Uniform light-cone gauge

The main idea is to fix the gauge and obtain 2dim massive integrable QFT

Uniform light-cone gauge

Polyakov action $$S = -\frac{g}{2}\int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}$$ $$p_M = \frac{\delta S}{\delta \dot{X}^M} = -g\gamma^{\tau\beta}\partial_\beta X^N G_{MN}$$

Action in the first-order formalism $$S = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \left(p_M \dot{X}^M + \frac{1}{2g\gamma^{\tau\tau}} C_1 + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_2\right)$$

$$C_1 = G^{MN} p_M p_N + g^2 X'^M X'^N G_{MN}\,, \qquad C_2 = p_M X'^M \qquad \longleftarrow$$ Virasoro constraints

$$ds^2 = \underbrace{f_{\mathrm{a}}(z)\mathrm{d}t^2 + g_{\mathrm{a}}(z)\mathrm{d}z_i^2}_{\mathrm{AdS}_5} + \underbrace{f_{\mathrm{s}}(y)\mathrm{d}\phi^2 + g_{\mathrm{s}}(y)\mathrm{d}y_i^2}_{\mathrm{S}^5}$$

Isometries $$t \to t + \mathrm{const} \qquad \phi \to \phi + \mathrm{const}$$

$$\downarrow \qquad\qquad \downarrow$$

Noether's charges $$E = -\int_{-r}^{r} \mathrm{d}\sigma\, p_t \ , \qquad J = \int_{-r}^{r} \mathrm{d}\sigma\, p_\phi$$

Uniform light-cone gauge

Introduce the light-cone coordinates

$$\begin{aligned} x_- &= \phi - t\,, & x_+ &= (1-a)t + a\phi \\ p_- &= p_\phi + p_t\,, & p_+ &= (1-a)p_\phi - a p_t \end{aligned}$$

parameter

$$S = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \left(p_- \dot{x}_+ + p_+ \dot{x}_- + p_\mu \dot{x}^\mu + \frac{1}{2g\,\gamma^{\tau\tau}} C_1 + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_2 \right)$$

no a

Uniform light – cone gauge $\quad x_+ = \tau\,, \qquad p_+ = 1$

light-cone momentum is uniformly distributed along the string

$$C_2 = x'_- + p_\mu x'^\mu = 0 \quad \Longrightarrow \quad x'_- = -\, p_\mu x'^\mu$$

$$C_1 \quad \Longrightarrow \quad p_- = p_-(p_\mu, x^\mu, x'^\mu)$$

$$\Downarrow$$

$$S = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \, (p_\mu \dot{x}^\mu - \mathcal{H})\,, \qquad \mathcal{H} = -p_-(p_\mu, x^\mu, x'^\mu)$$

Uniform light-cone gauge

$$\int_{-r}^{r} \mathrm{d}\sigma\, p_- = J - E\,, \qquad \int_{-r}^{r} \mathrm{d}\sigma\, \underbrace{p_+}_{1} = (1-a)\, J + a\, E$$

It is convenient to choose $a = 0$ (temporal gauge)

$$\Downarrow$$

$$r = \frac{J}{2}$$

the only dependence on J

$$E - J = \int_{-J/2}^{J/2} \mathrm{d}\sigma\, \mathcal{H} = \mathrm{H}$$

2dim Hamiltonian

string energy $= \Delta$

conf. dim. of CFT operators

Massive QFT on a cylinder of circumference J

Uniform light-cone gauge

$$x'_- = -p_\mu x'^\mu$$

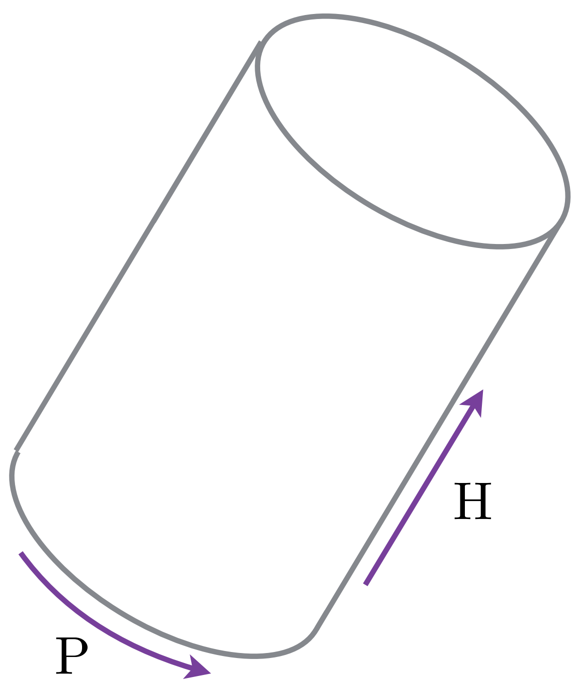
$$\Delta x_- = -\int_{-r}^{r} \mathrm{d}\sigma \, p_\mu x'^\mu = 0 \qquad \longleftarrow \qquad \text{Level – matching condition}$$

