Space-time dimensionality as complex variable[1]

Calculating loop integrals using dimensional recurrence relation and their analytic properties as functions of $\mathscr{D}$

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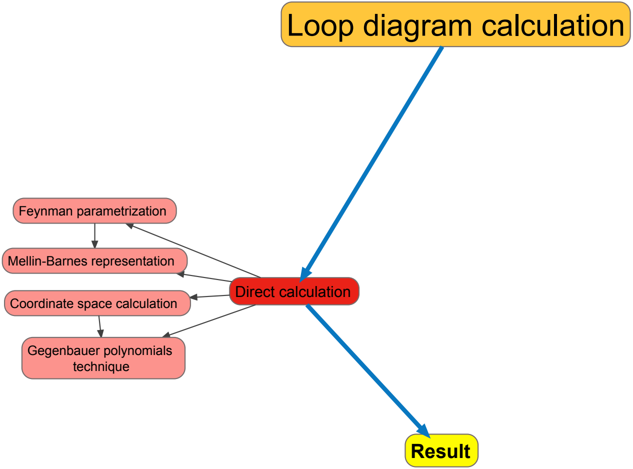
[1] Lee R.N., *Nucl. Phys. B*830 (2010) 474

Outline

Calculation path

Loop diagram calculation

Calculation path



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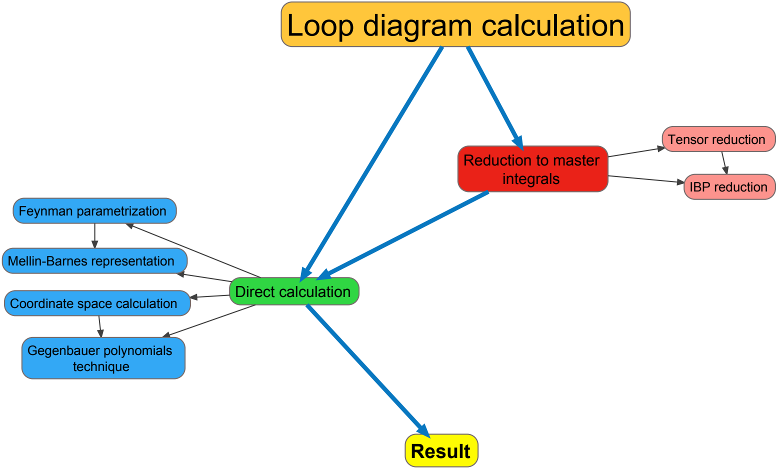

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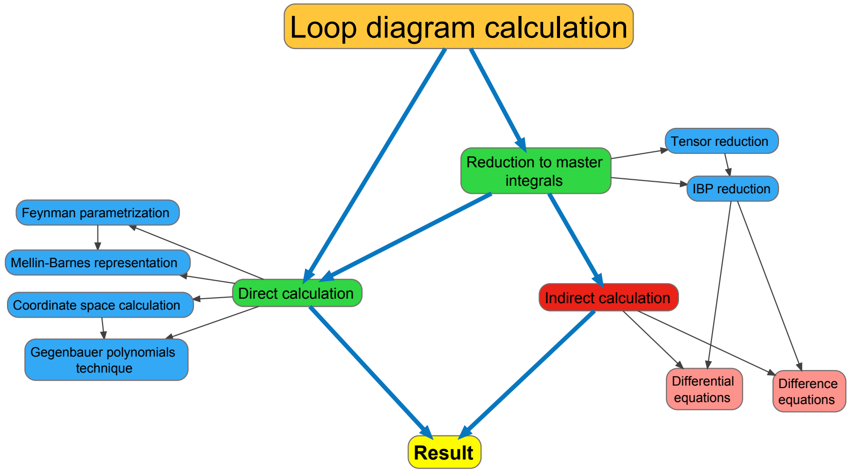

