Asymptotic spectrum
via
Bethe-Yang equations

Bethe wave function

On the plane momenta p_i are unrestricted. Compactifying the theory on a circle of large but finite circumference L, one naturally requires the Bethe wave function to be periodic. Periodicity condition leads to a system of equations for particle momenta which can be regarded as quantisation thereof.

$x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(N)}$

$p_1 > p_2 > \cdots > p_N$

Bethe wave – function in the σ – sector

$$\Psi^{i_1\cdots i_N}(x_1,\ldots,x_N|\sigma) = \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}^{i_1\cdots i_N}(\sigma|\tau)\, e^{ix_\sigma\cdot p_\tau}$$

flavor index is attached to a particle coordinate

The wave function satisfies the following symmetry condition

$$\Psi^{i_1\cdots i_k i_{k+1}\cdots i_N}(x_1,\ldots,x_k,x_{k+1},\ldots,x_N|\sigma) =$$
$$= (-)^{\epsilon_{i_k}\epsilon_{i_{k+1}}}\,\Psi^{i_1\cdots i_{k+1} i_k\cdots i_N}(x_1,\ldots,x_{k+1},x_k,\ldots,x_N|\alpha_{k,k+1}\sigma)$$

$\epsilon_i = 0$ for boson

$\epsilon_i = 1$ for fermion

Because of the symmetry condition it is sufficient to determine the wave function only in one domain which we take to be $x_1 < x_2 < \cdots < x_N$. To simplify the notations we omit the subscript $\sigma = e$, and write the wave function as

$$\Psi^{i_1\cdots i_N}(x_1,\ldots,x_N) = \sum_{\tau\in\mathfrak{S}_N} e^{i\,x\cdot p_\tau}\,\mathcal{A}^{i_1\cdots i_N}(\tau)\,, \quad x_1 < x_2 < \cdots < x_N$$

Bethe wave function

$$\Psi^{i_1\cdots i_N}(x_1,\ldots,x_N)=\sum_{\tau\in\mathfrak{S}_N} e^{i\,x\cdot p_\tau}\,\mathcal{A}^{i_1\cdots i_N}(\tau)\,,\qquad x_1<x_2<\cdots<x_N$$

The amplitudes $\mathcal{A}^{i_1\cdots i_N}(\tau)$ are related to $\mathcal{A}^{i_1\cdots i_N}(e)\equiv\mathcal{A}^{i_1\cdots i_N}$ through the S-matrix

ZF – vacuum

$$\Psi^{i_1\cdots i_N}(x_1,\ldots,x_N)=\langle 0|A^{i_N}(x_N)A^{i_{N-1}}(x_{N-1})\ldots A^{i_1}(x_1)\underbrace{|p_1,\ldots,p_N;\mathcal{A}\rangle}$$

$$A^i(x)=\int \mathrm{d}p\,e^{ipx}\,A^i(p)$$

$$\overbrace{|p_1,\ldots,p_N;\mathcal{A}\rangle}=\sum_{j_1\cdots,j_N}A^\dagger_{j_1}(p_1)\cdots A^\dagger_{j_N}(p_N)|0\rangle\,\mathcal{A}^{j_1\cdots j_N}$$

Bethe wave function

Example: two-body wave function

$$\begin{aligned}\Psi^{ij}(x_1,x_2) &= \langle 0|A^j(x_2)A^i(x_1)A_k^\dagger(p_1)A_l^\dagger(p_2)|\rangle\,\mathcal{A}^{kl}\\ &= \int \mathrm{d}k_2\mathrm{d}k_1 e^{ik_2x_2+ik_1x_1}\,\langle 0|A^j(k_2)A^i(k_1)A_k^\dagger(p_1)A_l^\dagger(p_2)|0\rangle\,\mathcal{A}^{kl}\end{aligned}$$

$$A^i(k_1)A_k^\dagger(p_1) = A_m^\dagger(p_1)S_{ks}^{mi}(p_1,k_1)A^s(k_1) + \delta_i^k\delta(k_1-p_1)$$

$$\Psi^{ij}(x_1,x_2) = \int \mathrm{d}k_2\mathrm{d}k_1 e^{ik_2x_2+ik_1x_1}$$
$$\times\,\langle 0|A^j(k_2)A_m^\dagger(p_1)S_{ks}^{mi}(p_1,k_1)A^s(k_1)A_l^\dagger(p_2) + \delta_i^k\delta(k_1-p_1)A^j(k_2)A_l^\dagger(p_2)|0\rangle\,\mathcal{A}^{kl}$$

$$\begin{aligned}\Psi^{ij}(x_1,x_2) &= \int \mathrm{d}k_2\mathrm{d}k_1 e^{ik_2x_2+ik_1x_1}\\ &\times\ \langle 0|S_{kl}^{ji}(p_1,k_1)\delta(k_2-p_1)\delta(k_1-p_2) + \delta_i^k\delta_j^\delta(k_1-p_1)\delta(k_2-p_2)|0\rangle\,\mathcal{A}^{kl}\\ &= \mathcal{A}^{ij}\,e^{ip_1x_1+ip_2x_2} + S_{kl}^{ji}(p_1,p_2)\mathcal{A}^{kl}\,e^{ip_2x_1+ip_1x_2}\end{aligned}$$

$$\Psi^{ij}(x_1,x_2) = \mathcal{A}^{ij}\,e^{ip_1x_1+ip_2x_2} + \underbrace{S_{kl}^{ji}(p_1,p_2)\mathcal{A}^{kl}}\,e^{ip_2x_1+ip_1x_2}$$

$\mathcal{A}^{ij}((12))$ $\qquad\qquad$ $\mathcal{A}^{ij}((21))$

Bethe wave function

Example: two-body wave function

We rederive the same result but with a different emphasis

$$\Psi^{ij}(x_1,x_2) = \left[\int_{k_1>k_2} + \int_{k_1<k_2}\right] \mathrm{d}k_1 \mathrm{d}k_2\, e^{ik_1x_1+ik_2x_2} \langle 0|A^j(k_2)A^i(k_1)A^\dagger_k(p_1)A^\dagger_l(p_2)|0\rangle\, \mathcal{A}^{kl}$$

$\delta(k_1-p_1)$

$$= \int_{k_1>k_2} \mathrm{d}k_1 \mathrm{d}k_2\, e^{ik_1x_1+ik_2x_2} \langle 0|A^j(k_2)A^i(k_1)A^\dagger_k(p_1)A^\dagger_l(p_2)|0\rangle\, \mathcal{A}^{kl}$$

$\delta(k_2-p_2)$

$\delta(k_1-p_1)$

$$+ \int_{k_1>k_2} \mathrm{d}k_1 \mathrm{d}k_2\, e^{ik_2x_1+ik_1x_2} \langle 0|A^j(k_1)A^i(k_2)A^\dagger_k(p_1)A^\dagger_l(p_2)|0\rangle\, \mathcal{A}^{kl}$$

$\delta(k_2-p_2)$

$$\Psi^{ij}(x_1,x_2) = e^{ip_1x_1+ip_2x_2} \langle 0|A^j(p_2)A^i(p_1)A^\dagger_k(p_1)A^\dagger_l(p_2)|0\rangle\, \mathcal{A}^{kl}$$
$$+ \; e^{ip_2x_1+ip_1x_2} \langle 0|A^j(p_1)A^i(p_2)A^\dagger_k(p_1)A^\dagger_l(p_2)|0\rangle\, \mathcal{A}^{kl}$$

Prescription: to compute this amplitude, use the ZF algebra to permute A and $A^\dagger$ with different momenta, and the rule that

$$A^i(p)A^\dagger_j(p) = \delta^i_j$$

$$\Psi^{ij}(x_1,x_2) = \mathcal{A}^{ij}\, e^{ip_1x_1+ip_2x_2} + S^{ji}_{kl}(p_1,p_2)\mathcal{A}^{kl}\, e^{ip_2x_1+ip_1x_2}$$

Periodicity condition for the Bethe wave function

General case

$$\Psi^{i_1\cdots i_N}(x_1,\ldots,x_N) = \sum_{\tau\in\mathfrak{S}_N} e^{i\,p_\tau\cdot x}\,\mathcal{A}^{i_1\cdots i_N}(\tau)\,,$$

where the amplitude $\mathcal{A}_\tau$ is

$$\mathcal{A}^{i_1\cdots i_N}(\tau) = \langle 0|A^{i_N}(p_{\tau(N)})\ldots A^{i_1}(p_{\tau(1)})A^\dagger_{j_1}(p_1)\ldots A^\dagger_{j_N}(p_N)|0\rangle\,\mathcal{A}^{j_1\cdots j_N}$$

Prescription: to compute this amplitude, use the ZF algebra to permute A and $A^\dagger$ with different momenta, and the rule that $A^i(p)A^\dagger_j(p) = \delta^i_j$

Periodicity condition for the Bethe wave function $\equiv$ Coordinate Bethe Ansatz

$$\Psi^{i_1\ldots i_k}(x_1,\ldots,x_k=0,\ldots,x_N) = \Psi^{i_1\ldots i_k}(x_1,\ldots,x_k=L,\ldots,x_N)$$

L is the size of the box (length of a circle)

Periodicity condition for the Bethe wave function

Two-body case

fundamental sector

$$\Psi^{ij}(0,x) = \Psi^{ij}(L,x) = (-1)^{\epsilon_i\epsilon_j}\Psi^{ji}(x,L)$$

$$\mathcal{A}^{ij}\, e^{ip_2 x} + S^{ji}_{kl}(p_1,p_2)\mathcal{A}^{kl}\, e^{ip_1 x} = (-1)^{\epsilon_i\epsilon_j}\Big(\mathcal{A}^{ji}\, e^{ip_1 x_1 + ip_2 L} + S^{ij}_{kl}(p_1 p_2)\mathcal{A}^{kl}\, e^{ip_2 x + ip_1 L}\Big)$$

These relations are equivalent to the following equations on the amplitude $\mathcal{A}^{kl}$

$$\begin{aligned}\Big[(-1)^{\epsilon_i\epsilon_j} S^{ij}_{kl}(p_1,p_2) - e^{-ip_1 L}\delta^i_k\delta^j_l\Big]\mathcal{A}^{kl} &= 0\,,\\ \Big[S^{ji}_{kl}(p_1,p_2) - (-1)^{\epsilon_i\epsilon_j}\delta^j_k\delta^i_l\, e^{ip_2 L}\Big]\mathcal{A}^{kl} &= 0\end{aligned}
\quad \underset{S_{12}(p_1,p_2)S_{21}(p_2,p_1)=\mathbb{1}}{\Longrightarrow} \quad
\begin{aligned}\Big[(-1)^{\epsilon_i\epsilon_j} S^{ij}_{kl}(p_1,p_2) - e^{-ip_1 L}\delta^i_k\delta^j_l\Big]\mathcal{A}^{kl} &= 0\,,\\ \Big[S^{ji}_{lk}(p_2,p_1)(-1)^{\epsilon_k\epsilon_l} - e^{-ip_2 L}\delta^i_k\delta^j_l\Big]\mathcal{A}^{kl} &= 0\end{aligned}$$