Conserved charge

$\sigma \to \sigma + a$

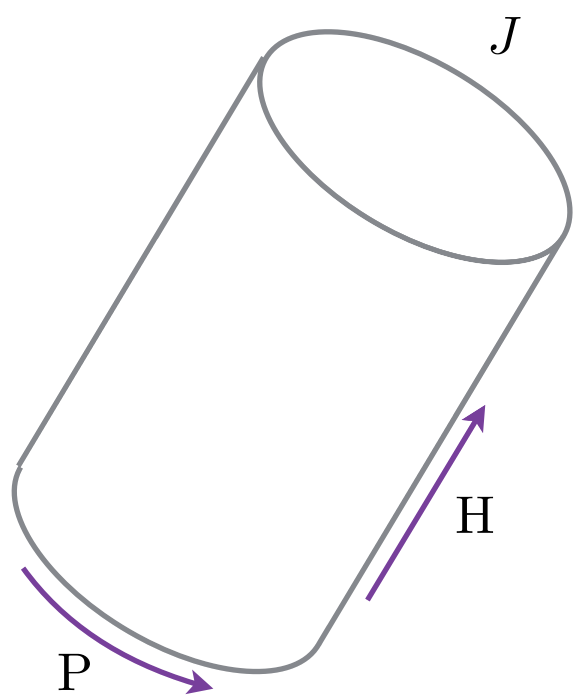
$$\mathrm{P} = -\int_{-r}^{r} \mathrm{d}\sigma \, p_\mu x'^\mu$$

$$\{\mathrm{H}, \mathrm{P}\} = 0$$



The theory with the condition $P = 0$ relaxed will be called off-shell

Decompactification limit



$$E - J = \int_{-J/2}^{J/2} \mathrm{d}\sigma\, \mathcal{H} = \mathrm{H}$$

Decompactification limit: $J \to \infty$, string tension g is kept fixed

Periodic boundary conditions for the world-sheet fields turn into vanishing boundary conditions to keep the world-sheet energy $\mathrm{H} = E - J$ finite

Light-cone symmetry algebra

Full symmetry algebra of the string sigma model is $\mathfrak{psu}(2,2|4) \supset \mathfrak{su}(2,2) \oplus \mathfrak{so}(6)$

Isometries of t and ϕ correspond to two $\mathfrak{u}(1)$'s inside $\mathfrak{su}(2,2)$ and $\mathfrak{so}(6)$, respectively

The centralizer of the $\mathfrak{u}(1)$-isometries corresponding to shifts of t and ϕ in $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$ is

$$\mathfrak{C} = \mathfrak{so}(4) \oplus \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

unbroken bosonic symmetry of the light-cone theory

The Noether charge of the symmetry algebra is 8×8 supermatrix Q

$$\mathrm{Q} = \int_{-r}^{r} \mathrm{d}\sigma\, J^\tau = g \int_{-r}^{r} \mathrm{d}\sigma\, \mathfrak{g}\Big[\gamma^{\tau\tau} A_\tau^{(2)} + \gamma^{\tau\sigma} A_\sigma^{(2)} - \frac{\kappa}{2}(A_\sigma^{(1)} - A_\sigma^{(3)})\Big]\mathfrak{g}^{-1}$$

Symmetry generators $\mathcal{M} \in \mathfrak{psu}(2,2|4)$ are obtained via projections

$$\mathrm{Q}_\mathcal{M} = \mathrm{str}(\mathrm{Q}\mathcal{M})$$

Charge conservation $$0 = \frac{d\mathrm{Q}_\mathcal{M}}{d\tau} = \frac{\partial \mathrm{Q}_\mathcal{M}}{\partial \tau} + \{\mathrm{H}, \mathrm{Q}_\mathcal{M}\}$$

Charges $\mathrm{Q}_\mathcal{M}$ independent of $x_+ = \tau$ leave H invariant

Light-cone symmetry algebra

Charges further classify as

- Kinematical – independent of x_-
- Dynamical – dependent on x_-

$$\mathcal{M} = \begin{pmatrix} \mathrm{k} & \mathrm{d} & \mathrm{d} & \mathrm{k} \\ \mathrm{d} & \mathrm{k} & \mathrm{k} & \mathrm{d} \\ \mathrm{d} & \mathrm{k} & \mathrm{k} & \mathrm{d} \\ \mathrm{k} & \mathrm{d} & \mathrm{d} & \mathrm{k} \end{pmatrix}$$

The red and blue blocks correspond to the two copies of $\mathfrak{psu}(2|2)$ subalgebra of $\mathfrak{su}(2,2|4)$ algebra.