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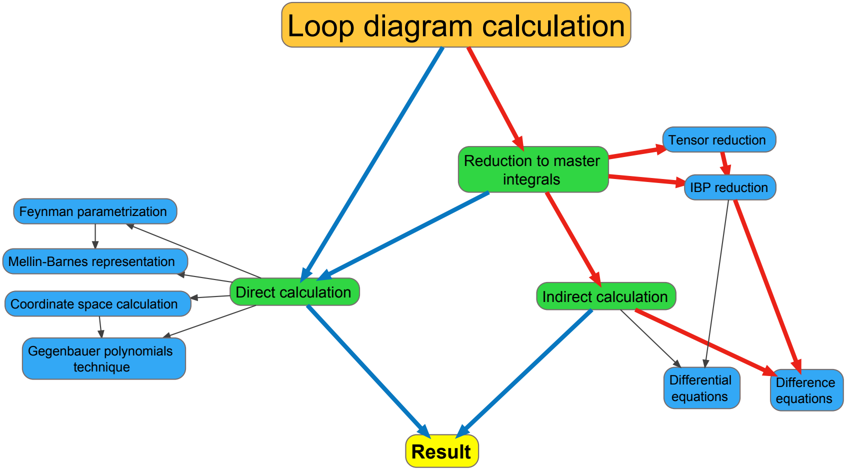

Loop Integral

L-loop diagram with E external momenta $p_1, \ldots p_E$:

Loop integral

$$J(\mathbf{n}) = J(n_1, \ldots, n_N) = \int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L j(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L}{D_1^{n_1} \ldots D_N^{n_N}}$$

where $D_1, \ldots, D_M$ are denominators of the diagram, and $D_{M+1}, \ldots, D_N$ are some additionally chosen numerators.

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Prerequisites

All denominators and numerators linearly depend on $l_i \cdot q_j$. Any product $l_i \cdot q_j$ can be expressed via D_k.

Notation

$$q_i = \begin{cases} l_i, & i \leqslant L \\ p_{i-L}, & i > L \end{cases}$$

The total number of denominators and numerators

$$N = L(L+1)/2 + LE, \quad N \geqslant M$$

Ordering of integrals

The goal of the reduction procedure

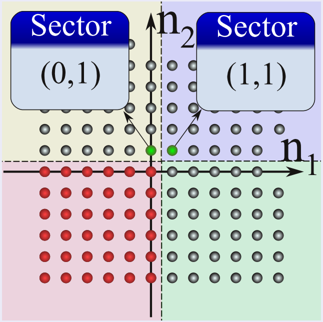
Any reduction procedure must have a goal, i.e., we have to know, what is simpler. Ordering of the integrals is required.

Common sense

Integrals with fewer denominators are simpler.

Sectors & Ordering

Integrals with the same set of denominators form a sector in $\mathbb{Z}^N$. Sectors can be labeled by their corner points.



Example

$$J(n_1,n_2) = \int \frac{d^{\mathscr{D}} l}{\left[l^2-m^2\right]^{n_1}\left[(l-p)^2-m^2\right]^{n_2}}$$

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1. The number of denominators.
2. Total power of denominators and numerators .
3. Number of numerators .
4. $n_1,\ n_2,\ \ldots$

Integration-by-part identities

The **integration-by-part identities** arise due to the fact, that, in dimensional regularization the integral of the total derivative is zero (Tkachov 1981, Chetyrkin and Tkachov 1981)

IBP identities

$$\int d^{\mathscr{D}}l_1 \ldots d^{\mathscr{D}}l_L \, O_{ij} j(\mathbf{n}) = 0 \qquad \text{(IBP)}$$

IBP operators

$$O_{ij} = \frac{\partial}{\partial l_i} \cdot q_j$$

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The **Lorentz-invariance identities** arise due to the fact that loop integrals are scalar functions of the external momenta (Gehrmann and Remiddi 2000).

LI identities

$$p_{1\mu} p_{2\nu} M^{\mu\nu} J = 0 \qquad \text{(LI)}$$

Lorentz generators

$$M^{\mu\nu} = \sum_e p_e^{[\mu} \partial_e^{\nu]}$$

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Huge redundancy of IBP&LI identities. In particular, it can be shown that LI identities are linear combinations of the IBP identities Lee (2008).

Operator representation

Let us introduce the operators, acting on the functions on $\mathbb{Z}^N$:

Operators $A_1,\ldots,A_N,B_1,\ldots,B_N$

$$(A_\alpha f)(n_1,\ldots,n_N) = n_\alpha f(n_1,\ldots,n_\alpha+1,\ldots,n_N),$$
$$(B_\alpha f)(n_1,\ldots,n_N) = f(n_1,\ldots,n_\alpha-1,\ldots,n_N).$$

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Commutator

$$[A_\alpha,B_\beta] = \delta_{\alpha\beta}$$

A and B well suited to sectors

For any polynomial $P(A,B)$ the result of action

$$P(A,B)J(\mathbf{n}) = \sum C_i J(\mathbf{n}_i)$$

contains only integrals of the same and lower sectors as $J(\mathbf{n})$.

