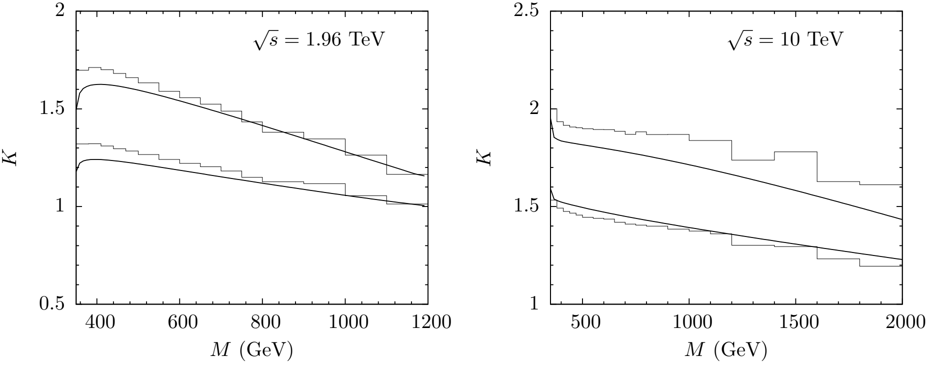
. . .

Part of the problem, by the way, is sociological: there are 300 string theory students at this school, but only $\sim$ 30 people in the world working on Standard Model calculations of basic importance for LHC discover."

LHC phenomenology for string theorists,
J. Lykken ('07)

Threshold Expansion vs Exact NLO

$$K = \frac{d\sigma^{\mathrm{NLO}}(\mu)/dM}{d\sigma^{\mathrm{LO}}(\mu = 2m_t)/dM}$$



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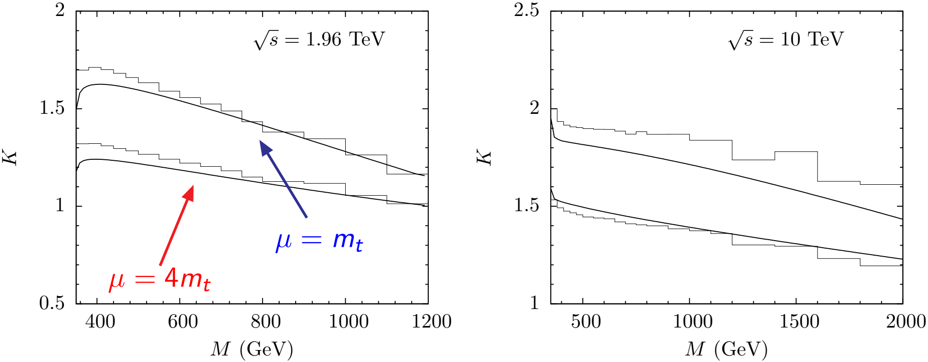

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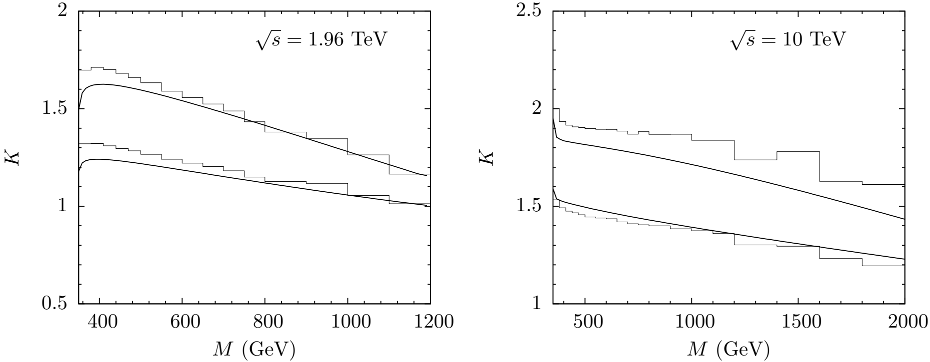

- Exact result obtained with MCFM (Campbell, Ellis)
- The threshold expansion agrees quite well with the exact result, even in the low invariant mass region