Symmetry algebra of H in the infinite-volume limit:

$$\underbrace{\mathfrak{psu}(2|2)}_{\text{rank 2}} \oplus \underbrace{\mathfrak{psu}(2|2)}_{\text{rank 2}} \oplus \mathrm{H} \oplus \mathrm{P}$$

Light-cone symmetry algebra

$$\underbrace{\mathfrak{psu}(2|2)}_{\text{rank } 2} \oplus \underbrace{\mathfrak{psu}(2|2)}_{\text{rank } 2} \oplus \mathrm{H} \oplus \mathrm{P}$$

In the off-shell theory each of $\mathfrak{psu}(2|2)$'s picks up the one and the same central extension

$\mathbf{L}_a{}^b$, $\mathbf{R}_\alpha{}^\beta$ generate $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ bosonic subalgebra

$\mathbf{Q}_\alpha{}^a$, $\mathbf{Q}^{\dagger\,\alpha}_a$ are the supersymmetry generators

$\mathbf{H}$, $\mathbf{C}$, $\mathbf{C}^\dagger$ are three central elements

$$\underline{\mathfrak{psu}(2|2)_{(\mathrm{H},\mathrm{C},\mathrm{C}^\dagger)} = \mathfrak{su}(2|2)_{(\mathrm{C},\mathrm{C}^\dagger)} \stackrel{\text{def}}{=} \mathfrak{su}(2|2)_{\mathcal{C}}}$$

$$\begin{aligned}
\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}^{\dagger\,\beta}_b\} &= \delta^a_b \mathbf{R}_\alpha{}^\beta + \delta^\beta_\alpha \mathbf{L}_b{}^a + \frac{1}{2}\delta^a_b \delta^\beta_\alpha \mathbf{H}\,, \\
\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\,\mathbf{C}\,, \qquad \{\mathbf{Q}^{\dagger\,\alpha}_a, \mathbf{Q}^{\dagger\,\beta}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\,\mathbf{C}^\dagger
\end{aligned}$$

Central charge

vanishes on $-$ shell

$$\mathbf{C} = ig(e^{i\mathbf{P}} - 1)$$

The origin of the central charge is related to the dynamical nature (x_--dependence) of the supercharges

Light-cone symmetry algebra

Introduce a basis of the four-dimensional fundamental representation of $\mathfrak{su}(2|2)_\mathcal{C}$

$$|e_M\rangle = \begin{cases} |e_a\rangle \\ |e_\alpha\rangle \,. \end{cases}$$

$$\mathrm{L}_a{}^b|e_c\rangle = \delta_c^b|e_a\rangle - \tfrac{1}{2}\delta_a^b|e_c\rangle \qquad \mathrm{R}_\alpha{}^\beta|e_a\rangle = 0$$

$$\mathrm{L}_a{}^b|e_\alpha\rangle = 0 \qquad \mathrm{R}_\alpha{}^\beta|e_\gamma\rangle = \delta_\gamma^\beta|e_\alpha\rangle - \tfrac{1}{2}\delta_\alpha^\beta|e_\gamma\rangle$$

$$\mathrm{Q}_\alpha{}^a|e_b\rangle = a\,\delta_a^b|e_\alpha\rangle \qquad \mathrm{Q}^{\dagger}{}_a^{\alpha}|e_b\rangle = c\,\epsilon_{ab}\epsilon^{\alpha\beta}|e_\beta\rangle$$

$$\mathrm{Q}_\alpha{}^a|e_\beta\rangle = b\,\epsilon_{\alpha\beta}\epsilon^{ab}|e_b\rangle \qquad \mathrm{Q}^{\dagger}{}_a^{\alpha}|e_\beta\rangle = d\,\delta_\beta^\alpha|e_a\rangle$$

$$\longrightarrow \qquad ad - bc = 1\,, \quad d^* = a\,, \quad c^* = b$$

$$\mathrm{H}|e_M\rangle = (ad+bc)|e_M\rangle\,, \qquad \mathrm{C}|e_M\rangle = ab\,|e_M\rangle\,, \qquad \mathrm{C}^\dagger|e_M\rangle = cd\,|e_M\rangle$$

$$H^2 - 4CC^* = a^2d^2 + b^2c^2 + 2adbc - 4abcd = (ad-bc)^2 = 1\,.$$

$$CC^* = \frac{g^2}{4}(1 - e^{ip} - e^{-ip} + 1) = \frac{g^2}{2}(1-\cos p) = g^2\sin^2\frac{p}{2} \qquad \longrightarrow \qquad H^2 = 1 + 4CC^* = 1 + 4g^2\sin^2\frac{p}{2}$$

$$\boxed{H = \sqrt{1 + 4g^2\sin^2\frac{p}{2}}}$$

$\leftarrow$ dispersion relation

The transverse fields of the light-cone string theory are 8 bosons and 8 fermions. They are organised into a bi-fundamental multiplet of $\mathfrak{su}(2|2)_\mathcal{C}$ as $2+2$ complex bosons and $2+2$ complex fermions.