Calculation of master integrals

IBP reduction

Using several available methods we can reduce all loop integrals emerging in our problem to a small set of master integrals.

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Reduction vs Calculation

- Reduction to masters requires some **algebraic** methods.
- Calculation of master integrals requires **analytic** methods.

Differential equations

Differentiating with respect to external parameter and performing IBP reduction of the result, we obtain differential equation for a given master integral(Kotikov 1991, Remiddi 1997).

Differential equation

$$\frac{\partial}{\partial a} J = f(a) J + h(a). \qquad \text{(DE)}$$

External parameter

$$s = \begin{cases} \text{mass} & \text{(Kotikov, 1991)} \\ \text{invariant of } p_e & \text{(Remiddi, 1997)} \end{cases}$$

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- **Scaleless integrals are zero in dimensional regularization.**
- n**-scale integrals (**$n \geq 2$**) can be investigated by the differential equation method.**
  Initial conditions for the differential equation are put in the point where the chosen parameter is expressed via the rest (or equal to $0, \infty$) $\Longrightarrow$ The problem is reduced to the calculation of integrals with $n-1$ parameter.

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  Initial conditions for the differential equation are put in the point where the chosen parameter is expressed via the rest (or equal to $0, \infty$) $\Longrightarrow$ The problem is reduced to the calculation of integrals with $n-1$ parameter.
- **One-scale integrals have obvious dependence on this scale. Differential equations cannot help.**
  Example: Massless propagator-type integrals, massive vacuum-type integrals, onshell massless vertices, onshell massive propagator

Laporta's difference equations

One-scale multiloop ($L \geqslant 2$) integrals:

Conventional approach: either direct calculation or by Laporta's difference equations (Laporta 2000).

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Derivation

Consider "generalized master" $J(x)$ obtained from the original $J(1)$ by rasing one denominator to power x. Perform Laporta algorithm near $J(x)$ in order to find the recurrence relation of the form

$$\sum_{k=0}^{n} c_k(x) J(x+k) = h(x) \tag{LΔE}$$

The left-hand side contains simpler integrals which are assumed to be known.

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Solution

Factorial series or Laplace transform (Laporta 2000). Homogeneous part can be fixed from large-x asymptotics.

Laporta's difference equations

Weak points

- Order of difference equation can be high $n \sim 10$
- Slow convergence of the factorial series at small $x \implies$ Calculate at sufficiently large x and then use recurrence to reach $x = 1 \implies$ loss of precision.

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Results

- Numerical and analytical (using `pslq`) results for four-loop massive tadpoles (Laporta 2002, Schroder and Vuorinen 2003, 2005, Bejdakic and Schroder 2006)
- Numerical results for three-loop onshell massive operators. Numerical and analytical results for three and four-loop onshell sunrise.(Laporta 2001, 2008)

Dimensional recurrence relation

Another variant of difference equation for the master integrals: Dimensional recurrence relation (Tarasov 1996).

Advantages

- Small order of dimensional recurence. Topologies with only one master $\Longrightarrow$ first-order equation.
- Fast convergence. In many cases the convergence is exponential $\Longrightarrow$ easy to obtain precise results and then use `pslq`.

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Why not use?

The homogeneous part of the solution depends on several (or one) periodic functions. Their determination appears extremely difficult!
Initial Tarasov's idea to fix them from the large-$\mathscr{D}$ does not work for multiloop integrals.

Dimensional recurrence relation

Dimensional recurrence relation

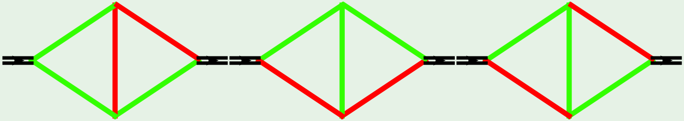
Raising DRR

Original Tarasov's formula was derived from the Feynman representation and is expressed via characteristics of the graph:

$$J^{(\mathscr{D}-2)}(\mathbf{n}) = \mu^L \sum_{\text{trees}} A_{i_1} \ldots A_{i_L} J^{(\mathscr{D})}(\mathbf{n}),$$

where $i_1, \ldots, i_L$ enumerate the chords of the given tree, and $\mu = \pm 1$ for the Eucledian/pseudoEucledian case.