Reviews

- M. Beneke *et al* *Top quark physics* hep-ph/0003033
- S. Dawson *The top quark, QCD, and new physics* hep-ph/0303191
- W. Wagner *Top quark physics in hadron collisions* hep-ph/0507207
- A. Quadt *Top quark physics at hadron colliders* EJPC (2006)
- W. Bernreuther *Top-quark physics at the LHC* 0805.1333
- J. R. Incandela *et al* *Status and Prospects of Top Quark Physics* 0904.2499

The Laporta Algorithm

The set of denominators $\mathcal{D}_1, \cdots, \mathcal{D}_t$ defines a topology; for each topology

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- The scalar integrals are related via Integration By Parts identities (10 identities per integral for a two-loop four-point function)

$$\int \mathfrak{D}^d k_1 \mathcal{D}^d k_2 \frac{\partial}{\partial k_i^\mu} \left[v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{r_1} \cdots \mathcal{D}_t^{r_t}} \right] = 0 \quad v = k_1, k_2, p_1, p_2, p_3$$

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- Building the IBPs for growing powers of the propagators and scalar products the number of equations grows faster that the number of unknown: one finds a system of equations which is apparently over-constrained
- Solving the system of IBPs (in a problem with a small number of scales) one finds that only a few of the scalar integrals above (if any) are independent: the MIs.

The Lorentz Invariance Identities (LIs)

T. Gehrmann, E. Remiddi ('99)

A scalar integral is invariant under Lorentz transformation of the external momenta:

$$p^\mu \to p^\mu + \delta p^\mu = p^\mu + \delta\epsilon^\mu_\nu p^\nu \quad \delta\epsilon^\mu_\nu = -\delta\epsilon^\nu_\mu$$
$$I(p_1, p_2, p_3) = I(p_1 + \delta p_1, p_2 + \delta p_2, p_3 + \delta p_3)$$

implying the following 3 identities for a 4-point functions

$$(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu) \sum_{n=1}^{3} \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) = 0$$

$$(p_2^\mu p_3^\nu - p_2^\nu p_3^\mu) \sum_{n=1}^{3} \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) = 0$$

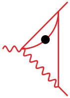
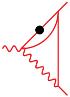
$$(p_1^\mu p_3^\nu - p_1^\nu p_3^\mu) \sum_{n=1}^{3} \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) = 0$$

The General Symmetry Relations Identities

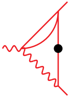
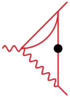
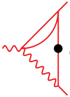
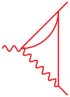
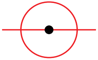
Further identities arise when a Feynman graph has symmetries

It is in those cases possible to perform a transformation of the loop momenta that does not change the value of the integral but allows to express the integrand as a combination of different integrands

Some relations are immediately seen:

$0 =$  $-$ 

others are more involved

$0 =$  $k_1 \cdot k_2 +$  $p_1 \cdot k_2 +$  $p_2 \cdot k_1 + \frac{s}{2}$  $+ \frac{1}{2}$ 

Equations and Unknowns

In two-loop $2 \to 2$ processes there are two integration momenta and three external momenta $\longrightarrow$ 9 possible scalar products

Consider the integrals $I_{t,r,s}$ where

- t $\to$ # of propagators
- 9-t $\to$ # of irreducible scalar products
- r $\to$ sum of the powers of all propagators
- s $\to$ sum of the powers of all irreducible scalar products

The number of integrals belonging to the $I_{t,r,s}$ set is

$$N(I_{t,r,s}) = \binom{r-1}{r-t}\binom{8-t+s}{s}$$

It is possible to build $(N_{\text{IBP}} + N_{\text{LI}})N(I_{t,r,s})$ identities

Calculation of the MIs: Differential Equation Method

For each Master Integral belonging to a given topology
$F_l^{(q)} \to \{\mathcal{D}_1, \cdots, \mathcal{D}_q\}$

- Take the derivative of a given integral with respect to the external momenta p_i

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} F_l^{(q)} = p_j^\mu \int \mathfrak{D}^d k_1 \mathfrak{D}^d k_2 \, \frac{\partial}{\partial p_i^\mu} \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{r_1} \cdots \mathcal{D}_q^{r_q}}$$