Note on parametrization

$x^{\pm}$-variables

The momentum p can be replaced with two new but related parameters $x^{\pm}$

$$\frac{x^+}{x^-} = e^{ip}\,,$$

satisfying the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g} \qquad \Longrightarrow \qquad x^{\pm}(p) = \frac{e^{\pm ip/2}}{2g\sin\frac{p}{2}}\left[1+\sqrt{1+4g^2\sin^2\frac{p}{2}}\right]$$

Rapidity u

$$u = x^+ + \frac{1}{x^+} - \frac{i}{g} = x^- + \frac{1}{x^-} + \frac{i}{g} \qquad \leftarrow \text{rapidity}$$

$$u(p) = \frac{1}{g}\cot\frac{p}{2}\sqrt{1+4g^2\sin^2\frac{p}{2}}$$

rapidity of XXX model

$$H \;=\; 1 + \frac{ig}{x^+} - \frac{ig}{x^-} = igx^- - igx^+ - 1$$

Note on parametrization

Elliptic parametrization

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}$$ $\longleftarrow$ defines an elliptic curve

$$H^2 = 1 + 4g^2 \sin^2 \frac{p}{2}$$ $k = -4g^2 < 0$ $\longleftarrow$ elliptic modulus

$$\mathrm{dn}^2 z + k\,\mathrm{sn}^2 z = 1$$

Set $$\sin\frac{p}{2} = \mathrm{sn}(z,k)\,, \qquad H = \mathrm{dn}(z,k)$$

The variable z takes values on the elliptic curve (torus) with periods $2\omega_1$ and $2\omega_2$, where

$$2\omega_1 = 4\mathrm{K}(k)\,, \qquad 2\omega_2 = 4i\mathrm{K}(1-k) - 4\mathrm{K}(k)$$

$$\boxed{p = 2\,\mathrm{am} z\,, \qquad x^\pm = \frac{1}{2g}\left(\frac{\mathrm{cn}\,z}{\mathrm{sn}\,z} \pm i\right)(1 + \mathrm{dn}\,z)\,, \qquad u = \frac{\mathrm{cn}\,z\,\mathrm{dn}\,z}{g\,\mathrm{sn}\,z}}$$

Summary

We obtained the massive non-Lorentz invariant QFT on a plane. The fields are $2+2$ complex bosons and $2+2$ complex fermions transforming in the bifundamental atypical irrep of $\mathfrak{su}(2|2)_{\mathcal{C}}$

Dispersion $$H^2 = 1 + 4g^2 \sin^2 \frac{p}{2}\,, \qquad g = \frac{\sqrt{\lambda}}{2\pi}$$

Limit $g \to \infty$ can be taken in different ways

$$H = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \quad \Longrightarrow \quad \begin{cases} H = \frac{\sqrt{\lambda}}{\pi}|\sin\frac{p}{2}| & \textit{giant magnon} \quad p \text{ finite} \\ H = \sqrt{1+p^2} & \textit{plane-wave type} \quad p \to \frac{p}{\sqrt{\lambda}} \\ H = \lambda^{1/4}|p| & \textit{flat-space} \quad p \to \frac{p}{\sqrt[4]{\lambda}} \end{cases}$$

Bootstrap approach to the string S-matrix

Factorised scattering theory

$$-\frac{1}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\Psi(x_1,\ldots,x_N)+\sum_{i\neq j}v(x_i-x_j)\Psi(x_1,\ldots,x_N)=E\Psi(x_1,\ldots,x_N)$$

$\{I_m\}_{m=1}^N$ - set of mutually commuting operators

$$I_m\Psi(x_1,\ldots,x_N)=h_m\Psi(x_1,\ldots,x_N)\,,\qquad m=1,\ldots,N$$

Consider a kinematic domain

$$x_1<x_2<\ldots<x_N$$

fundamental sector.

$$I_m^{(0)}\Psi(x_1,\ldots,x_N)=h_m\Psi(x_1,\ldots,x_N)\,,\qquad m=1,\ldots,N$$

where $I_m^{(0)}$ are free conservation laws.

$$\Psi\sim e^{ip_1x_1+\ldots ip_Nx_N}\qquad\rightarrow\qquad I_m^{(0)}(p_i)=h_m\,,\quad m=1,\ldots,N$$

asymptotic momenta

$$p_1>p_2>\ldots>p_N$$

$$\Psi(x_1,\ldots,x_N)=\sum_{\tau\in\mathfrak{S}_N}\mathcal{A}(\tau)e^{ix_1p_{\tau(1)}+\ldots+ix_Np_{\tau(N)}}$$

Factorised scattering theory

In general, the configuration space $\mathbb{R}^N$ can be divided into $N!$ disconnected domains, each domain corresponds to a certain ordering of coordinates

$$x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(N)} \,,$$

where the latter are labelled by permutations $\sigma \in \mathfrak{S}_N$.