Trees



Derivation is based on the analysis of the graph.

Dimensional recurrence relation

Derivation from Baikov's formula

Baikov's approach (to reduction)

Pass from the integration over the loop momenta to the integration over loop-momenta dependent scalar products (or the denominators $\mathscr{D}$)

$$d^{\mathscr{D}}l_1 \dots d^{\mathscr{D}}l_L \longrightarrow ds_{11}ds_{12}\dots ds_{L,L+E}, \qquad s_{ij} = l_i \cdot q_j$$

Jacobian is expressed via

Gram determinant

$$V\left(q_1,\dots,q_M\right) = \det\{q_i \cdot q_j\}$$

Gram determinant is polynomial in D_i

$$V(l_1,\dots l_L,p_1,\dots,p_E) = P(D_1,\dots D_N)$$

Dimensional recurrence relation

Derivation from Baikov's formula

Master formula

$$\int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L}{\pi^{L\mathscr{D}/2} D_1^{n_1} \ldots D_N^{n_N}} = \frac{\mu^L \pi^{-LE/2-L(L-1)/4}}{\Gamma\left[\left(\mathscr{D}-E-L+1\right)/2, \ldots, \left(\mathscr{D}-E\right)/2\right]} \times \int \left(\prod_{i=1}^{L} \prod_{j=i}^{L+E} ds_{ij} \right) \frac{\left[V\left(l_1, \ldots l_L, p_1, \ldots, p_E\right)\right]^{(\mathscr{D}-E-L-1)/2}}{\left[V\left(p_1, \ldots, p_E\right)\right]^{(\mathscr{D}-E-1)/2} D_1^{n_1} \ldots D_N^{n_N}}$$

Lowering DRR

$$J^{(\mathscr{D}+2)}(\mathbf{n}) = \frac{(2\mu)^L \left[V\left(p_1, \ldots, p_E\right)\right]^{-1}}{\left(\mathscr{D}-E-L+1\right)_L} \left(P\left(B_1, \ldots, B_N\right) J^{(\mathscr{D})} \right)(\mathbf{n}).$$

Advantages

This formula has no reference to the graph and therefore can be easily implemented.

Dimensional recurrence relation

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Lowering DRR

$$J^{(\mathscr{D}+2)}\left(\mathbf{n}\right) = \frac{(2\mu)^L \left[V\left(p_1, \ldots, p_E\right)\right]^{-1}}{\left(\mathscr{D}-E-L+1\right)_L} \left(P\left(B_1, \ldots, B_N\right) J^{(\mathscr{D})} \right) \left(\mathbf{n}\right).$$

Disadvantages

The shift of indices is $L+E$ to be compared with L for the raising DRR $\Longrightarrow$ IBP reduction is more difficult.

Dimensional recurrence relation

Useful identity I

Let

$$X=\begin{vmatrix} x_1^1 & \cdots & x_1^N \\ \vdots & \ddots & \vdots \\ x_N^1 & \cdots & x_N^N \end{vmatrix}$$

be determinant of a **general matrix** understood as function of its elements.

"Differentiation with respect to minor"

$$\begin{vmatrix} \frac{\partial}{\partial x_1^1} & \cdots & \frac{\partial}{\partial x_1^L} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_L^1} & \cdots & \frac{\partial}{\partial x_L^L} \end{vmatrix} X^\alpha = \alpha(\alpha+1)\ldots(\alpha+L-1)\begin{vmatrix} x_{L+1}^{L+1} & \cdots & x_{L+1}^N \\ \vdots & \ddots & \vdots \\ x_N^{L+1} & \cdots & x_N^N \end{vmatrix} X^{\alpha-1}$$

Can be proved from generalized Dodgson's (better known as Lewis Caroll) identity.

Dimensional recurrence relation

Useful identity II

Let

$$S = \begin{vmatrix} s_{11} & \cdots & s_{1N} \\ \vdots & \ddots & \vdots \\ s_{1N} & \cdots & s_{NN} \end{vmatrix}$$

be determinant of a **symmetric matrix** understood as function of its elements.