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- Take the derivative of a given integral with respect to the external momenta p_i
- The integrals are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs

$$p_j^\mu \int \mathfrak{D}^d k_1 \mathfrak{D}^d k_2 \frac{\partial}{\partial p_i^\mu} \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{r_1} \cdots \mathcal{D}_q^{r_q}} = \sum c_i F_i^{(q)} + \sum_{r \neq q} \sum_j k_j F_j^{(r)}$$

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- Rewrite the diff. eq. in terms of derivatives with respect to s and t

$$\frac{\partial}{\partial s} F_l^{(q)}(s,t) = \sum_j c_j(s,t) F_j^{(q)}(s,t) + \sum_{r \neq q} \sum_l k_l(s,t) F_l^{(r)}(s,t)$$

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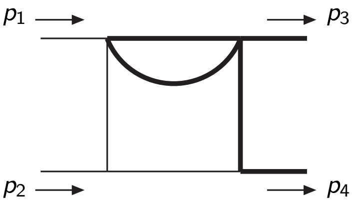
- The integrals are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs
- Rewrite the diff. eq. in terms of derivatives with respect to s and t
- Fix somehow the initial condition(s) (ex. knowing the behavior of the integral at $s = 0$) and solve the system of DE(s)

Five Denominator MIs

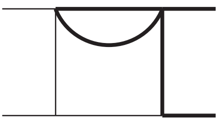
The most complicated irreducible topology in the calculation of the heavy fermion loop corrections is a five denominator box with two MIs
(thick lines indicate massive propagators, thin lines massless ones)

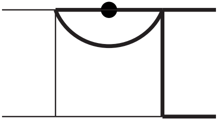


$$M(\alpha_1,\ldots,\alpha_9)=\int \frac{\mathfrak{D}^d k_1 \mathfrak{D}^d k_2\,(p_2\cdot k_1)^{\alpha_6}(p_1\cdot k_2)^{\alpha_7}(p_2\cdot k_2)^{\alpha_8}(p_3\cdot k_2)^{\alpha_9}}{P_0^{\alpha_1}(k_1+p_1)P_0^{\alpha_2}(k_1+p_1+p_2)P_m^{\alpha_3}(k_2)P_m^{\alpha_4}(k_1-k_2)P_m^{\alpha_5}(k_1+p_3)}$$

$$P_0(q) = q^2 \qquad\qquad P_m(q) = q^2 + m^2$$

Five Denominator MIs-II

$M_1 =$ 

$M_2 =$ 

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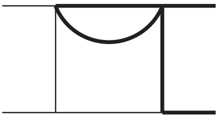
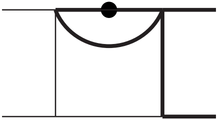
$M_1 =$ 

$M_2 =$ 

- the two MIs satisfy two independent first order differential equations

$$\frac{d M_i(s,t)}{dt} = C_i(s,t) M_i(s,t) + \Omega_i(s,t)$$

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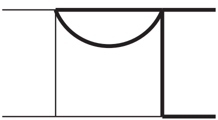
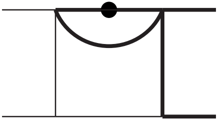
$M_1 =$ 

$M_2 =$ 

- the two MIs satisfy two independent first order differential equations

$$\frac{dM_i(s,t)}{dt} = C_i(s,t)M_i(s,t) + \Omega_i(s,t)$$

- One of the two needed initial conditions can be fixed by imposing the regularity of the integrals in $t = 0$
  The second integration constant can be fixed by calculating the integral in $t = 0$ with MB techniques

Euler Method

First Order Differential Equation

$$\frac{d}{dx}f(x) + C(x)f(x) = \Omega(x)$$

1. find the solution of the homogeneous equation

$$\frac{d}{dx}h(x) + C(x)h(x) = 0$$

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2. build the general solution of the non-homogeneous equation

$$f(x) = h(x)\left[k + \int dx \frac{\Omega(x)}{h(x)}\right]$$

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2. build the general solution of the non-homogeneous equation

$$f(x) = h(x) \left[k + \int dx \frac{\Omega(x)}{h(x)} \right]$$

3. fix the integration constant k

Euler Method-II

Second Order Differential Equation

$$\frac{d^2}{dx^2} f(x) + A(x) \frac{d}{dx} f(x) + B(x) f(x) = \Omega(x)$$

1. find the two solution of the homogeneous equation

$$\frac{d^2}{dx^2} h_{1,2}(x) + A(x) \frac{d}{dx} h_{1,2}(x) + B(x) h_{1,2}(x) = 0$$