Bethe wave function

$$\Psi(x_1, \ldots, x_N|\sigma) = \sum_{\tau \in \mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{ix_{\sigma(1)}p_{\tau(1)} + \ldots + ix_{\sigma(N)}p_{\tau(N)}}$$

$N! \times N!$ matrix depending on particle momenta

- Exact in integrable quantum-mechanical theories
- Asymptotic in 2dim integrable quantum field theories

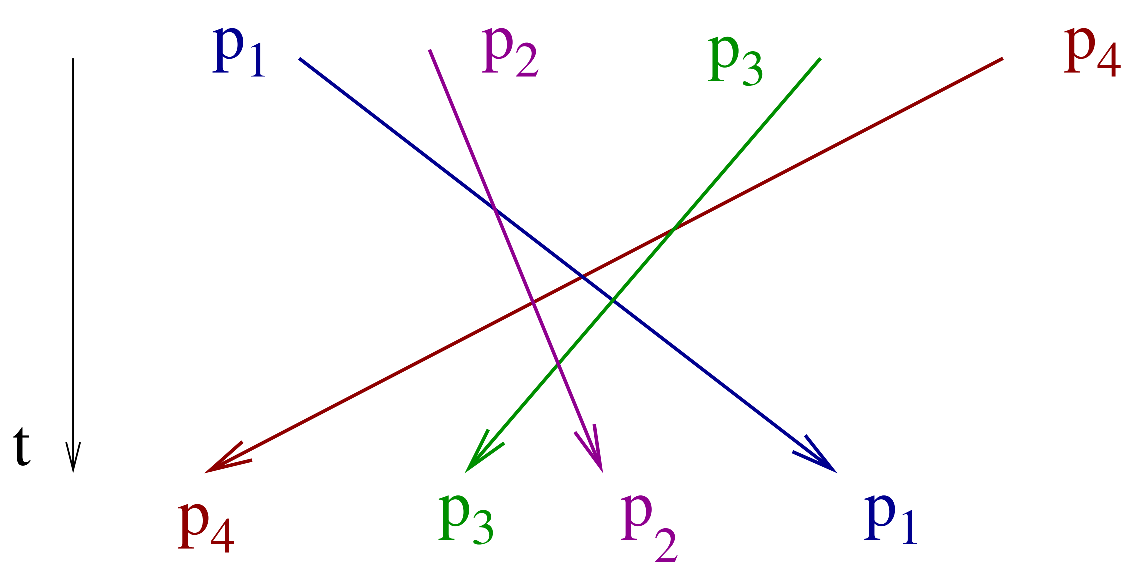
Factorised scattering theory

$\mathscr{J}$ symmetry algebra

$A_i^\dagger(p)$ creates a multiplet $\mathscr{V}$ of particles out of the vacuum with momentum p transforming in a linear irreducible representation of $\mathscr{J}$

Introduce the *in*-basis and *out*-basis as

$$|p_1, p_2, \cdots, p_N\rangle^{(in)}_{i_1,\dots,i_N} = A^\dagger_{i_1}(p_1)\cdots A^\dagger_{i_N}(p_N)|0\rangle\,, \quad p_1 > p_2 > \cdots > p_N$$
$$|p_1, p_2, \cdots, p_N\rangle^{(out)}_{i_1,\dots,i_N} = A^\dagger_{i_N}(p_N)\cdots A^\dagger_{i_1}(p_1)|0\rangle\,, \quad p_1 > p_2 > \cdots > p_N$$



In the scattering process the in-state goes to the out-state

$$|p_1, \dots, p_N\rangle^{(in)}_{i_1,\dots,i_N} \to |p_1, \dots, p_N\rangle^{(out)}_{i_1,\dots,i_N}$$

Factorised scattering theory

Scattering

$$|p_1,\ldots,p_N\rangle^{(in)}_{i_1,\ldots,i_N} \rightarrow |p_1,\ldots,p_N\rangle^{(out)}_{i_1,\ldots,i_N}$$

We expand initial states on a basis of final states

In particular, the two-particle in- and out-states are related as follows:

$$|p_1,p_2\rangle^{(in)}_{i,j} = \mathbf{S}\cdot|p_1,p_2\rangle^{(out)}_{i,j} = \underbrace{S^{kl}_{ij}(p_1,p_2)}_{\mathrm{two-body\ S-matrix}} \quad |p_1,p_2\rangle^{(out)}_{k,l}$$

or by using the explicit basis

$$A^\dagger_i(p_1)A^\dagger_j(p_2)|0\rangle = \mathbf{S}\cdot A^\dagger_j(p_2)A^\dagger_i(p_1)|0\rangle = S^{kl}_{ij}(p_1,p_2)A^\dagger_l(p_2)A^\dagger_k(p_1)|0\rangle$$