"Differentiation with respect to minor"

$$\begin{vmatrix} \frac{\partial}{\partial s_{11}} & \cdots & \frac{\partial}{2\partial s_{1L}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{2\partial s_{1L}} & \cdots & \frac{\partial}{\partial s_{LL}} \end{vmatrix} S^{\alpha} = \alpha\left(\alpha+\frac{1}{2}\right)\ldots\left(\alpha+\frac{L-1}{2}\right) \begin{vmatrix} s_{11} & \cdots & s_{1L} \\ \vdots & \ddots & \vdots \\ s_{1L} & \cdots & s_{LL} \end{vmatrix} S^{\alpha-1}$$

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$$\times \int \left(\prod_{i=1}^{L}\prod_{j=i}^{L+E} ds_{ij}\right) \frac{\left[V\left(l_1, \ldots l_L, p_1, \ldots, p_E\right)\right]^{(\mathscr{D}-E-L-1)/2}}{\left[V\left(p_1, \ldots, p_E\right)\right]^{(\mathscr{D}-E-1)/2} D_1^{n_1} \ldots D_N^{n_N}}$$

Raising DRR

$$J^{(\mathscr{D}-2)}\left(\mathbf{n}\right) = \left(-\mu\right)^L \begin{vmatrix} \frac{\partial}{\partial s_{11}} & \cdots & \frac{\partial}{2\partial s_{1L}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{2\partial s_{1L}} & \cdots & \frac{\partial}{\partial s_{LL}} \end{vmatrix} J^{(\mathscr{D})}\left(\mathbf{n}\right).$$

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Raising DRR

$$J^{(\mathscr{D}-2)}\left(\mathbf{n}\right) = \mu^L \det\left[\sum_k \frac{\partial D_k}{\partial s_{ij}} A_k \bigg|_{i,j=1,\dots L}\right] J^{(\mathscr{D})}\left(\mathbf{n}\right).$$

Solution of dimensional recurrence relation

Dimensional recurrence relation

$$J^{(\mathscr{D}-2)} = C(\mathscr{D})\,J^{(\mathscr{D})} + R(\mathscr{D}),$$

If there is no other master integral of the same topology, $R(\mathscr{D})$ contain only integrals of the simpler topologies and is assumed to be known.

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Solution

- Determine summing factor $\Sigma(\mathscr{D})$ from the equation $\frac{\Sigma(\mathscr{D})}{\Sigma(\mathscr{D}-2)} = C(\mathscr{D})$
  Summing factor permits multiplication by periodic function.

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Solution

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  Summing factor permits multiplication by periodic function.
- The general solution reads:

$$J^{(\mathscr{D})} = \Sigma^{-1}(\mathscr{D}) \left[\omega(z) - \sum_{k=0}^{\infty} \Sigma(\mathscr{D}-2k-2)\, R(\mathscr{D}-2k) \right],$$

where $\omega(z) = \omega(\exp[i\pi\mathscr{D}])$ is arbitrary function to be fixed.

Mittag-Leffler's theorem

Mittag-Leffler's theorem from complex analysis

Meromorphic function $f(z)$ can be restored from its singular parts up to the holomorphic function $h(z)$. If $f(z)$ is bounded at infinity, $h(z)$ is constant.

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Idea!

We can fix $\omega(z)$ by considering the analytical properties of $J^{(\mathscr{D})}$. Let us express $\omega(z)$ from the general solution

$$\omega(z) = \Sigma(\mathscr{D}) J^{(\mathscr{D})} + \sum_{k=0}^{\infty} \Sigma(\mathscr{D}-2k-2) R(\mathscr{D}-2k),$$

Suppose that we know all singularities of $\Sigma(\mathscr{D}) J^{(\mathscr{D})}$ on some **basic stripe** $S = \{\mathscr{D}, \operatorname{Re}\mathscr{D} \in (d, d+2]\}$ and know that $\Sigma(\mathscr{D}) J^{(\mathscr{D})}$ behaves well when $\operatorname{Im}\mathscr{D} \to \pm\infty$. Then we can use Mittag-Leffler's theorem to fix $\omega(z)$.

Analytical properties from parametric representation

Parametric representation

If I is the number of internal lines of the integral, parametric representation reads

$$J^{(\mathscr{D})} = \Gamma\left(I - L\mathscr{D}/2\right) \int dx_1 \ldots dx_I \delta\left(1 - \sum x_i\right) \frac{\left[Q\left(x\right)\right]^{\mathscr{D}L/2-I}}{\left[P\left(x\right)\right]^{\mathscr{D}(L+1)/2-I}}$$

$P(x) > 0$ and $Q(x) > 0$ are determined in terms of trees and 2-trees of the graph. Dependence on $\mathscr{D}$ is explicit here.