Euler Method-II

Second Order Differential Equation

$$\frac{d^2}{dx^2}f(x) + A(x)\frac{d}{dx}f(x) + B(x)f(x) = \Omega(x)$$

1. find the two solution of the homogeneous equation

$$\frac{d^2}{dx^2}h_{1,2}(x) + A(x)\frac{d}{dx}h_{1,2}(x) + B(x)h_{1,2}(x) = 0$$

2. build the Wronskian

$$W(x) = h_1(x)\frac{d}{dx}h_2(x) - h_2(x)\frac{d}{dx}h_1(x)$$

Euler Method-II

Second Order Differential Equation

$$\frac{d^2}{dx^2} f(x) + A(x) \frac{d}{dx} f(x) + B(x) f(x) = \Omega(x)$$

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$$\frac{d^2}{dx^2} h_{1,2}(x) + A(x) \frac{d}{dx} h_{1,2}(x) + B(x) h_{1,2}(x) = 0$$

2. build the Wronskian

$$W(x) = h_1(x) \frac{d}{dx} h_2(x) - h_2(x) \frac{d}{dx} h_1(x)$$

3. build the solution

$$f(x) = h_1(x) \left(k_1 - \int_0^x \frac{dw}{W(w)} h_2(w) \Omega(w) \right) + h_2(x) \left(k_2 + \int_0^x \frac{dw}{W(w)} h_1(w) \Omega(w) \right)$$

Euler Method-II

Second Order Differential Equation

$$\frac{d^2}{dx^2} f(x) + A(x)\frac{d}{dx} f(x) + B(x) f(x) = \Omega(x)$$

1. find the two solution of the homogeneous equation

$$\frac{d^2}{dx^2} h_{1,2}(x) + A(x)\frac{d}{dx} h_{1,2}(x) + B(x) h_{1,2}(x) = 0$$

2. build the Wronskian

$$W(x) = h_1(x)\frac{d}{dx} h_2(x) - h_2(x)\frac{d}{dx} h_1(x)$$

3. build the solution

$$f(x) = h_1(x)\left(k_1 - \int_0^x \frac{dw}{W(w)} h_2(w)\Omega(w)\right) + h_2(x)\left(k_2 + \int_0^x \frac{dw}{W(w)} h_1(w)\Omega(w)\right)$$

4. fix the integration constants k_1 and k_2

2-dimensional Harmonic Polylogarithms (2dHPL)-II

The 2dHPLs share the properties of the HPLs

The analytic properties of both HPLs & 2dHPLs are know
Codes for their numerical evaluation are available

E. Remiddi, T. Gehrmann (2001-2002)

Up to $w = 3$ (our case) the 2dHPLs can be expressed in terms of
ln, Li_2, Li_3, $S_{1,2}$

$$
\begin{aligned}
G(-1/y;x) &= \ln(1+xy)\,, \\
G(-1/y,0;x) &= \ln(x)\ln(xy+1) + \mathrm{Li}_2(-xy)\,, \\
G(-y,1;x) &= \frac{1}{2}\ln^2(y+1) - \ln(1-x)\ln(y+1) - \ln(y)\ln(y+1) \\
&\quad + \ln(1-x)\ln(x+y) - \mathrm{Li}_2(-y) + \mathrm{Li}_2\left(\frac{1-x}{y+1}\right) - \frac{\pi^2}{6}\,, \\
G(-y,1,0;x) &= -\frac{1}{3}\ln^3(1-x) - \ln(x)\ln^2(1-x) + \cdots
\end{aligned}
$$