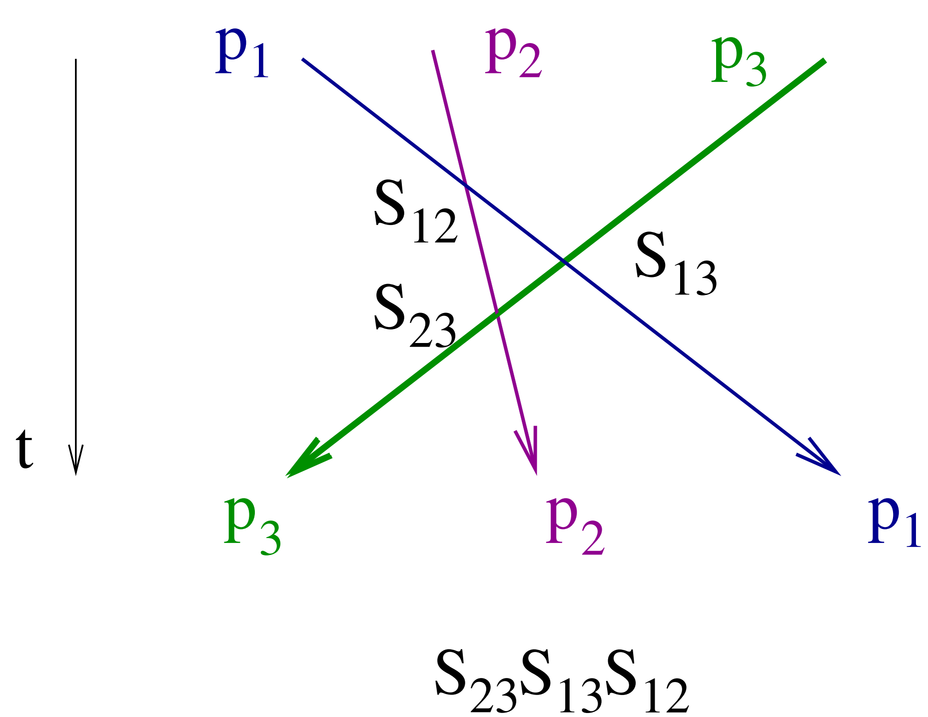
Factorised scattering theory

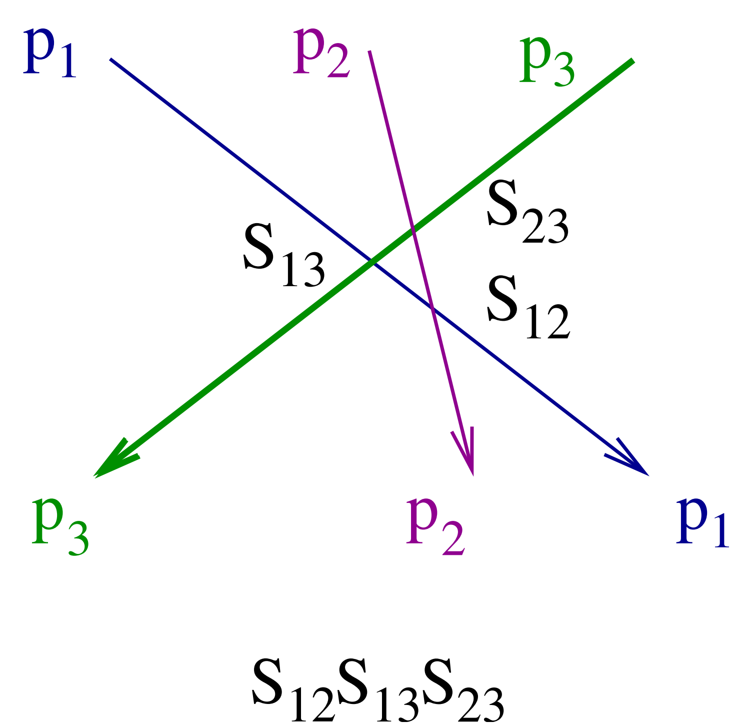
The Zamolodchikov algebra is

$$A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger S_{12}$$

Following two different ways of reordering $A_1^\dagger A_2^\dagger A_3^\dagger$ to $A_3^\dagger A_2^\dagger A_1^\dagger$ we obtain

$$A_1^\dagger A_2^\dagger A_3^\dagger = A_3^\dagger A_2^\dagger A_1^\dagger S_{12} S_{13} S_{23} \qquad A_1^\dagger A_2^\dagger A_3^\dagger = A_3^\dagger A_2^\dagger A_1^\dagger S_{23} S_{13} S_{12}$$





and derive the Yang-Baxter equation

$$S_{23}(p_2,p_3)S_{13}(p_1,p_3)S_{12}(p_1,p_2) = S_{12}(p_1,p_2)S_{13}(p_1,p_3)S_{23}(p_2,p_3)$$

Assume that the Hamiltonian H commutes with generators $\mathbf{J^a}$ of $\mathscr{J}$

$$\mathbf{J^a} \cdot |0\rangle = 0$$

$$\mathbf{J^a} \cdot A_i^\dagger(p)|0\rangle = J^{\mathbf{a}\,j}_{\;\;i}(p) A_j^\dagger(p)|0\rangle$$

$$\mathbf{J^a} \cdot A_i^\dagger(p_1) A_j^\dagger(p_2)|0\rangle = J^{\mathbf{a}\,kl}_{\;\;ij}(p_1, p_2) A_k^\dagger(p_1) A_l^\dagger(p_2)|0\rangle$$