It is convenient to introduce the graded identity $I^g = (-1)^{\epsilon_i\epsilon_j} E^i_i \otimes E^j_j$ and introduce

column

$$S_{12}(p_1,p_2) = S^{kl}_{ij}(p_1,p_2)\, E^i_k \otimes E^j_l\,, \qquad \mathcal{A} = \mathcal{A}^{mn} E_m \otimes E_n\,, \qquad E^i_k E_m = \delta^i_m E_k$$

$$\boxed{\begin{aligned} I^g_{12} S_{12}(p_1,p_2)\mathcal{A} &= e^{-ip_1 L}\,\mathcal{A}\\ S_{21}(p_2,p_1) I^g_{21}\mathcal{A} &= e^{-ip_2 L}\,\mathcal{A}\end{aligned}}$$

matrix Bethe – Yang equations

Periodicity condition for the Bethe wave function

Three-body case

$$\begin{aligned}
\Psi^{i_1 i_2 i_3}(x_1,x_2,x_3) &= \mathcal{A}^{i_1 i_2 i_3}\, e^{ip_1x_1+ip_2x_2+ip_3x_3} & (123)\\
&+ S^{i_2 i_1}_{j_1 j_2}(p_1,p_2)\delta^{i_3}_{j_3}\mathcal{A}^{j_1 j_2 j_3}\, e^{ip_2x_1+ip_1x_2+ip_3x_3} & (213)\\
&+ S^{m i_1}_{j_1 j_2}(p_1,p_2)S^{i_3 i_2}_{m j_3}(p_1,p_3)\,\mathcal{A}^{j_1 j_2 j_3}\, e^{ip_2x_1+ip_3x_2+ip_1x_3} & (231)\\
&+ S^{i_3 i_2}_{kl}(p_1,p_2)S^{k i_1}_{j_1 m}(p_1,p_3)S^{lm}_{j_2 j_3}(p_2,p_3)\,\mathcal{A}^{j_1 j_2 j_3}\, e^{ip_3x_1+ip_2x_2+ip_1x_3} & (321)\\
&+ S^{i_2 i_1}_{j_1 k}(p_1,p_3)S^{i_3 k}_{j_2 j_3}(p_2,p_3)\,\mathcal{A}^{j_1 j_2 j_3}\, e^{ip_3x_1+ip_1x_2+ip_2x_3} & (312)\\
&+ \delta^{i_1}_{j_1}S^{i_3 i_2}_{j_2 j_3}(p_2,p_3)\,\mathcal{A}^{j_1 j_2 j_3}\, e^{ip_1x_1+ip_3x_2+ip_2x_3} & (132)
\end{aligned}$$

$$\Psi^{i_1 i_2 i_3}(0,x_2,x_3) = \Psi^{i_1 i_2 i_3}(L,x_2,x_3) = (-1)^{\epsilon_{i_1}\epsilon_{i_2}}(-1)^{\epsilon_{i_1}\epsilon_{i_3}}\,\Psi^{i_2 i_1 i_3}(x_2,x_3,L)$$

This yields the matrix Bethe-Yang equations

$$\boxed{\begin{aligned}
I^g_{13}I^g_{12}S_{13}(p_1,p_3)S_{12}(p_1,p_2)\,\mathcal{A} &= e^{-ip_1L}\,\mathcal{A}\\
S_{21}(p_2,p_1)I^g_{21}I^g_{23}S_{23}(p_2,p_3)\,\mathcal{A} &= e^{-ip_2L}\,\mathcal{A}\\
S_{32}(p_3,p_2)S_{31}(p_3,p_1)I^g_{32}I^g_{31}\,\mathcal{A} &= e^{-ip_3L}\,\mathcal{A}
\end{aligned}}$$

$$\begin{aligned}
T_1 &= I^g_{13}I^g_{12}S_{13}(p_1,p_3)S_{12}(p_1,p_2)\\
T_2 &= S_{21}(p_2,p_1)I^g_{21}I^g_{23}S_{23}(p_2,p_3)\\
T_3 &= S_{32}(p_3,p_2)S_{31}(p_3,p_1)I^g_{32}I^g_{31}
\end{aligned}$$

For $\mathcal{A}$ to exist, T_1, T_2, T_3 must mutually commute

Periodicity condition for the Bethe wave function

$$T_1 = I^g_{13} I^g_{12} S_{13}(p_1,p_3) S_{12}(p_1,p_2)$$
$$T_2 = S_{21}(p_2,p_1) I^g_{21} I^g_{23} S_{23}(p_2,p_3)$$
$$T_3 = S_{32}(p_3,p_2) S_{31}(p_3,p_1) I^g_{32} I^g_{31}$$

Compatibility of scattering with statistics implies that

$$S^{kl}_{ij}(p_1,p_2) = (-1)^{\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_l} S^{kl}_{ij}(p_1,p_2)$$

$$I^g_{12} I^g_{23} S_{13} = S_{13} I^g_{12} I^g_{23}$$

Exercise

Check, for instance, commutativity of T_1 and T_2

$$\begin{aligned}
[T_1, T_2] &= I^g_{13} I^g_{12} S_{13} \overbrace{S_{12} S_{21}}^{\mathbb{1}} I^g_{21} I^g_{23} S_{23} - S_{21} I^g_{21} I^g_{23} S_{23} \underbrace{I^g_{13} I^g_{12}}_{} S_{13} S_{12} \\
&= I^g_{13} I^g_{12} S_{13} \underbrace{I^g_{21} I^g_{23}}_{} S_{23} - S_{21} \underbrace{I^g_{23} I^g_{13}}_{} S_{23} S_{13} S_{12} \\
&= I^g_{13} I^g_{23} S_{13} S_{23} - I^g_{13} I^g_{23} S_{21} \underbrace{S_{23} S_{13} S_{12}}_{\text{YB}} \\
&= I^g_{13} I^g_{23} S_{13} S_{23} - I^g_{13} I^g_{23} \overbrace{S_{21} S_{12}}^{\mathbb{1}} S_{13} S_{23} = 0
\end{aligned}$$

Analogously,

$$[T_1, T_3] = 0 = [T_2, T_3]$$

Periodicity condition for the Bethe wave function

General case

$$\Psi^{i_1\dots i_k\,\dots\, i_N}(x_1,\dots,x_k=0,\dots,x_N)=\Psi^{i_1\dots i_k\,\dots\, i_N}(x_1,\dots,x_k=L,\dots,x_N)$$

Generalisation to the twist boundary conditions

$$\Psi^{i_1\dots i_k\,\dots\, i_N}(x_1,\dots,x_k=0,\dots,x_N)=\Psi^{i_1\dots n\,\dots\, i_N}(x_1,\dots,x_k=L,\dots,x_N)W_n^{i_k}$$

twist

where the diagonal matrix W is equal to the identity matrix if the fermions are periodic, and it is $W=(-1)^{\epsilon_i}E_i{}^i$ if the fermions are anti-periodic.

$$\boxed{\Big[S_{k,k-1}\cdots S_{k1}I^g_{k,k-1}\cdots I^g_{k1}W_kI^g_{kN}\cdots I^g_{k,k+1}S_{kN}\cdots S_{k,k+1}\Big]\,\mathcal{A}=e^{-ip_kL}\mathcal{A}}$$

matrix Bethe – Yang equations

$$I^g_{13}I^g_{12}S_{13}(p_1,p_3)S_{12}(p_1,p_2)\,\mathcal{A}=e^{-ip_1L}\,\mathcal{A}$$
$$S_{21}(p_2,p_1)I^g_{21}I^g_{23}S_{23}(p_2,p_3)\,\mathcal{A}=e^{-ip_2L}\,\mathcal{A}$$
$$S_{32}(p_3,p_2)S_{31}(p_3,p_1)I^g_{32}I^g_{31}\,\mathcal{A}=e^{-ip_3L}\,\mathcal{A}$$

Why the operators

$$T_k\equiv S_{k,k-1}\cdots S_{k1}I^g_{k,k-1}\cdots I^g_{k1}W_kI^g_{kN}\cdots I^g_{k,k+1}S_{kN}\cdots S_{k,k+1}$$

mutually commute?

Periodicity condition for the Bethe wave function

$$T_k \equiv S_{k,k-1} \cdots S_{k1} I^g_{k,k-1} \cdots I^g_{k1} W_k I^g_{kN} \cdots I^g_{k,k+1} S_{kN} \cdots S_{k,k+1}$$

The matrices T_k are related to the transfer matrix

$$T(p_A) = -\mathrm{str}_A \, W_A S^f_{AN}(p_A, p_N) S^f_{A,N-1}(p_A, p_{N-1}) \cdots S^f_{A1}(p_A, p_1)$$

auxiliary

where $A = N+1$, and S^f_{jk} is the so-called fermionic R-operator defined as follows

$$S^f_{jk}(p_j, p_k) = \begin{cases} I^g_{j\cdots N} I^g_{k\cdots N} \, I^g_{jk} S_{jk}(p_j, p_k) \, I^g_{j\cdots N} I^g_{k\cdots N} & \text{if } j < k\,; \\ I^g_{j\cdots N} I^g_{k\cdots N} \, S_{jk}(p_j, p_k) I^g_{jk} \, I^g_{j\cdots N} I^g_{k\cdots N} & \text{if } j > k\,. \end{cases}$$

$$I^g_{j\cdots N} \equiv I^g_{j,j+1} I^g_{j,j+2} \cdots I^g_{jN}$$

The fermionic R-operator satisfies the graded Yang-Baxter equation which formally looks the same as the ordinary Yang-Baxter equation

As a consequence of the graded Yang-Baxter equation, the transfer matrix obeys

$$\boxed{T(u)T(v) = T(v)T(u)\,, \quad \forall u, v}$$

One can show that

$$T(p_A) = -\mathrm{str}_A \, W_A \, S_{AN} \cdots S_{A1} \, I^g_{AN} \cdots I^g_{A1}$$

Periodicity condition for the Bethe wave function

$$T(p_A) = -\mathrm{str}_A\, W_A\, S_{AN} \cdots S_{A1}\, I^g_{AN} \cdots I^g_{A1}$$

Choose $p_A = p_k$ and use the fact that $S_{Ak}(p_k, p_k) = -P_{Ak}$ to show that

permutation

$$T(p_k) = T_k$$

Since $\quad T(u)T(v) = T(v)T(u)\,, \quad \forall u, v$

$$[T_k, T_m] = 0\,, \qquad \forall\, k, m$$

Denoting the eigenvalues of the monodromy matrix $T(p_A)$ by $\Lambda(p_A, \{p_i\})$, the set of Bethe-Yang equations can be written as

$$\boxed{e^{ip_k L}\Lambda(p_k, \{p_i\}) = 1}$$

Bethe – Yang equations

Example: scalar S-matrices $\qquad e^{ip_k L}\prod_{j\neq k}^{N} S(p_k, p_j) = 1 \quad \longleftarrow \quad$ conventional Bethe equations

Finding the eigenvalues of $T(p_A)$ is a complicated problem which can be solved by using either the algebraic Bethe ansatz or the nested Bethe ansatz technique.