HPLs as Multiple Sums

S. Weinzierl and J. Vollinga ('04)

We defined (2-d)HPLs in terms of iterated integrations

$$G(z_1, \cdots, z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \cdots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$

but they can be written also in terms of multiple sums

$$G(\underbrace{0, \cdots, 0}_{m_1-1}, z_1, \cdots, z_{k-1}, \underbrace{0, \cdots, 0}_{m_k-1}, z_k; y) \equiv G_{m_1,\cdots,m_k}(z_1, \cdots, z_k; y)$$

$$\begin{aligned} G_{m_1,\cdots,m_k}(z_1, \cdots, z_k; y) &= \sum_{j_1=1}^{\infty} \cdots \sum_{j_1=1}^{\infty} \frac{1}{(j_1 + \cdots + j_k)^{m_1}} \left(\frac{y}{z_1}\right)^{j_1} \times \\ &\times \frac{1}{(j_2 + \cdots + j_k)^{m_2}} \left(\frac{y}{z_1}\right)^{j_2} \cdots \frac{1}{j_k^{m_k}} \left(\frac{y}{z_1}\right)^{j_k} \end{aligned}$$

IR Poles in QCD Amplitudes

IR poles in QCD amplitudes can be removed by a multiplicative renormalization

Becher and Neubert ('09)

$$\mathbf{Z}^{-1}(\epsilon, \{p\}, \{m\}) \, |\mathcal{M}_n(\epsilon, \{p\}, \{m\})\rangle_{\alpha_s^{QCD} \to \xi \alpha_s} = \text{FINITE}$$

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$$\mathbf{Z}^{-1}(\epsilon, \{p\}, \{m\}) \, \mathcal{M}_n(\epsilon, \{p\}, \{m\})\big|_{\alpha_s^{QCD} \to \xi \alpha_s} = \text{FINITE}$$

$$\mathbf{Z}^{-1} = \mathbf{1} - \alpha_s \mathbf{Z}_{(1)} + \alpha_s^2 (\mathbf{Z}_{(1)}^2 - \mathbf{Z}_{(2)}) + \mathcal{O}(\alpha_s^3)$$

$$\mathcal{M} = \alpha_s \mathcal{M}^{(0)} + \alpha_s^2 \mathcal{M}^{(1)} + \alpha_s^3 \mathcal{M}^{(2)} + \mathcal{O}(\alpha_s^4)$$

IR Poles in QCD Amplitudes

IR poles in QCD amplitudes can be removed by a multiplicative renormalization

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$$\mathbf{Z}^{-1}(\epsilon, \{p\}, \{m\}) \, |\mathcal{M}_n(\epsilon, \{p\}, \{m\})\rangle_{\alpha_s^{QCD} \to \xi \alpha_s} = \text{FINITE}$$

therefore

$$\begin{aligned} |\mathcal{M}_n^{(1),\,\mathrm{sing}}\rangle &= \mathbf{Z}^{(1)} \, |\mathcal{M}_n^{(0)}\rangle \\ |\mathcal{M}_n^{(2),\,\mathrm{sing}}\rangle &= \left[\mathbf{Z}^{(2)} - \left(\mathbf{Z}^{(1)} \right)^2 \right] |\mathcal{M}_n^{(0)}\rangle + \left(\mathbf{Z}^{(1)} \, |\mathcal{M}_n^{(1)}\rangle \right)_{\mathrm{poles}} \end{aligned}$$

But what is **Z**?

Evolution Matrix

$\mathbf{Z}$ satisfies the evolution equation

$$\mathbf{Z}^{-1}(\epsilon,\{\underline{p}\},\{\underline{m}\},\mu)\,\frac{d}{d\ln\mu}\,\mathbf{Z}(\epsilon,\{\underline{p}\},\{\underline{m}\},\mu) = -\mathbf{\Gamma}(\{\underline{p}\},\{\underline{m}\},\mu)$$