The invariance condition for the S-matrix is derived from

$$\mathbf{J^a} \cdot A_i^\dagger(p_1) A_j^\dagger(p_2)|0\rangle = S_{ij}^{kl}(p_1, p_2)\, \mathbf{J^a} \cdot A_l^\dagger(p_2) A_k^\dagger(p_1)|0\rangle$$

This results into the invariance condition

$$S_{12}(p_1, p_2) J^{\mathbf{a}}_{12}(p_1, p_2) = J^{\mathbf{a}}_{21}(p_2, p_1) S_{12}(p_1, p_2)$$

Full set of conditions for an invariant S-matrix

A $\mathscr{J}$-invariant S-matrix $S(p_1, p_2)$, which depends on real momenta p_1 and p_2 of scattering particles, should obey

- the symmetry condition

$$S_{12}(p_1, p_2) J^{\mathbf{a}}_{12}(p_1, p_2) = J^{\mathbf{a}}_{21}(p_2, p_1) S_{12}(p_1, p_2)$$

- the Yang-Baxter equation

$$S_{23} S_{13} S_{12} = S_{12} S_{13} S_{23}$$

- the unitarity condition

$$S_{12}(p_1, p_2) S_{21}(p_2, p_1) = \mathbb{1}$$

- the physical unitarity condition

$$S_{12}(p_1, p_2) S^{\dagger}_{12}(p_1, p_2) = \mathbb{1}$$

- the requirement of crossing symmetry

$$\mathscr{C}_1^{-1} S^{t_1}_{12}(p_1, p_2) \mathscr{C}_1 S_{12}(-p_1, p_2) = \mathbb{1} \, ,$$

  where $\mathscr{C}$ is the charge conjugation matrix.

Full set of conditions for an invariant S-matrix

A $\mathscr{J}$-invariant S-matrix $S(p_1, p_2)$, which depends on real momenta p_1 and p_2 of scattering particles, should obey

- the symmetry condition

$$S_{12}(p_1, p_2) J_{12}^{\mathbf{a}}(p_1, p_2) = J_{21}^{\mathbf{a}}(p_2, p_1) S_{12}(p_1, p_2)$$

- the Yang-Baxter equation

$$S_{23} S_{13} S_{12} = S_{12} S_{13} S_{23}$$

- the unitarity condition

$$S_{12}(p_1, p_2) S_{21}(p_2, p_1) = \mathbb{1}$$

- the physical unitarity condition

$$S_{12}(p_1, p_2) S_{12}^{\dagger}(p_1, p_2) = \mathbb{1}$$

- the requirement of crossing symmetry

$$\mathscr{C}_1^{-1} S_{12}^{t_1}(p_1, p_2) \mathscr{C}_1 S_{12}(-p_1, p_2) = \mathbb{1}\,,$$

  where $\mathscr{C}$ is the charge conjugation matrix.

$$
\begin{aligned}
S(p_1,p_2) \;=\;& \Big(E_1^1\otimes E_1^1+E_1^2\otimes E_2^1+E_2^1\otimes E_1^2+E_2^2\otimes E_2^2\Big)\\
+\;& \frac{(x_1^- - x_2^-)(x_1^+x_2^- - 1)x_2^+}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)x_2^-}\Big(E_1^1\otimes E_2^2 - E_1^2\otimes E_2^1 - E_2^1\otimes E_1^2 + E_2^2\otimes E_1^1\Big)\\
-\;& \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\,\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2}\Big(E_3^3\otimes E_3^3+E_3^4\otimes E_4^3+E_4^3\otimes E_3^4+E_4^4\otimes E_4^4\Big)\\
-\;& \frac{(x_1^- - x_2^-)(x_1^-x_2^+ - 1)x_1^+}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)x_1^-}\,\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2}\Big(E_3^3\otimes E_4^4 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4 + E_4^4\otimes E_3^3\Big)\\
+\;& \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,\frac{\tilde\eta_2}{\eta_2}\Big(E_1^1\otimes E_3^3+E_1^1\otimes E_4^4+E_2^2\otimes E_3^3+E_2^2\otimes E_4^4\Big)\\
+\;& \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\,\frac{\tilde\eta_1}{\eta_1}\Big(E_3^3\otimes E_1^1+E_3^3\otimes E_2^2+E_4^4\otimes E_1^1+E_4^4\otimes E_2^2\Big)\\
-\;& \frac{i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- - x_2^-)}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)\eta_1\eta_2}\,\frac{x_1^+x_2^+}{x_1^-x_2^-}\Big(E_1^3\otimes E_2^4 - E_1^4\otimes E_2^3 - E_2^3\otimes E_1^4 + E_2^4\otimes E_1^3\Big)\\
-\;& \frac{i(x_1^- - x_2^-)\tilde\eta_1\tilde\eta_2}{(x_1^- - x_2^+)(x_1^+x_2^+ - 1)}\Big(E_3^1\otimes E_4^2 - E_3^2\otimes E_4^1 - E_4^1\otimes E_3^2 + E_4^2\otimes E_3^1\Big)\\
+\;& \frac{x_1^- - x_1^+}{x_1^- - x_2^+}\,\frac{\tilde\eta_2}{\eta_1}\Big(E_1^3\otimes E_3^1+E_1^4\otimes E_4^1+E_2^3\otimes E_3^2+E_2^4\otimes E_4^2\Big)\\
+\;& \frac{x_2^- - x_2^+}{x_1^- - x_2^+}\,\frac{\tilde\eta_1}{\eta_2}\Big(E_3^1\otimes E_1^3+E_3^2\otimes E_2^3+E_4^1\otimes E_1^4+E_4^2\otimes E_2^4\Big)
\end{aligned}
$$