Bethe -Yang equations for $\mathrm{AdS}_5 \times \mathrm{S}^5$ *superstring*

$x_k^{\pm} \equiv x^{\pm}(p_k)$

$$1 = e^{iJp_k} \prod_{l \neq k}^{K^{\mathrm{I}}} S_{\mathfrak{sl}(2)}(p_k, p_l) \prod_{l=1}^{K_-^{\mathrm{II}}} \frac{x_k^- - y_l^{(-)}}{x_k^+ - y_l^{(-)}} \sqrt{\frac{x_k^+}{x_k^-}} \prod_{l=1}^{K_+^{\mathrm{II}}} \frac{x_k^- - y_l^{(+)}}{x_k^+ - y_l^{(+)}} \sqrt{\frac{x_k^+}{x_k^-}}$$

These equations are supplied with auxiliary Bethe equations for the roots $y^{(\alpha)}$ and $w^{(\alpha)}$, $\alpha = \pm$,

$\nu_k^{(\alpha)} = y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}}.$

$$\prod_{i=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_i^-}{y_k^{(\alpha)} - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} = \prod_{i=1}^{K_\alpha^{\mathrm{III}}} \frac{w_i^{(\alpha)} - \nu_k^{(\alpha)} - \frac{i}{g}}{w_i^{(\alpha)} - \nu_k^{(\alpha)} + \frac{i}{g}},$$

$$\prod_{i=1}^{K_\alpha^{\mathrm{II}}} \frac{w_k^{(\alpha)} - \nu_i^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - \nu_i^{(\alpha)} - \frac{i}{g}} = -\prod_{i=1}^{K_\alpha^{\mathrm{III}}} \frac{w_k^{(\alpha)} - w_i^{(\alpha)} + \frac{2i}{g}}{w_k^{(\alpha)} - w_i^{(\alpha)} - \frac{2i}{g}}.$$

Five excitation numbers

number of momentum − carrying particles

$$(K_-^{\mathrm{III}}, K_-^{\mathrm{II}}, K^{\mathrm{I}}, K_+^{\mathrm{II}}, K_+^{\mathrm{III}})$$

weights of four light − cone $\mathfrak{su}(2)'$s

$$q_1 = K_-^{\mathrm{II}} - 2K_-^{\mathrm{III}}$$
$$p = J - \tfrac{1}{2}(K_-^{\mathrm{II}} + K_+^{\mathrm{II}}) + K_-^{\mathrm{III}} + K_+^{\mathrm{III}}$$
$$q_2 = K_+^{\mathrm{II}} - 2K_+^{\mathrm{III}}$$

$$s_1 = K^{\mathrm{I}} - K_-^{\mathrm{II}}$$
$$s_2 = K^{\mathrm{I}} - K_+^{\mathrm{II}}$$

Large spin anomalous dimensions of operators from the sl(2)-sector

Bethe-Yang equations for sl(2)-sector

$$S_{\mathfrak{sl}(2)}(x_1,x_2)=\sigma_{12}^{-2}s_{12}\,,\qquad s_{12}=\frac{x_1^+-x_2^-}{x_1^--x_2^+}\frac{1-\frac{1}{x_1^-x_2^+}}{1-\frac{1}{x_1^+x_2^-}}$$

$$K^{\mathrm{I}}=S$$

$$e^{iJp_k}=\prod_{l\neq k}^{S}\frac{x_k^--x_l^+}{x_k^+-x_l^-}\frac{1-\frac{1}{x_k^+x_l^-}}{1-\frac{1}{x_k^-x_l^+}}\sigma_{kl}^2$$

Excitations correspond to gauge-theory operators

$$\mathrm{Tr}(\mathcal{D}^{\overset{\text{spin}}{S}}Z^{\overset{\text{charge = twist}}{J}})+\ldots$$

The energy is

$$E=J+H=\overbrace{J+S}^{\text{canonical}}+\underbrace{\sum_{k=1}^{S}\Big(\frac{ig}{x_k^+}-\frac{ig}{x_k^-}\Big)}_{\text{anomalous}}$$

Bethe-Yang equations for sl(2)-sector

$$e^{iJp_k} = \prod_{l\neq k}^{S} \frac{x_k^- - x_l^+}{x_k^+ - x_l^-} \frac{1-\frac{1}{x_k^+ x_l^-}}{1-\frac{1}{x_k^- x_l^+}} \sigma_{kl}^2$$

Technicalities

$$\left\{ \begin{array}{ll} u_k - \frac{i}{g} = x_k^- + \frac{1}{x_k^-}\,, & u_k + \frac{i}{g} = x_k^+ + \frac{1}{x_k^+}\,, \\ u_k + \frac{i}{g} = x_l^+ + \frac{1}{x_l^+}\,, & u_l - \frac{i}{g} = x_l^- + \frac{1}{x_l^-} \end{array} \right\}$$

$$u_k - u_l - \frac{2i}{g} = x_k^- + \frac{1}{x_k^-} - x_l^+ - \frac{1}{x_l^+} = (x_k^- - x_l^+)\left(1 - \frac{1}{x_k^- x_l^+}\right)$$

$$u_k - u_l + \frac{2i}{g} = x_k^+ + \frac{1}{x_k^+} - x_l^- - \frac{1}{x_l^-} = (x_k^+ - x_l^-)\left(1 - \frac{1}{x_k^+ x_l^-}\right)$$

$$\frac{x_k^- - x_l^+}{x_k^+ - x_l^-} = \frac{u_k - u_l - \frac{2i}{g}}{u_k - u_l + \frac{2i}{g}} \left(\frac{1-\frac{1}{x_k^+ x_l^-}}{1-\frac{1}{x_k^- x_l^+}} \right)$$

Bethe-Yang equations for sl(2)-sector

$$e^{iJp_k} = \prod_{l\neq k}^{S} \frac{x_k^- - x_l^+}{x_k^+ - x_l^-} \frac{1 - \frac{1}{x_k^+ x_l^-}}{1 - \frac{1}{x_k^- x_l^+}} \sigma_{kl}^2$$

Technicalities

$$u + \frac{i}{g} = x^+ + \frac{1}{x^+}$$
$$u - \frac{i}{g} = x^- + \frac{1}{x^-}$$

Dividing one equation by the other one gets

$$\frac{u+\frac{i}{g}}{u-\frac{i}{g}} = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}} = \frac{x^+}{x^-} \frac{1+\frac{1}{x^{+2}}}{1+\frac{1}{x^{-2}}}$$

$$e^{ip} = \frac{x^+}{x^-} = \frac{u+\frac{i}{g}}{u-\frac{i}{g}} \frac{1+\frac{1}{x^{-2}}}{1+\frac{1}{x^{+2}}}$$

Bethe-Yang equations for sl(2)-sector

The all-loop Bethe ansatz takes the form

$$\left(\frac{u_k+\frac{i}{g}}{u_k-\frac{i}{g}}\right)^J\left(\frac{1+\frac{1}{x_k^{-2}}}{1+\frac{1}{x_k^{+2}}}\right)^J=\prod_{l\neq k}^{S}\frac{u_k-u_l-\frac{2i}{g}}{u_k-u_l+\frac{2i}{g}}\left(\frac{1-\frac{1}{x_k^+x_l^-}}{1-\frac{1}{x_k^-x_l^+}}\,\sigma_{kl}\right)^2$$

Rescale $u\to u/g$

$$x(u)=\frac{u}{2}\left(1+\sqrt{1-\frac{4}{u^2}}\right)\quad\longrightarrow\quad x(u)=\frac{u}{2g}\left(1+\sqrt{1-\frac{4g^2}{u^2}}\right)$$

$$\left(\frac{u_k+i}{u_k-i}\right)^J\left(\frac{1+\frac{1}{x_k^{-2}}}{1+\frac{1}{x_k^{+2}}}\right)^J=\prod_{l\neq k}^{S}\frac{u_k-u_l-2i}{u_k-u_l+2i}\left(\frac{1-\frac{1}{x_k^+x_l^-}}{1-\frac{1}{x_k^-x_l^+}}\,\sigma_{kl}\right)^2$$

Bethe-Yang equations for sl(2)-sector at one loop

$$x(u) = \frac{u}{2g}\left(1 + \sqrt{1 - \frac{4g^2}{u^2}}\right)$$

First one needs to find a distribution of roots at one loop.

In the limit $g \to 0$ one has

$$x^{\pm} \to \frac{u \pm i}{g}$$

The Bethe-Yang equations turn into

$$\left(\frac{u_k + i}{u_k - i}\right)^J = \prod_{l \neq k}^{S} \frac{u_k - u_l - 2i}{u_k - u_l + 2i}$$

The one-loop energy is

$$\Delta^{(1)} = g^2 \sum_{k=1}^{S} \frac{2}{u_k^2 + 1}$$

and the level matching condition is

$$\prod_{k=1}^{S} \frac{u_k + i}{u_k - i} = 1$$

Bethe-Yang equations for sl(2)-sector at one loop and S → ∞

- For all J and S the roots are always real
- The case $J = 2$ can be solved exactly
- The roots are real and symmetrically distributed around zero
- The root distribution density has a peak at the origin, no gap around zero
- The outermost roots grow linearly with the spin $\{|u_k|\} \to S/2$

Take the logarithm of the Bethe equations

$$-iJ \log \frac{u_k + i}{u_k - i} = 2\pi n_k - i \sum_{l \neq k}^{S} \log \frac{u_k - u_l - 2i}{u_k - u_l + 2i}$$

$n_k \in \mathbb{Z}$ – quantum mode numbers

Rescale the roots $\dfrac{u_k}{S} \to v_k$ and expand the Bethe equations in the limit $S \to \infty$

$$\frac{2J}{v_k S} = 2\pi n_k - \sum_{l \neq k}^{S} \frac{1}{S} \frac{1}{v_k - v_l} + \mathcal{O}(1/S^2)$$

for the lowest state all roots have $n_k = \pm 1$ for any J

Introduce the normalised root density

$$\rho(v) = \frac{1}{S} \sum_{k=1}^{S} \delta(v - v_k)\,, \qquad \int_{-a}^{a} dv \rho(v) = 1$$

Bethe-Yang equations for sl(2)-sector at one loop

$$\cancel{\frac{2J}{v_k S}} = 2\pi n_k - \sum_{l\neq k}^{S} \frac{1}{S}\frac{1}{v_k - v_l} + \cancel{\mathcal{O}(1/S^2)}$$

J drops out!

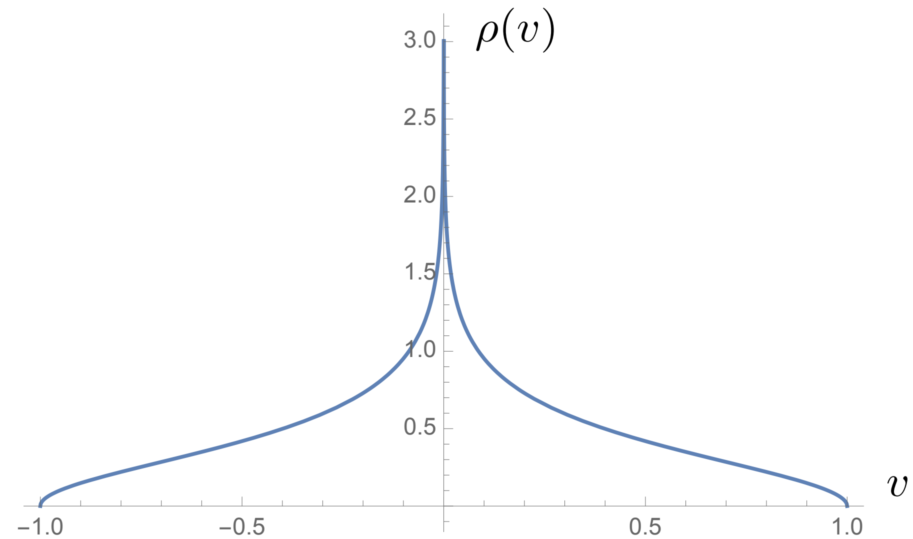
$n_k \to \epsilon(v)$

$\sum_{l\neq k}^{S}\frac{1}{S}\dots \to \fint_{-a}^{a}\rho(v')\dots$

$$0 = 2\pi\epsilon(v) - 4\fint_{-a}^{a} \mathrm{d}v' \, \frac{\rho(v')}{v - v'}$$

Solution

$$\rho(v) = \frac{1}{\pi}\operatorname{arctanh}\sqrt{1 - v^2}$$



$a = 1$

The anomalous one-loop energy is

$$\Delta^{(1)} = g^2 \sum_{k=1}^{S} \frac{2}{u_k^2 + 1} \xrightarrow{S\to\infty} \frac{2}{S}\int_{-1}^{1} \mathrm{d}v \, \frac{\rho(v)}{v^2 + 1/S^2} \xrightarrow{S\to\infty} 2g^2 \log S$$

regularization

[Korchemsky 1995]
[Eden & Staudacher 2006]

Exercise

A general solution of the singular integral equation

$$\fint_a^b \frac{\rho(t)\mathrm{d}t}{t-v} = f(v)\,, \qquad a < v < b$$

is given by

$$\rho(v) = \frac{1}{\pi^2\sqrt{(v-a)(b-v)}}\left[\fint_a^b \mathrm{d}t \frac{\sqrt{(t-a)(b-t)}}{v-t} f(t) + \pi c\right], \quad c = \int_a^b \mathrm{d}t\, \rho(v)$$