where, in the color space formalism

$$\begin{aligned}
\mathbf{\Gamma} \;=\; & \sum_{(i,j)} \frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\,\gamma_{\rm cusp}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s) \\
- & \sum_{(I,J)} \frac{\mathbf{T}_I\cdot\mathbf{T}_J}{2}\,\gamma_{\rm cusp}(\beta_{IJ},\alpha_s) + \sum_I \gamma^I(\alpha_s) + \sum_{I,j} \mathbf{T}_I\cdot\mathbf{T}_j\,\gamma_{\rm cusp}(\alpha_s)\,\ln\frac{m_I\mu}{-s_{Ij}} \\
+ & \sum_{(I,J,K)} if^{abc}\,\mathbf{T}_I^a\,\mathbf{T}_J^b\,\mathbf{T}_K^c\,F_1(\beta_{IJ},\beta_{JK},\beta_{KI}) \\
+ & \sum_{(I,J)}\sum_k if^{abc}\,\mathbf{T}_I^a\,\mathbf{T}_J^b\,\mathbf{T}_k^c\,f_2\!\left(\beta_{IJ},\ln\frac{-\sigma_{Jk}\,v_J\cdot p_k}{-\sigma_{Ik}\,v_I\cdot p_k}\right) + \mathcal{O}(\alpha_s^3)
\end{aligned}$$

Becher and Neubert ('09)

Evolution Matrix

Z satisfies the evolution equation

$$\mathbf{Z}^{-1}(\epsilon,\{\underline{p}\},\{\underline{m}\},\mu)\,\frac{d}{d\ln\mu}\,\mathbf{Z}(\epsilon,\{\underline{p}\},\{\underline{m}\},\mu) = -\boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{m}\},\mu)$$

where, in the color space formalism

$$\boldsymbol{\Gamma} = \sum_{(i,j)} \frac{\mathbf{T}_i\cdot\mathbf{T}_j}{\ldots} \ldots \mu^2 \ldots \sum_i \ldots$$

$$-\sum_{(I,J)} \ldots \sum_I \ldots \sum_{I,j} \ldots \alpha_s)\ln\frac{m_I\mu}{-s_{Ij}}$$

$$+\sum_{(I,J,K)} if^{abc}\,\mathbf{T}_I^a\,\mathbf{T}_J^b\,\mathbf{T}_K^c\,F_1(\beta_{IJ},\beta_{JK},\beta_{KI})$$

$$+\sum_{(I,J)}\sum_k if^{abc}\,\mathbf{T}_I^a\,\mathbf{T}_J^b\,\mathbf{T}_k^c\,f_2\left(\beta_{IJ},\ln\frac{-\sigma_{Jk}\,v_J\cdot p_k}{-\sigma_{Ik}\,v_I\cdot p_k}\right)+\mathcal{O}(\alpha_s^3)$$

$\mathbf{T}_i$ ($\mathbf{T}_I$) are color generators associated to massless (massive) particles

Becher and Neubert ('09)

Evolution Matrix

Z satisfies the evolution equation

$$\mathbf{Z}^{-1}(\epsilon, \{\underline{p}\}, \{\underline{m}\}, \mu) \frac{d}{d \ln \mu} \mathbf{Z}(\epsilon, \{\underline{p}\}, \{\underline{m}\}, \mu) = -\boldsymbol{\Gamma}(\{\underline{p}\}, \{\underline{m}\}, \mu)$$

where, in the c

$$\begin{aligned} \boldsymbol{\Gamma} &= \sum_{(i,j)} \\ &- \sum_{(I,J)} \quad \alpha_s) \ln \frac{m_I \mu}{-s_{Ij}} \\ &+ \sum_{(I,J,K)} if^{abc}\, \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_K^c\, F_1(\beta_{IJ}, \beta_{JK}, \beta_{KI}) \\ &+ \sum_{(I,J)} \sum_k if^{abc}\, \mathbf{T}_I^a \mathbf{T}_J^b \mathbf{T}_k^c\, f_2\left(\beta_{IJ}, \ln \frac{-\sigma_{Jk}\, v_J \cdot p_k}{-\sigma_{Ik}\, v_I \cdot p_k}\right) + \mathcal{O}(\alpha_s^3) \end{aligned}$$

Staring at two-loop order one finds
three particle correlators
The explicit expression for the coefficient functions F_1 and f_2 was recently obtained
AF, Neubert, Pecjak, Yang ('09)

Becher and Neubert ('09)

Harmonic Polylogarithms (HPLs)