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$P(x) > 0$ and $Q(x) > 0$ are determined in terms of trees and 2-trees of the graph. Dependence on $\mathscr{D}$ is explicit here.

Observations

- If the integral converges on the real interval $\mathscr{D} \in (d_1, d_2]$, it is a holomorphic function on the whole stripe $\{\mathscr{D}, \operatorname{Re}\mathscr{D} \in (d_1, d_2]\}$.
- When $\operatorname{Im}\mathscr{D} \to \pm\infty$ the integral can be estimated as

$$J^{(\mathscr{D})} \lesssim \text{const} \times e^{-\pi L|\operatorname{Im}\mathscr{D}|/4} \left|\operatorname{Im}\mathscr{D}\right|^{I-1/2-L\operatorname{Re}(\mathscr{D})/2}$$

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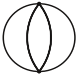
6. If needed, fix the remaining constants by conventional methods.

Example 1

Three-loop sunrise tadpole

$$J^{(\mathscr{D})} =$$



$$= \int \frac{d^{\mathscr{D}}k\, d^{\mathscr{D}}l\, d^{\mathscr{D}}r}{\pi^{3\mathscr{D}/2}\left[k^2+1\right]\left[l^2+1\right]\left[r^2+1\right]\left[(k+l+r)^2+1\right]}$$

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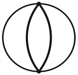

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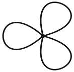

$$= \Gamma^3\left(1-\mathscr{D}/2\right)$$

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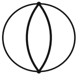
$$J^{(\mathscr{D})} = $$  $$ = \int \frac{d^{\mathscr{D}}k\, d^{\mathscr{D}}l\, d^{\mathscr{D}}r}{\pi^{3\mathscr{D}/2}\left[k^2+1\right]\left[l^2+1\right]\left[r^2+1\right]\left[(k+l+r)^2+1\right]}$$

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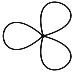
$$J_a^{(\mathscr{D})} = $$  $$ = \Gamma^3\left(1-\mathscr{D}/2\right)$$

2. The integral $J^{(\mathscr{D})}$ is a holomorphic function in the stripe $S = \{\mathscr{D}, \operatorname{Re}\mathscr{D} \in [-2,0)\}$ which can be deduced from its parametric representation. It is easy to check that any Euclidean integral with all lines massive is holomorphic in the whole half-plane $\operatorname{Re}\mathscr{D} < 0$.

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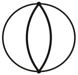

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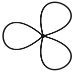

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3. Dimensional recurrence reads

$$J^{(\mathscr{D}-2)} = -\frac{(3\mathscr{D}-10)_3\,(\mathscr{D}-2)}{128(\mathscr{D}-4)} J^{(\mathscr{D})} - \frac{(11\mathscr{D}-38)(\mathscr{D}-2)^3}{64(\mathscr{D}-4)} J_a^{(\mathscr{D})}$$

Example 1

Three-loop sunrise tadpole

4. We choose the summing factor as

$$\Sigma(\mathscr{D}) = \frac{4^{-\mathscr{D}}\Gamma(2-\mathscr{D}/2)}{\Gamma(3/2-\mathscr{D}/2)\Gamma(3-3\mathscr{D}/2)}$$

The general solution has the form

$$\Sigma(\mathscr{D})J^{(\mathscr{D})} = \omega(z) + \sum_{k=1}^{\infty} t(\mathscr{D}-2k), \quad t(\mathscr{D}) = \frac{4^{-\mathscr{D}-2}(11\mathscr{D}-16)\Gamma^4(1-\mathscr{D}/2)}{\Gamma(3/2-\mathscr{D}/2)\Gamma(3-3\mathscr{D}/2)}$$

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   Both $\Sigma(\mathscr{D})J^{(\mathscr{D})}$ and $\sum_{k=1}^{\infty} t(\mathscr{D}-2k)$ grow slower than $|z|^{\mp 1}$ when $\operatorname{Im} D \to \pm\infty \implies$ the function $\omega(z)$ is constant!

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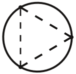
6. From $J^{(0)} = 1$ we obtain

$$\omega(z) = -\sum_{k=0}^{\infty} t(-2k) \overset{\text{pslq}}{=} \frac{3\pi^{3/2}}{16}$$

Example 2

Four-loop tadpole: Dealing with masless internal lines.