By using this general formula reconstruct the solution of the one-loop Bethe equation

$$0 = 2\pi\epsilon(v) - 4\fint_{-a}^{a} \mathrm{d}v'\, \frac{\rho(v')}{v-v'}$$

Bethe-Yang equations for sl(2)-sector at all loops

Recall

$$\left(\frac{u_k+i}{u_k-i}\right)^J\left(\frac{1+\frac{1}{x_k^{-2}}}{1+\frac{1}{x_k^{+2}}}\right)^J=\prod_{l\neq k}^{S}\frac{u_k-u_l-2i}{u_k-u_l+2i}\left(\frac{1-\frac{1}{x_k^+x_l^-}}{1-\frac{1}{x_k^-x_l^+}}\sigma_{kl}\right)^2$$

dressing factor

We change the choice of the branches of the logarithm by means of the formulae

$$\log\frac{u+i}{u-i}=i\pi-2i\arctan u\,,\qquad u>0\,,$$
$$\log\frac{u+i}{u-i}=-i\pi-2i\arctan u\,,\quad u<0\,,$$

and relabel k so it runs the set $\{-S/2,\dots,-1,1,\dots,S/2\}$

Taking the logarithm of the Bethe equations and multiplying by i yields

$\tilde n_k=k+\frac{J-3}{2}\epsilon(k)$

for the lowest state

$$2J\arctan(u_k)+iJ\log\left(\frac{1+\frac{1}{x_k^{-2}}}{1+\frac{1}{x_k^{+2}}}\right)\;=\;2\pi\tilde n_k-2\sum_{\substack{j=-S/2\\ j\neq 0}}^{S/2}\arctan\tfrac{1}{2}(u_k-u_l)$$
$$+2i\sum_{\substack{j=-S/2\\ j\neq 0}}^{S/2}\log\left(\frac{1-\frac{1}{x_k^+x_l^-}}{1-\frac{1}{x_k^-x_l^+}}\sigma_{kl}\right)$$

$\tilde n_k=k'+\frac{J-2}{2}\epsilon(k)$ for $k'=\pm\frac{1}{2},\pm\frac{3}{2},\dots,\pm\frac{S-1}{2}$

Bethe-Yang equations for sl(2)-sector at all loops

After replacing $k' \to k$

$$2J\arctan(u_k) + iJ\log\left(\frac{1+\frac{1}{x_k^{-2}}}{1+\frac{1}{x_k^{+2}}}\right) = 2\pi k + \pi(J-2)\epsilon(k) - 2\sum_{\substack{j=-S/2\\ j\neq 0}}^{S/2} \arctan\tfrac{1}{2}(u_k - u_l)$$
$$+ 2i\sum_{\substack{j=-S/2\\ j\neq 0}}^{S/2} \log\left(\frac{1-\frac{1}{x_k^+ x_l^-}}{1-\frac{1}{x_k^- x_l^+}}\sigma_{kl}\right),$$

Introduce the so-called counting function

$$\eta(u_k) = \frac{k}{S}$$

the quantum number of this solution

solution of the Bethe equations

η is monotonically increasing, in the limit $S \to \infty$ the variable $x = k/S$ becomes continuous

$$u_k \longrightarrow u(x) \quad \Longrightarrow \quad x = x(u)$$

Introduce the density as derivative of the counting function

$$\rho(u) = \lim_{S\to\infty} \frac{\eta(u_{k+1}) - \eta(u_k)}{u_{k+1} - u_k} = \frac{d\eta(u)}{du} = \frac{dx}{du}$$

Bethe-Yang equations for sl(2)-sector at all loops

Divide Bethe equation by S and take the limit $S \to \infty$

$$\frac{2J}{S}\frac{1}{u^2+1} + \frac{iJ}{S}\frac{d}{du}\log\cancel{\left(\frac{1+\frac{1}{x^{-2}(u)}}{1+\frac{1}{x^{+2}(u)}}\right)} = 2\pi\rho(u) + \frac{2\pi}{S}(J-2)\delta(u) - 4\int_{-b}^{b} \mathrm{d}v\,\frac{\rho(v)}{(u-v)^2+4}$$

$$+\,2i\int_{-b}^{b}\mathrm{d}v\,\rho(v)\,\frac{d}{du}\log\left(\frac{1-\frac{1}{x^{+}(u)x^{-}(v)}}{1-\frac{1}{x^{-}(u)x^{+}(v)}}\,\sigma(u,v)\right)$$

$$\rho(u) = \underset{\text{one-loop}}{\underbrace{\rho_0(u)}} + g^2\underset{\text{higher-loop}}{\underbrace{\tilde{\sigma}(u)}}$$

The one-loop contribution splits off

$$\frac{2J}{S}\frac{1}{u^2+1} = 2\pi\rho_0(u) + \frac{2\pi}{S}(J-2)\delta(u) - 4\int_{-S/2}^{S/2}\mathrm{d}v\,\frac{\rho_0(v)}{(u-v)^2+4}$$

In the limit $S \to \infty$ this equation turns into the derivative of the one-loop equation

Exercise check that this is the case

For the fluctuation density one has

$$
\begin{aligned}
0 \;=\; & 2\pi\tilde{\sigma}(u) - 4\int_{-\infty}^{+\infty} \mathrm{d}v\, \frac{\tilde{\sigma}(v)}{(u-v)^2+4} \\
& \underbrace{+\,\frac{2i}{g^2}\int_{-S/2}^{S/2} \mathrm{d}v\, \rho_0(v)\, \frac{d}{du}\log\left(\frac{1-\frac{1}{x^+(u)x^-(v)}}{1-\frac{1}{x^-(u)x^+(v)}}\,\sigma(u,v)\right)}_{=Q} \\
& +\,2i\int_{-\infty}^{\infty} \mathrm{d}v\, \tilde{\sigma}(v)\, \frac{d}{du}\log\left(\frac{1-\frac{1}{x^+(u)x^-(v)}}{1-\frac{1}{x^-(u)x^+(v)}}\,\sigma(u,v)\right)
\end{aligned}
$$

BES dressing phase
starts to contribute at four loops

sets up logarithmic scale

$$
Q \to \frac{2}{g}\frac{\log S}{S}\frac{d}{du}\left[\frac{1}{x^+(u)} + \frac{1}{x^-(u)}\right] - \frac{1}{g^2}\int_{-1}^{1} \mathrm{d}\bar{v}\, \bar{\rho}_0(\bar{v})\, \frac{d}{du}\theta(u, \tfrac{2}{S}\bar{v})
$$

The equation can be solved perturbatively in g^2 by the Fourier transform

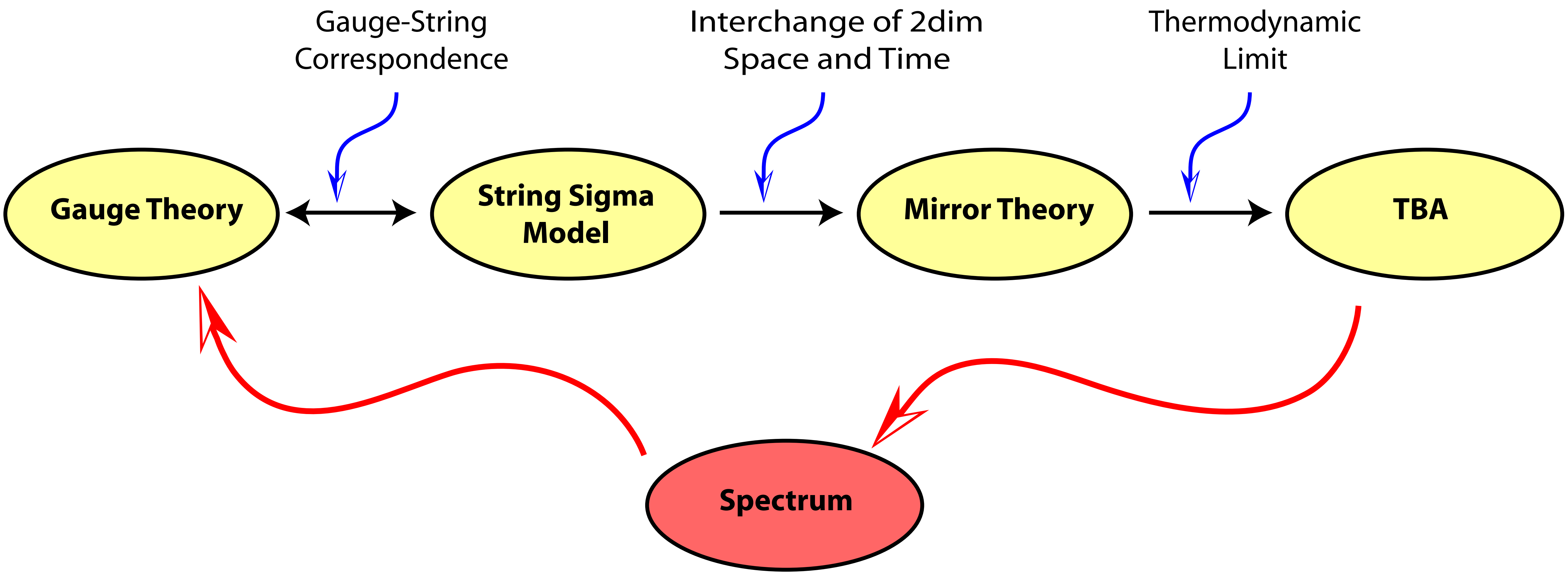
Energy in the large spin limit $E = S + f(g)\log S + \mathcal{O}(S^0)$

$$
\begin{aligned}
f(g) \;=\; & 2g^2 - \frac{1}{6}\pi^2 g^4 + \frac{11}{360}\pi^4 g^6 - \frac{1}{16}\left(\frac{73}{630}\pi^6 + 4\zeta(3)^2\right)g^8 \\
& + \frac{1}{32}\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\zeta(3)^2 + 40\zeta(3)\zeta(5)\right)g^{10} + \ldots
\end{aligned}
$$

[Beisert, Eden & Staudacher 2006]

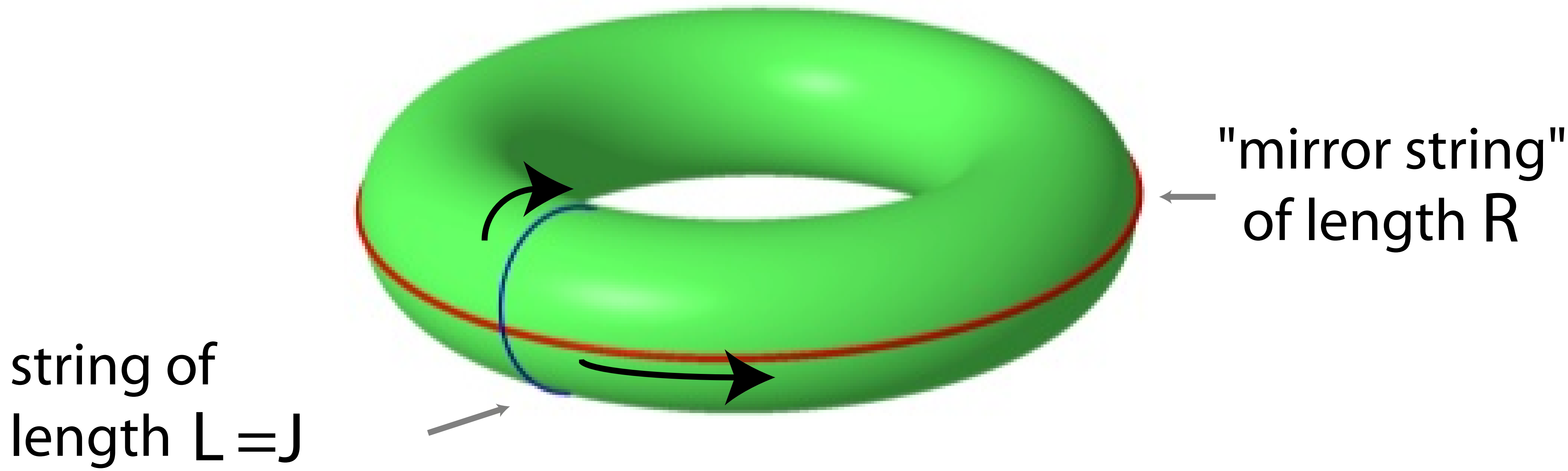
Mirror theory

The main idea



The mirror TBA - a tool to solve the spectral problem of AdS/CFT

Mirror theory



- One Euclidean theory – two Minkowski theories. One is related to the other by the double Wick rotation:

$$\tilde{\sigma} = -i\tau\,, \qquad \tilde{\tau} = i\sigma$$

The Hamiltonian $\tilde{\mathrm{H}}$ w.r.t. $\tilde{\tau}$ defines the *mirror theory*.