E. Remiddi, J. Vermaseren (1999)

E. Remiddi, T. Gehrmann (2001)

Functions of the variable x and a set of indices $\vec{a}$ with weight w; each index can assume values $1, 0, -1$

$$H(\mathbf{a}; x)$$

Definitions: $w = 1$

$$\begin{aligned} H(1;x) &= \int_0^x \frac{dt}{1-t} = -\ln(1-x) \\ H(0;x) &= \ln x \\ H(-1;x) &= \int_0^x \frac{dt}{1+t} = \ln(1+x) \end{aligned}$$

$$\frac{d}{dx} H(a;x) = f(a;x) \quad f(1;x) = \frac{1}{1-x} \quad f(0;x) = \frac{1}{x} \quad f(-1;x) = \frac{1}{1+x}$$

HPLs: Definitions

Definitions: $w > 1$

$$\text{if } \vec{a} = 0, 0, \ldots, 0 \ (w \text{ times}) \quad H(\vec{0}_w; x) = \frac{1}{w!} \ln^w x$$

$$\text{else} \quad H(i, \vec{a}; x) = \int_0^x dt f(i; t) H(\vec{a}; t)$$

$$\text{consequences:} \quad \frac{d}{dx} H(i, \vec{a}; x) = f(i; x) H(\vec{a}; x) \quad H(\vec{a} \notin \vec{0}; 0) = 0$$

a few examples @ $w = 2$

$$H(0, 1; x) = \int_0^x dt f(0; t) H(1; t) = -\int_0^x dt \frac{1}{t} \ln(1 - t) = \mathrm{Li}_2(x)$$

$$H(1, 0; x) = \int_0^x dt f(1; t) H(0; t) = \int_0^x dt \frac{1}{1 - t} \ln t$$

$$= -\ln x \ln(1 - x) + \mathrm{Li}_2(x)$$

HPLs as a Generalization of the Nielsen's PolyLogs

The HPLs include the Nielsen's PolyLogs

$$S_{n,p(x)} = \frac{(-1)^{n+p-1}}{(n+p)!p!}\int_0^1 \frac{dt}{t}\ln^{n-1} t \ln^p(1-xt) \quad \mathrm{Li}_n(x) = S_{n-1,1}(x)$$

$$\begin{aligned} \mathrm{Li}_n(x) &= H(\vec{0}_{n-1}, 1; x) \\ S_{n,p}(x) &= H(\vec{0}_n, \vec{1}_p; x) \end{aligned}$$

but the HPLs are a larger set of functions: from $w = 4$ one finds things as

$$H(-1,0,0,1;x) = \int_0^x \frac{dt}{1+t}\mathrm{Li}_3(x) \notin \sum \text{Nielsen's PolyLogs}$$

The HPLs Algebra

- Shuffle Algebra:

$$H(\vec{p};x)H(\vec{q};x) = \sum_{\vec{r}=\vec{p}\uplus\vec{q}} H(\vec{r};x)$$

some examples

$$\begin{aligned} H(a;x)H(b;x) &= H(a,b;x) + H(b,a;x) \\ H(a;x)H(b,c;x) &= H(a,b,c;x) + H(b,a,c;x) + H(b,c,a;x) \end{aligned}$$

- Product Ids:

$$\begin{aligned} H(m_1,\ldots,m_q;x) &= H(m_1;x)H(m_2,\ldots,m_q;x) \\ &- H(m_2,m_1;x)H(m_3,\ldots,m_q;x) \\ &+ \cdots + (-1)^{q+1}H(m_q,\ldots,m_1;x) \end{aligned}$$

2-DIMENSIONAL HARMONIC POLYLOGARITHMS (2dHPLs)

E. Remiddi, T. Gehrmann (2000)

As for the HPLs, they are obtained by repeated integration over a new set of factors depending on a second variable.

$$f(-y;x) = \frac{1}{x+y} \quad f(-1/y;x) = \frac{1}{x+1/y}$$

$$G(i,\vec{a};x) = \int_0^x dt f(i;t) G(\vec{a};t)$$

a few examples:

$$G(-y;x) = \int_0^x \frac{dz}{z+y} = \ln\left(1+\frac{x}{y}\right) \quad G(-1/y;x) = \int_0^x \frac{dz}{z+1/y} = \ln(1+xy)$$

$$G(-y,0;x) = \ln x \ln\left(1+\frac{x}{y}\right) + \mathrm{Li}_2\left(-\frac{x}{y}\right)$$