$J^{(\mathscr{D})} =$  $= \int \frac{d^{\mathscr{D}}k\, d^{\mathscr{D}}l\, d^{\mathscr{D}}r\, d^{\mathscr{D}}p}{\pi^{2\mathscr{D}} k^2 l^2 r^2 \left[(k+p)^2+1\right]\left[(l+p)^2+1\right]\left[(r+p)^2+1\right]}$

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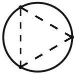

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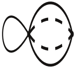

$$= -\frac{\Gamma(4-3\mathscr{D}/2)\,\Gamma^2(3-\mathscr{D})\,\Gamma^2(1-\mathscr{D}/2)\,\Gamma(\mathscr{D}/2-1)}{\Gamma(6-2\mathscr{D})}$$

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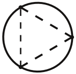
$$J^{(\mathscr{D})} =$$  $$= \int \frac{d^{\mathscr{D}}k\, d^{\mathscr{D}}l\, d^{\mathscr{D}}r\, d^{\mathscr{D}}p}{\pi^{2\mathscr{D}} k^2 l^2 r^2 \left[(k+p)^2+1\right]\left[(l+p)^2+1\right]\left[(r+p)^2+1\right]}$$

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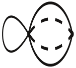
$$J_a^{(\mathscr{D})} =$$  $$= -\frac{\Gamma(4-3\mathscr{D}/2)\,\Gamma^2(3-\mathscr{D})\,\Gamma^2(1-\mathscr{D}/2)\,\Gamma(\mathscr{D}/2-1)}{\Gamma(6-2\mathscr{D})}$$

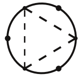
2.

Divergencies

- Infrared divergence at $\mathscr{D}=2$.
- Ultraviolet divergence at $\mathscr{D}=3$.

Stripe $S = \{\mathscr{D}, \quad \operatorname{Re}\mathscr{D} \in (2,3)\}$ is too narrow.

Choose another master

$$\tilde{J}^{(\mathscr{D})} =$$ 

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Four-loop tadpole: Dealing with masless internal lines.

Original master via new master

$$J^{(\mathscr{D})} = -\frac{3(3\mathscr{D}-11)(3\mathscr{D}-10)}{4(\mathscr{D}-4)(\mathscr{D}-3)^3(2\mathscr{D}-7)}\tilde{J}^{(\mathscr{D})} - \frac{3(\mathscr{D}-2)(3\mathscr{D}-8)\left(13\mathscr{D}^2-88\mathscr{D}+148\right)}{128(\mathscr{D}-3)^2(2\mathscr{D}-7)^2}J_a^{(\mathscr{D})}$$

Usefull consequence

$$\tilde{J}^{(3-2\varepsilon)} = \frac{\pi^2}{4} + \varepsilon\frac{\pi^2}{4}(11-4\gamma-8\ln 2) + O(\varepsilon)$$

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- Ultraviolet divergence at $\mathscr{D}=4.5$.

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$$\tilde{J}_2^{(\mathscr{D})} = -\frac{4(2\mathscr{D}-8)_4(\mathscr{D}-3)^3(\mathscr{D}-1)_2}{3(3\mathscr{D}-11)_5}\tilde{J}_2^{(\mathscr{D}+2)} + R(\mathscr{D})$$

Example 2

Four-loop tadpole: Dealing with masless internal lines.

4. We choose the summing factor as

$$\Sigma(\mathscr{D}) = \frac{\Gamma\left(7-\frac{3\mathscr{D}}{2}\right)\Gamma^2\left(\frac{\mathscr{D}}{2}-\frac{3}{2}\right)\Gamma(\mathscr{D}-1)}{2^{\mathscr{D}}\Gamma(9-2\mathscr{D})\Gamma(5-\mathscr{D})\Gamma\left(\frac{3\mathscr{D}}{2}-\frac{11}{2}\right)} \frac{\sin\left(\frac{\pi}{2}\mathscr{D}-\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{2}\mathscr{D}-\frac{\pi}{3}\right)\sin^2\left(\frac{\pi\mathscr{D}}{2}\right)}{\sin\left(\frac{\pi}{2}\mathscr{D}-\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{2}\mathscr{D}-\frac{\pi}{6}\right)}.$$

General solution

$$\Sigma(\mathscr{D})\tilde{J}^{(\mathscr{D})} = \omega(z) + \sum_{k=0}^{\infty} t(\mathscr{D}+2k), \quad t(\mathscr{D}) = \ldots$$