[Frolov & G.A. 2007]

Mirror theory

$$\textit{string part. func.}\quad Z(R,L) \equiv \sum_n \langle \psi_n | e^{-HR} | \psi_n \rangle = \sum_n e^{-E_n R}$$

$$\textit{mirror part. func.}\quad \widetilde{Z}(L,R) \equiv \sum_n \langle \widetilde{\psi}_n | e^{-\widetilde{H}L} | \widetilde{\psi}_n \rangle$$

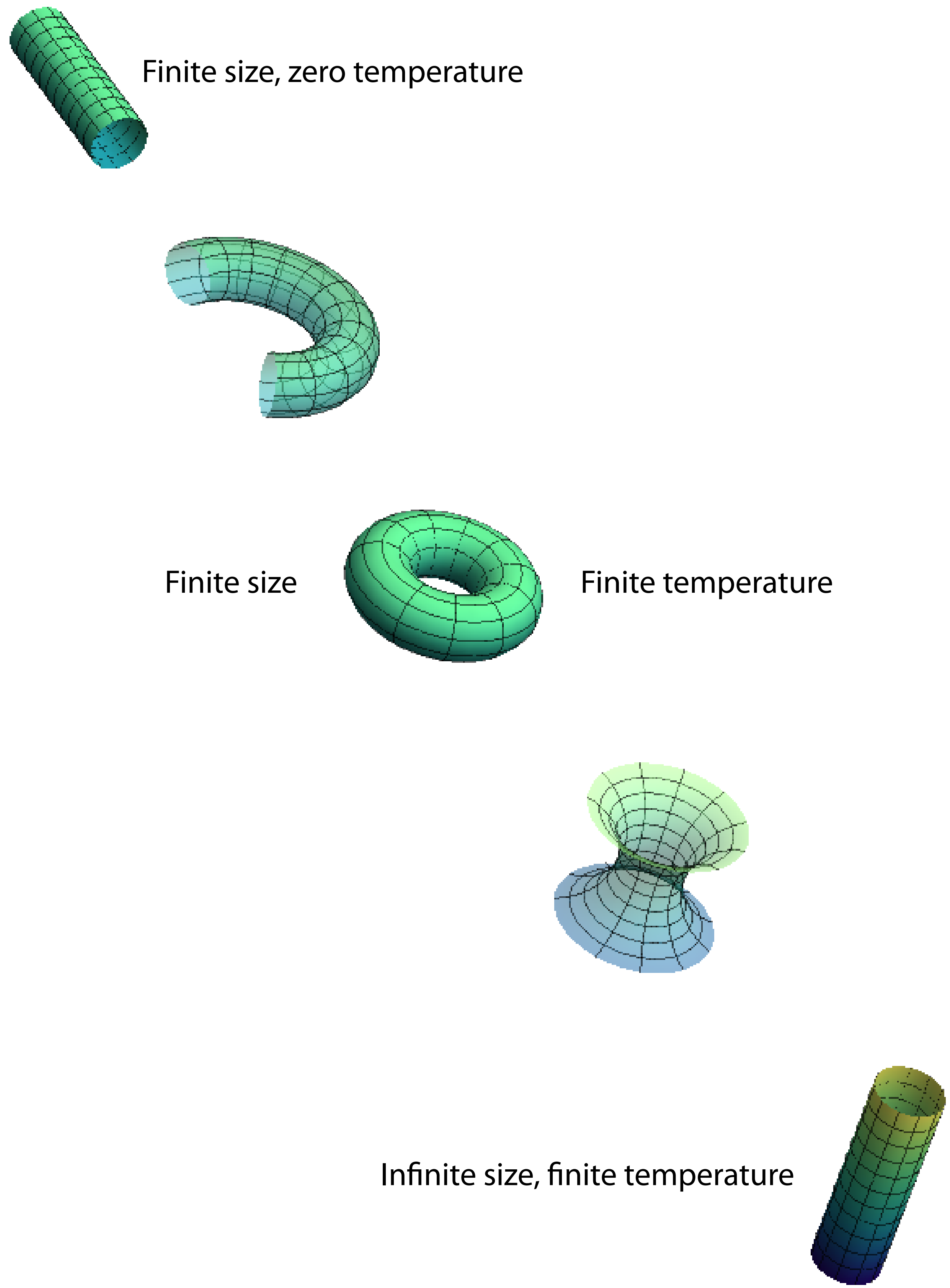
$1/T_{\mathrm{mirror}}$

$$\boxed{Z(R,L) = \widetilde{Z}(L,R)}$$

$1/T$ size size

- When $R \to \infty$ one gets $\log Z(R,L) \sim -RE(L)$, where $E(L)$ is the ground state energy
- $\log \widetilde{Z}(R,L) = -LF(L)$, where $F(L)$ is the free energy of the mirror theory at the temperature $T = 1/L$
- Ground state energy is related to the free energy of its mirror

Mirror theory



Example: 2dim relativistic QFT

Minkowski dispersion relation is

$$H^2 - p^2 = m^2 \qquad\qquad p_0 = H$$

The Wick rotation to the Euclidean theory is

$$H = iH_E\,, \quad p = p_E$$

The dispersion relation of the Euclidean theory

$$H_E^2 + p_E^2 + m^2 = 0$$

The passage to the mirror theory is performed by declaring that

$$H_E = \widetilde{p}\,, \qquad p_E = i\widetilde{H}$$

inverse Wick rotation

We have

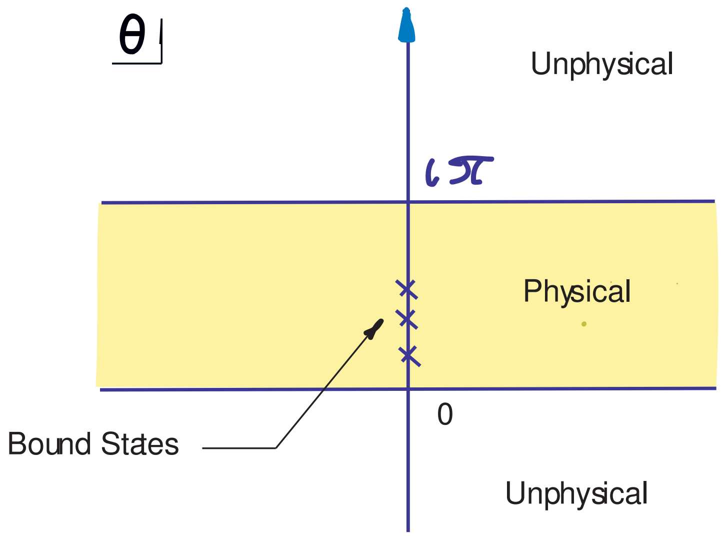
$$\widetilde{p}^2 - \widetilde{H}^2 + m^2 = 0 \quad \Longrightarrow \quad \widetilde{H}^2 - \widetilde{p}^2 = m^2$$

The dispersion relation of a relativistic two-dimensional theory and of its mirror has the one and the same form. The passage to the mirror theory can be considered as the analytic continuation

$$p \to i\sqrt{\widetilde{p}^2 + m^2}\,, \qquad H \to i\widetilde{p}$$

Example: 2dim relativistic QFT

The dispersion relation of the original theory is uniformised on the sphere by means of a rapidity variable θ



$$H = m\cosh\theta\,, \qquad p = m\sinh\theta$$

where for real H, p the range of θ is $-\infty < \theta < +\infty$

For the mirror theory

$$\widetilde{H} = \frac{1}{i} m\sinh\theta\,, \qquad \widetilde{p} = \frac{1}{i} m\cosh\theta$$

Allow for complex θ and consider the shift

$$\theta = \boldsymbol{\theta} + \frac{i\pi}{2}$$

where $\boldsymbol{\theta}$ on the right hand side is real, then

$$\widetilde{H} = \frac{1}{i} m\sinh(\boldsymbol{\theta} + \tfrac{i\pi}{2}) = m\cosh\boldsymbol{\theta}\,, \qquad \widetilde{p} = \frac{1}{i} m\cosh(\boldsymbol{\theta} + \tfrac{i\pi}{2}) = m\sinh\boldsymbol{\theta}$$

Thus, transition to the mirror theory corresponds to the shift of the real rapidity variable of the original theory by the *quarter of imaginary period*: $\frac{i\pi}{2} = \frac{2\pi i}{4}$.

Dispersion of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror theory

$$H^2 = 1 + 4g^2 \sin^2 \frac{p}{2}$$

passage to Eucllidean theory

$H = iH_E,\ p = p_E$

$$-H_E^2 = 1 + 4g^2 \sin^2 \frac{p_E}{2}$$

passage to mirror theory

$H_E = \widetilde{p}\,, \qquad p_E = i\widetilde{H}$

$$-\widetilde{p}^2 = 1 - 4g^2 \sinh^2 \frac{\widetilde{H}}{2}$$

$$\boxed{\widetilde{H} = 2\mathrm{arcsinh}\frac{1}{2g}\sqrt{1+\widetilde{p}^2}}$$

Thus, the passage from the original (Minkowski) theory to the mirror (Minkowski) theory can be considered as the following analytic continuation of momentum

$$p \to 2i\,\mathrm{arcsinh}\frac{1}{2g}\sqrt{1+\widetilde{p}^2}$$

Some notations

- J – momentum carried by string along the equator of $\mathbb{S}^5$,
  L – “length” (will be related to J)

- p – momentum of a string particle

- $\mathcal{E}$ – energy of a string particle: $\mathcal{E} = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$

- $\tilde{p}$ – momentum of a mirror particle

- $\tilde{\mathcal{E}}$ – energy of a mirror particle: $\tilde{\mathcal{E}} = 2\,\mathrm{arcsinh}\left(\frac{1}{2g}\sqrt{1 + \tilde{p}^2}\right)$

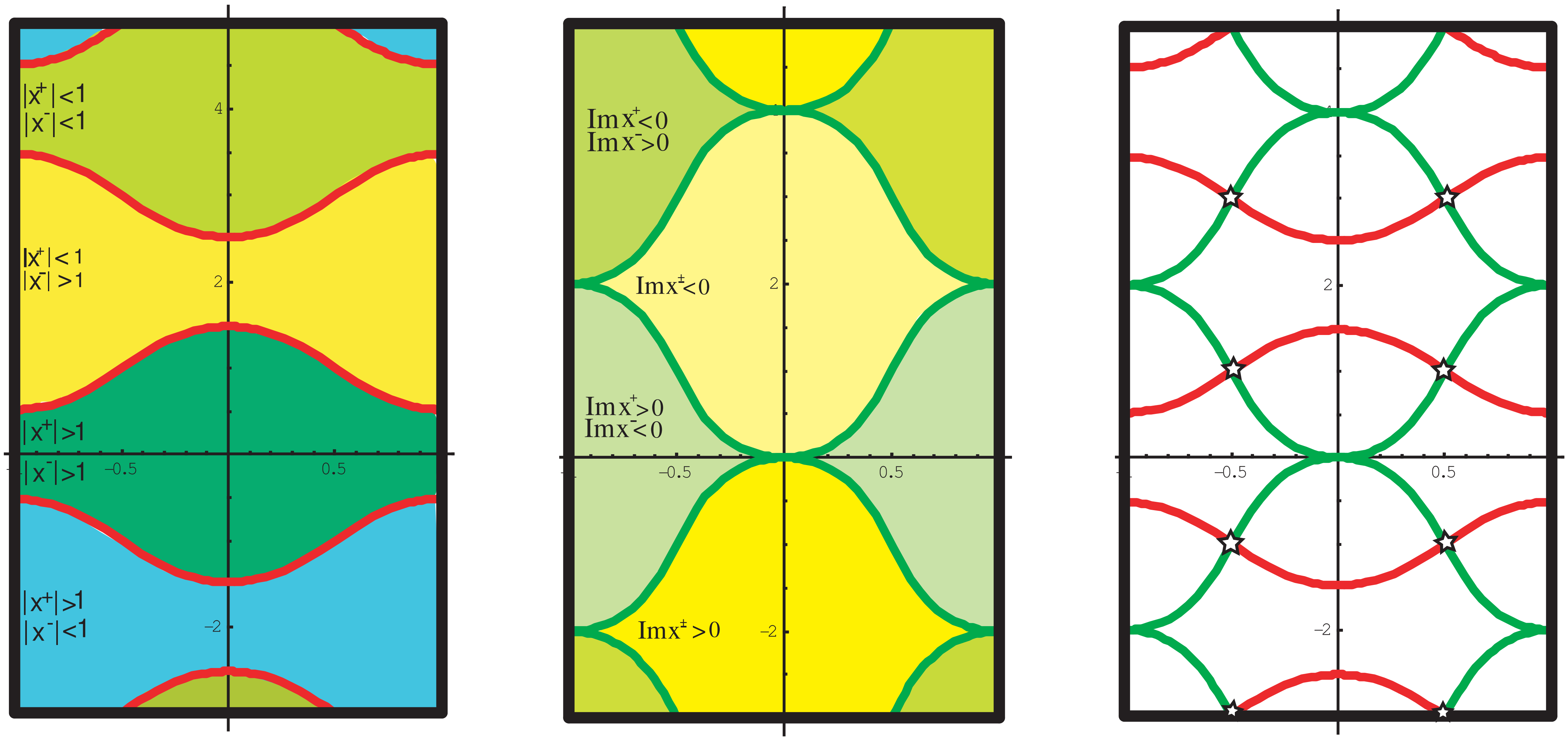
- String S-matrix $S(p_1, p_2)$

- Mirror S-matrix $\tilde{S}(\tilde{p}_1, \tilde{p}_2)$

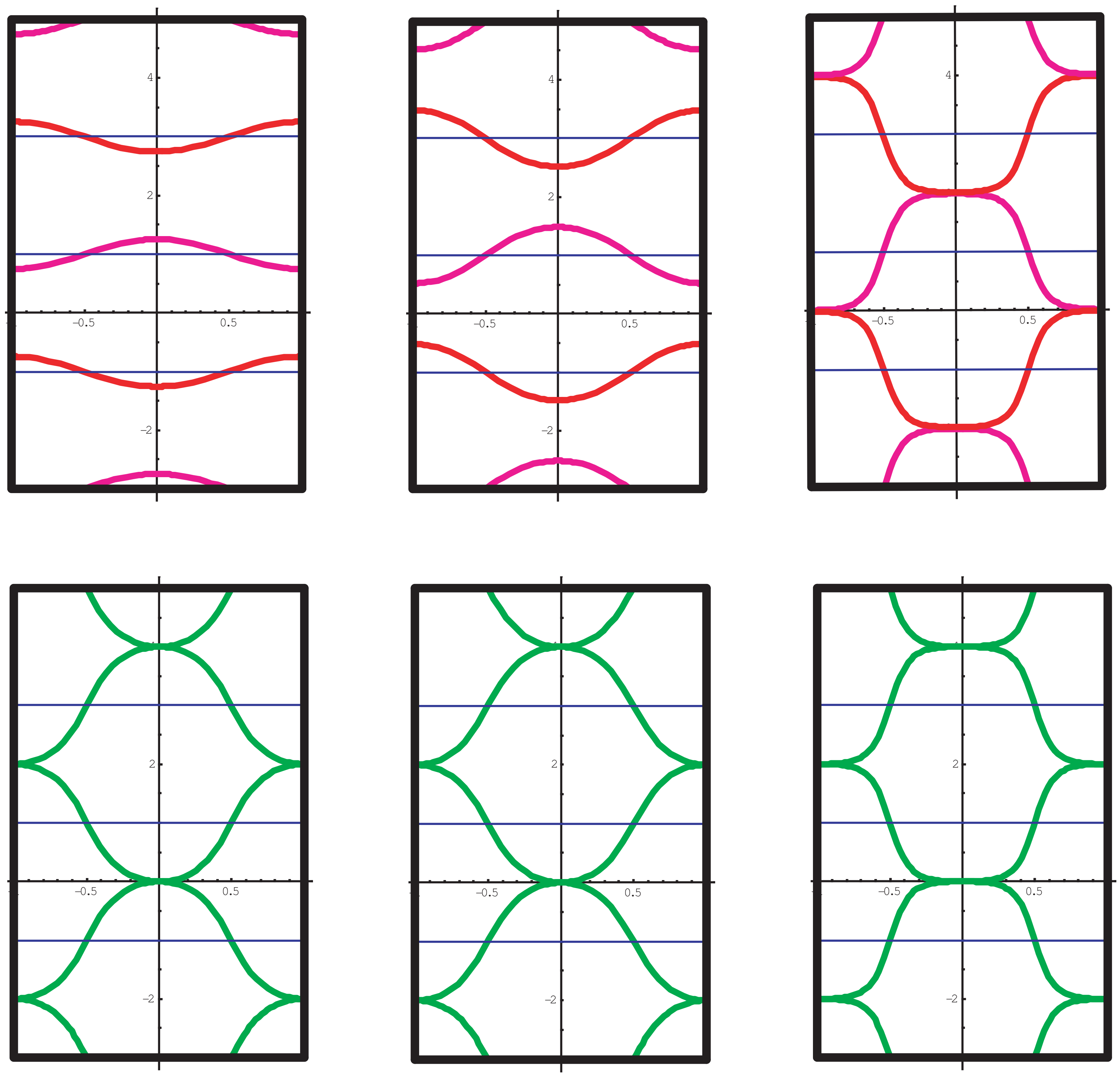
S-matrix on z-torus

$$
\begin{aligned}
S(z_1,z_2) \;=\;& \Big(E^1_1\otimes E^1_1+E^2_1\otimes E^1_2+E^1_2\otimes E^2_1+E^2_2\otimes E^2_2\Big)\\
&+\frac{(x_1^- - x_2^-)(x_1^+x_2^- - 1)x_2^+}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)x_2^-}\Big(E^1_1\otimes E^2_2-E^2_1\otimes E^1_2-E^1_2\otimes E^2_1+E^2_2\otimes E^1_1\Big)\\
&-\frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\,\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2}\Big(E^3_3\otimes E^3_3+E^4_3\otimes E^3_4+E^3_4\otimes E^4_3+E^4_4\otimes E^4_4\Big)\\
&-\frac{(x_1^- - x_2^-)(x_1^-x_2^+ - 1)x_1^+}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)x_1^-}\,\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2}\Big(E^3_3\otimes E^4_4-E^4_3\otimes E^3_4-E^3_4\otimes E^4_3+E^4_4\otimes E^3_3\Big)\\
&+\frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,\frac{\tilde\eta_2}{\eta_2}\Big(E^1_1\otimes E^3_3+E^1_1\otimes E^4_4+E^2_2\otimes E^3_3+E^2_2\otimes E^4_4\Big)\\
&+\frac{x_1^- - x_2^-}{x_1^- - x_2^+}\,\frac{\tilde\eta_1}{\eta_1}\Big(E^3_3\otimes E^1_1+E^3_3\otimes E^2_2+E^4_4\otimes E^1_1+E^4_4\otimes E^2_2\Big)\\
&-\frac{i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- - x_2^-)}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)\eta_1\eta_2}\,\frac{x_1^+x_2^+}{x_1^-x_2^-}\Big(E^3_1\otimes E^4_2-E^4_1\otimes E^3_2-E^3_2\otimes E^4_1+E^4_2\otimes E^3_1\Big)\\
&-\frac{i(x_1^- - x_2^-)\tilde\eta_1\tilde\eta_2}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)}\Big(E^1_3\otimes E^2_4-E^2_3\otimes E^1_4-E^1_4\otimes E^2_3+E^2_4\otimes E^1_3\Big)\\
&+\frac{x_1^- - x_1^+}{x_1^- - x_2^+}\,\frac{\tilde\eta_2}{\eta_1}\Big(E^3_1\otimes E^1_3+E^4_1\otimes E^1_4+E^3_2\otimes E^2_3+E^4_2\otimes E^2_4\Big)\\
&+\frac{x_2^- - x_2^+}{x_1^- - x_2^+}\,\frac{\tilde\eta_1}{\eta_2}\Big(E^1_3\otimes E^3_1+E^2_3\otimes E^3_2+E^1_4\otimes E^4_1+E^2_4\otimes E^4_2\Big)
\end{aligned}
$$

Foliation of z-torus



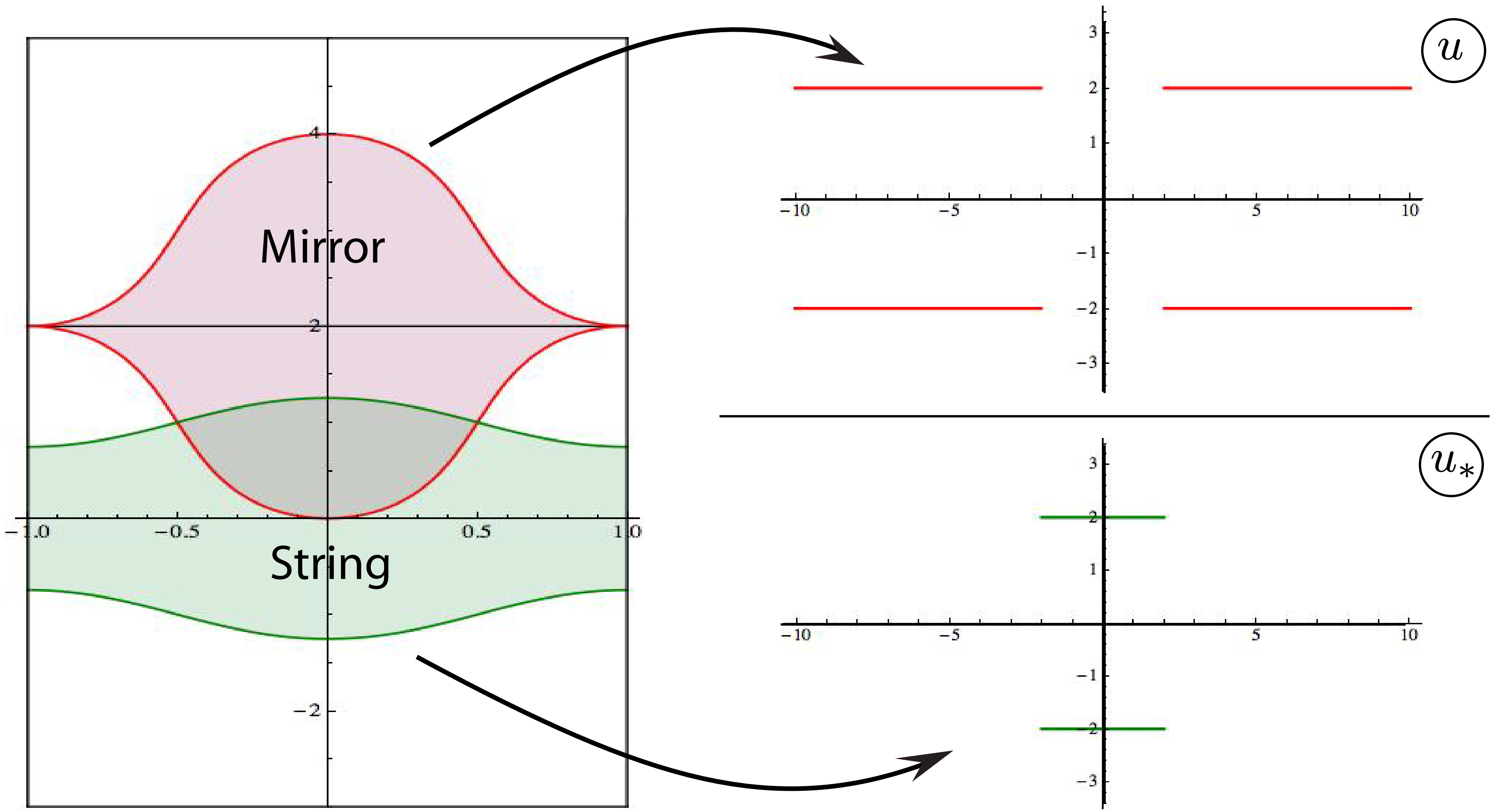
$$x = \mathrm{Re}(\frac{2}{\omega_1}z)\,, \quad y = \mathrm{Re}(\frac{4}{\omega_2}z)$$