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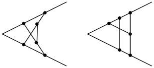
5. Taking into account singularities of $\sum_{k=0}^{\infty} t(\mathscr{D}+2k)$ and the only singularity of $\Sigma(\mathscr{D})\tilde{J}^{(\mathscr{D})}$ at $\mathscr{D}=3$, which is known, we obtain

$$\omega(z) \overset{\text{pslq}}{=} -\frac{3\pi^{7/2}}{32}\left(\cot\tfrac{\pi}{2}\left(\mathscr{D}-2\tfrac{1}{3}\right) - \cot\tfrac{\pi}{2}\left(\mathscr{D}-2\tfrac{2}{3}\right) - \cot\tfrac{\pi}{2}\left(\mathscr{D}-3\tfrac{1}{3}\right) + \cot\tfrac{\pi}{2}\left(\mathscr{D}-3\tfrac{2}{3}\right) - 4\cot\tfrac{\pi}{2}(\mathscr{D}-4)\right)$$

Summary and Outlook

- The missing ingredient for the application of dimensional recurrence relation to the calculation of the loop integrals is the analytical properties of the integrals as functions of complex variable $\mathscr{D}$.
- Derived method has already been successfully applied to the calculation of three-loop quark and gluon form factors. All masters are obtained **exactly in** $\mathscr{D}$. In particular the missing terms of ε-expansion of the two most complex integrals  are obtained in analytic form (Lee, Smirnov and Smirnov 2010). $\Longrightarrow$ V.Smirnov's talk Friday
- Work on three-loop static quark potential is in progress. The method of dealing with several-masters topologies is being derived.
- Outlook
  - Application to the four-loop tadpoles.
  - Application to the three- and four-loop $g-2$ master integrals
  - Other suggestions are welcome.

References I

Bejdakic, E. and Schroder, Y.: 2006, Hypergeometric representation of a four-loop vacuum bubble.

Chetyrkin, K. G. and Tkachov, F. V.: 1981, Integration by parts: The algorithm to calculate beta functions in 4 loops, Nucl. Phys. **B192***, 159–204.*

Gehrmann, T. and Remiddi, E.: 2000, Differential Equations for Two-Loop Four-Point Functions, Nucl. Phys. B **580***, 485.*

Kotikov, A. V.: 1991, Differential equations method: New technique for massive Feynman diagrams calculation, Phys. Lett. **B254***, 158–164.*

Laporta, S.: 2000, High precision calculation of multiloop feynman integrals by difference equations., Int. J. Mod. Phys. A **15***, 5087.*

Laporta, S.: 2001, High-precision epsilon-expansions of three-loop master integrals contributing to the electron g-2 in qed, Phys. Lett. **B523***, 95–101.*

Laporta, S.: 2002, High-precision e-expansions of massive four-loop vacuum bubbles.

Laporta, S.: 2008, Analytical expressions of 3 and 4-loop sunrise feynman integrals and 4-dimensional lattice integrals, Int. J. Mod. Phys. **A23***, 5007–5020.*

Lee, R. N.: 2008, Group structure of the integration-by-part identities and its application to the reduction of multiloop integrals, Journal of High Energy Physics **2008***(07), 031.*
***URL:** http://stacks.iop.org/1126-6708/2008/i=07/a=031*

Lee, R., Smirnov, A. and Smirnov, V.: 2010, Analytic results for massless three-loop form factors, Journal of High Energy Physics **2010***(4), 1–12.*
***URL:** http://dx.doi.org/10.1007/JHEP04(2010)020*

Remiddi, E.: 1997, Differential equations for Feynman graph amplitudes, Nuovo Cim. **A110***, 1435–1452.*

Schroder, Y. and Vuorinen, A.: 2003, High-precision evaluation of four-loop vacuum bubbles in three dimensions.

Schroder, Y. and Vuorinen, A.: 2005, High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles.

Tarasov, O. V.: 1996, Connection between feynman integrals having different values of the space-time dimension, Phys. Rev. D **54***, 6479.*

Tkachov, F. V.: 1981, A theorem on analytical calculability of 4-loop renormalization group functions, Physics Letters B **100***(1), 65–68.*
***URL:** http://www.sciencedirect.com/science/article/B6TVN-46YSNNV-109/1/886c25adc81acf1d171b80d7b1e7cb7f*