g=1/2 g=1 g=50

String and mirror regions on z-torus

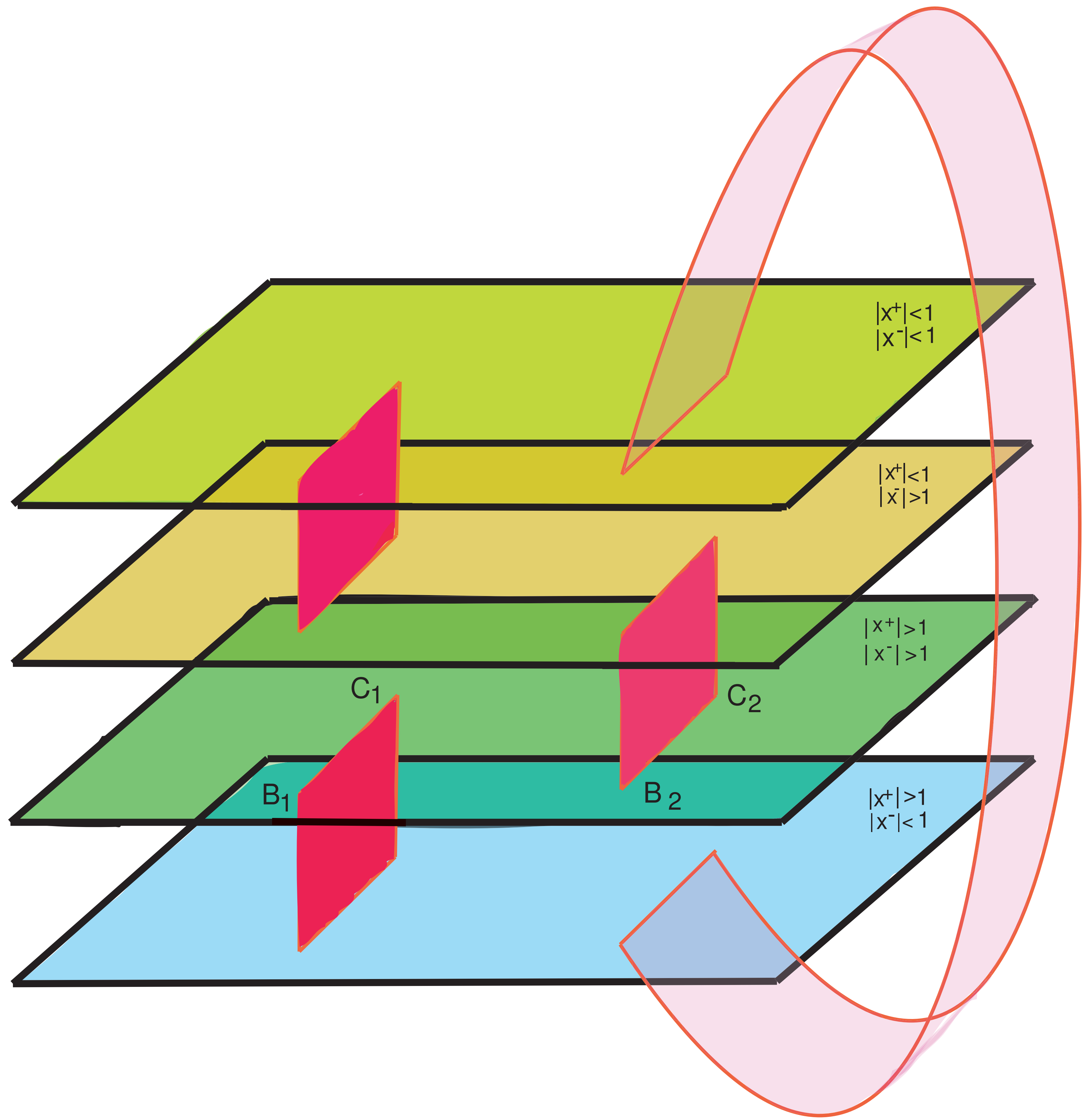
$$x(u) = \frac{1}{2}(u - i\sqrt{4-u^2}), \quad \mathrm{Im}x(u) < 0$$



$$x_s(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4}{u^2}}\right) \quad |x_s(u)| \geq 1$$

Mirror and string regions on the z-torus. The boundaries are mapped onto the cuts on the u- and u_*-planes, respectively

Gluing torus out of four planes



Derivation of the TBA equation

- <u>Step 1:</u> *Construct the mirror Bethe equations for fundamental particles. Apply the Wick rotation to the world-sheet S-matrix:*

$$S(p_1, p_2) \to \tilde{S}(\tilde{p}_1, \tilde{p}_2)$$

$$1 = e^{i\tilde{p}_k R} \prod_{j \neq k} \tilde{S}(\tilde{p}_k, \tilde{p}_j)$$

- <u>Step 2:</u> *Identify configurations of Bethe roots which contribute in the thermodynamic limit* $R \to \infty$

  This is known as "the string hypothesis", but a priory it has nothing to do with string theory!

- <u>Step 3:</u> *Construct TBA equations for excited states*

Bethe-Yang equations for mirror particles

$$1 = e^{i\widetilde{p}_k R} \prod_{\substack{l=1 \\ l\neq k}}^{K^{\mathrm{I}}} S(\tilde{p}_k, \tilde{p}_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}}$$

$$-1 = \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}$$

$$1 = \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1 \\ l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}$$

follow from $\tilde{S}(\tilde{p}_1, \tilde{p}_2)$

Auxiliary roots – w^{α}, y^{α}; $\quad v = y + 1/y$

Bound states

In terms of particle rapidities u_1 and u_2 the S-matrix involves

$$S(\tilde{p}_1, \tilde{p}_2) \sim \frac{u_1 - u_2 + \frac{2i}{g}}{u_1 - u_2 - \frac{2i}{g}}$$

which has a pole at $u_1 - u_2 - \frac{2i}{g} = 0$

In general, a multi-particle S-matrix will have poles at

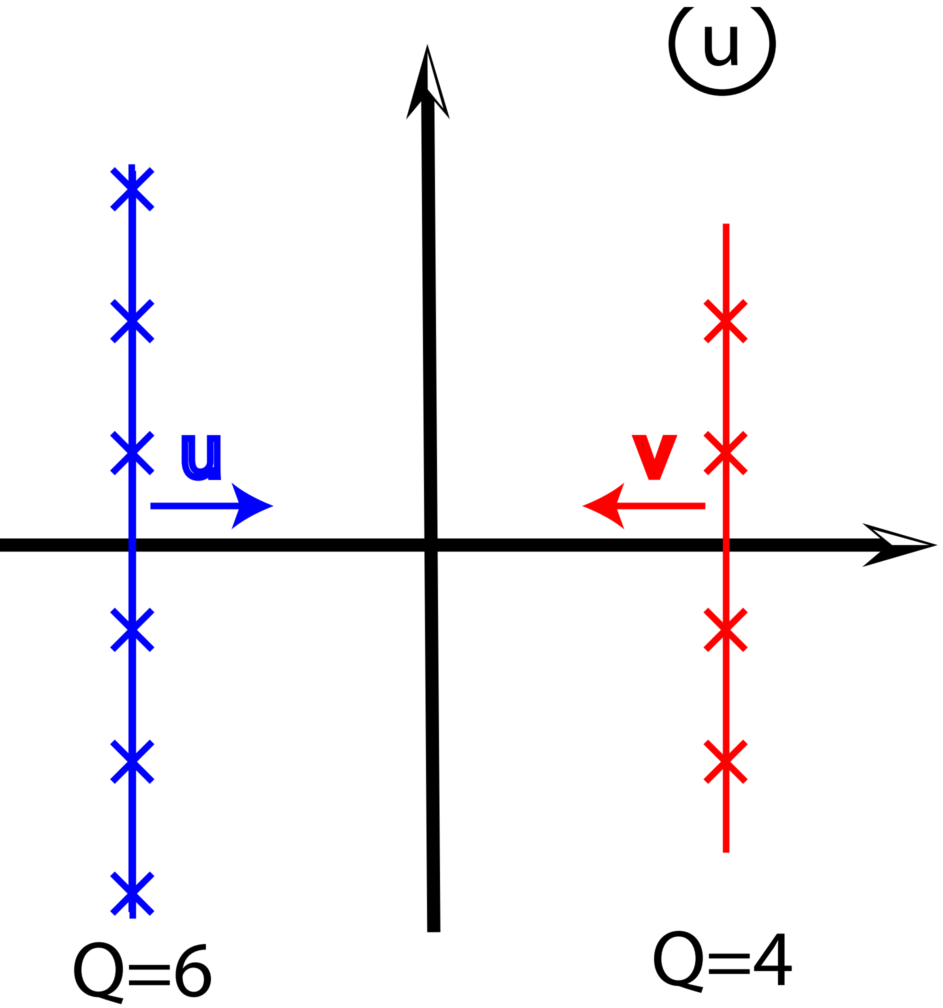
$$u_j - u_{j+1} - \frac{2i}{g} = 0\,, \quad i, j = 1, \ldots, Q-1$$

which gives a pattern of a "Bethe string"

$$u_j = u + (Q + 1 - 2j)\frac{i}{g}\,, \quad j = 1, \ldots, Q, \quad u \in \mathbf{R}$$

regarded as the Q-particle bound state.

Bethe strings



"Bethe strings" $\longrightarrow$ Quasi-particles with real rapidities

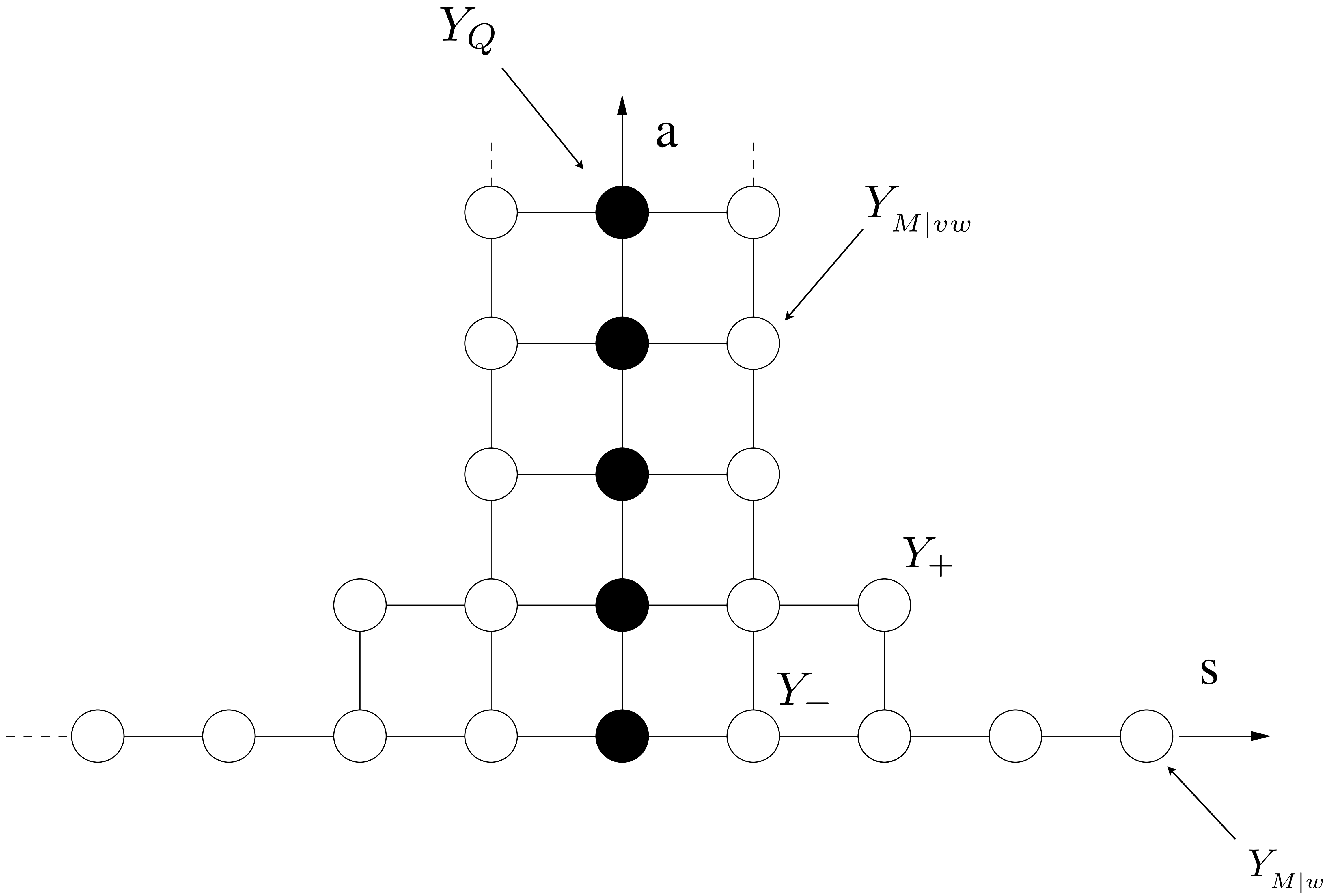
$$u_j = u + (Q + 1 - 2j)\frac{i}{g}\,, \quad j = 1, \ldots, Q, \quad u \in \mathbf{R}$$

Auxiliary roots $v = y + 1/y$ and w participate in building up Bethe strings!

String hypothesis suggests the existence of
nine types of TBA vacuum particles ($\alpha = 1, 2$):

- Q-particles (Q-particle bound states) carrying momentum $\tilde{p}_Q \implies Y_Q^{(\alpha)}(u)$
- $y^{\pm(\alpha)}$-particles corresponding to fermionic Bethe roots $\implies Y_{\pm}^{(\alpha)}(u), \quad |u| < 2$
- $M|vw^{(\alpha)}$-strings $\implies Y_{M|vw}^{(\alpha)}(u)$
- $M|w^{(\alpha)}$-strings $\implies Y_{M|w}^{(\alpha)}(u)$

Y-functions of the mirror TBA



Mirror TBA for the ground state

- $M|w$-strings: $\quad \log Y^{(\alpha)}_{M|w} = \log(1+Y^{(\alpha)}_{M-1|w})(1+Y^{(\alpha)}_{M+1|w})\star s + \delta_{M1}\,\log\frac{1-\frac{1}{Y^{(\alpha)}_{-}}}{1-\frac{1}{Y^{(\alpha)}_{+}}}\,\hat{\star}\, s$

- $M|vw$-strings:

$$\log Y^{(\alpha)}_{M|vw} = \log(1+Y^{(\alpha)}_{M-1|vw})(1+Y^{(\alpha)}_{M+1|vw})\star s - \log(1+Y_{M+1})\star s + \delta_{M1}\log\frac{1-Y^{(\alpha)}_{-}}{1-Y^{(\alpha)}_{+}}\,\hat{\star}\, s$$

- y-particles $\quad \log\frac{Y^{(\alpha)}_{+}}{Y^{(\alpha)}_{-}} = \log(1+Y_Q)\star K_{Qy}\,,$

$$\log Y^{(\alpha)}_{+}Y^{(\alpha)}_{-} = \log\,(1+Y_Q)\star(-K_Q+2K^{Q1}_{xv}\star s) + 2\log\frac{1+Y_{1|vw}}{1+Y_{1|w}}\star s$$

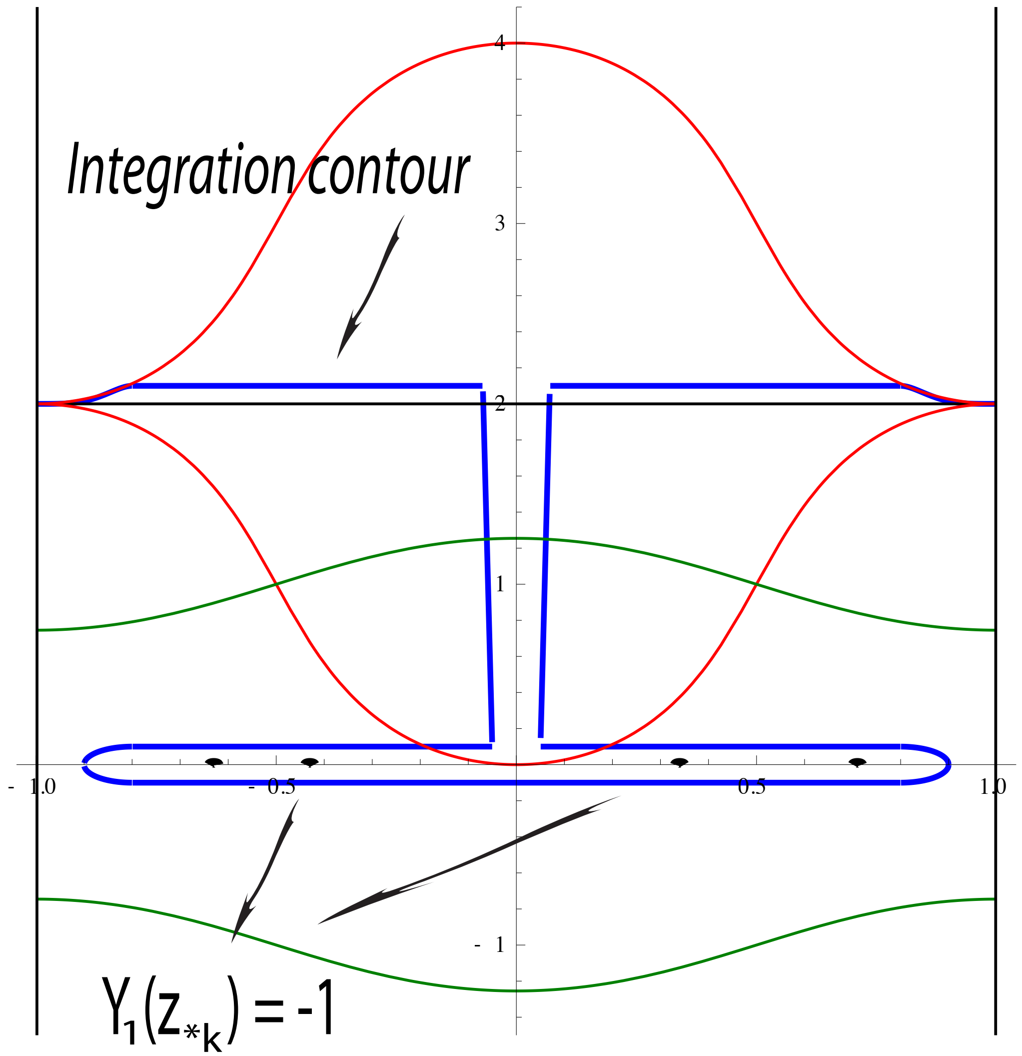
- Q-particles for $Q\geq 2$ $\quad \log Y_Q = \log\frac{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)}{(1+\frac{1}{Y_{Q-1}})(1+\frac{1}{Y_{Q+1}})}\star s$

- $Q=1$-particle $\quad \log Y_1 = \log\frac{\left(1-\frac{1}{Y^{(1)}_{-}}\right)\left(1-\frac{1}{Y^{(2)}_{-}}\right)}{1+\frac{1}{Y_2}}\star s - \Delta(L)\,\check{\star}\, s\,, \qquad s(u) = \frac{g}{4\cosh\frac{g\pi u}{2}}$

$$E(L) = J - \frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{d\tilde{p}^Q}{du}\log(1+Y_Q)$$

Frolov and G.A. '09(b)

Excited states



- **TBA's for excited states differ only by a choice of the integration contour**

- Taking the contour back to the real mirror line produces extra contributions $-\log S(z_*, z)$ from $\log(1+Y_1)\star K$, where $K(w,z)=\frac{1}{2\pi i}\frac{d}{dw}\log S(w,z)$

Large J (asymptotic) solution

$L \to \infty$: Bethe-Yang (all $1/L$ powers) + Lüscher corrections (leading e^{-mL} corrections)

(standing 1-particle states) Lüscher '86

(general N-particle states) Bajnok, Janik '08

$$Y^o_Q(v) \;=\; \Upsilon_Q(v)\, T_{Q,-1}(v)\, T_{Q,1}(v)$$

- Transfer matrix

$$T_{Q,1}(u) = \mathrm{Tr}_Q \Big[S_{Q,1}(u,u_1) \ldots S_{Q,N}(u,u_N) \Big]$$

- The prefactor

$$\Upsilon^+_Q \Upsilon^-_Q = \Upsilon_{Q-1} \Upsilon_{Q+1}\,, \qquad \Upsilon_Q(v) \sim e^{-J\widetilde{\mathcal{E}}_Q(v)}$$

- Bethe-Yang equations are equivalent to

$$Y^o_{1_*}(u_k) = -1\,, \qquad k = 1, \ldots, N$$

Excited states TBA

1. Solve the BY equations for a fixed set of integers

$$J\,,\qquad N=K^{\mathrm{I}}\,,\qquad (K_-^{\mathrm{III}},\,K_-^{\mathrm{II}},\,K_+^{\mathrm{II}},\,K_+^{\mathrm{III}})$$

   Pick up a solution. It is characterized by a definite set of g-dependent momenta.

   Auxiliary roots are completely fixed by the momenta p_k and play no independent role in the description of a state

2. Compute asymptotic Y-functions and find zeroes and poles of $1+Y$ and Y

3. Choose contours and engineer TBA equations for the state so that they reproduce the asymp. solution

4. Exact momenta p_k are found from the *exact Bethe equations* (quantization cond.)

$$Y_{1_*}^0(p_k)=-1\qquad\Longrightarrow\qquad Y_{1_*}(p_k)=-1$$

5. Energy spectrum: $$E=J+\underbrace{\sum_{i=1}^{N}\mathcal{E}(p_i)}_{\mathrm{Bethe-Yang}}-\underbrace{\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\frac{d\tilde{p}^Q}{du}\log(1+Y_Q)}_{\mathrm{finite-size\quad corr.}}$$