$$\eta = e^{\frac{i}{4}p}\sqrt{ix^- - ix^+}$$

$$\eta_1=\eta(p_1)\,,\quad \eta_2=e^{\frac{i}{2}p_1}\eta(p_2)\,,\quad \tilde\eta_1=e^{\frac{i}{2}p_2}\eta(p_1)\,,\quad \tilde\eta_2=\eta(p_2)$$

String S-matrix

On the z-torus

$$x^{\pm} = \frac{1}{2g}\left(\frac{\operatorname{cn} z}{\operatorname{sn} z} \pm i\right)(1 + \operatorname{dn} z)$$

$$\eta(z) = \frac{\sqrt{2}}{\sqrt{g}} \frac{\operatorname{dn}\frac{z}{2}\left(\operatorname{cn}\frac{z}{2} + i \operatorname{sn}\frac{z}{2}\operatorname{dn}\frac{z}{2}\right)}{1 + 4g^2 \operatorname{sn}^4\frac{z}{2}}$$

$$\eta_1 = \eta(z_1)\,, \quad \eta_2 = (\operatorname{cn} z_1 + i \operatorname{sn} z_1)\eta(z_2)\,, \quad \tilde{\eta}_1 = (\operatorname{cn} z_2 + i \operatorname{sn} z_2)\eta(z_1)\,, \quad \tilde{\eta}_2 = \eta(z_2)$$

$$\Big\downarrow$$

$$S(p_1, p_2) \quad \rightarrow \quad S(z_1, z_2)$$

spin structure

$$S(z_1 + 2\omega_1, z_2) = \Sigma_1\, S(z_1, z_2)\Sigma_1 = \Sigma_2\, S(z_1, z_2)\Sigma_2$$
$$S(z_1 + 2\omega_2, z_2) = \Sigma_1\, S(z_1, z_2)\Sigma_1 = \Sigma_2\, S(z_1, z_2)\Sigma_2$$

$\Sigma_1 = \Sigma \otimes \mathbb{I}$ and $\Sigma_2 = \mathbb{I} \otimes \Sigma$ $\qquad$ $\Sigma = \operatorname{diag}(1, 1, -1, -1)$ $\qquad$ $[S, \Sigma \otimes \Sigma] = 0$

String S-matrix at one loop

In the elliptic parametrisation the S-matrix becomes a function on the product of two elliptic curves

$$S = S(z_1, z_2)$$

Consider the limit $g \to 0$ and obtain the one-loop S-matrix

$$\begin{aligned}
S(z_1,z_2) &= \left(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\right)\\
&+\frac{2i}{\cot z_1-\cot z_2-2i}\left(E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1 - E_1^2\otimes E_2^1 - E_2^1\otimes E_1^2\right)\\
&-e^{-i(z_1-z_2)}\frac{\cot z_1-\cot z_2+2i}{\cot z_1-\cot z_2-2i}\left(E_3^3\otimes E_3^3 + E_4^4\otimes E_4^4 + E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3\right)\\
&+e^{-i(z_1-z_2)}\frac{2i}{\cot z_1-\cot z_2-2i}\left(E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4\right)\\
&+e^{-iz_1}\frac{\cot z_1-\cot z_2}{\cot z_1-\cot z_2-2i}\left(E_1^1\otimes E_3^3 + E_1^1\otimes E_4^4 + E_2^2\otimes E_3^3 + E_2^2\otimes E_4^4\right)\\
&+e^{iz_2}\frac{\cot z_1-\cot z_2}{\cot z_1-\cot z_2-2i}\left(E_3^3\otimes E_1^1 + E_4^4\otimes E_1^1 + E_3^3\otimes E_2^2 + E_4^4\otimes E_2^2\right)\\
&-e^{-\frac{i}{2}(z_1-z_2)}\frac{2i}{\cot z_1-\cot z_2-2i}\left(E_1^3\otimes E_3^1 + E_1^4\otimes E_4^1 + E_2^3\otimes E_3^2 + E_2^4\otimes E_4^2\right)\\
&-e^{-\frac{i}{2}(z_1-z_2)}\frac{2i}{\cot z_1-\cot z_2-2i}\left(E_3^1\otimes E_1^3 + E_4^1\otimes E_1^4 + E_3^2\otimes E_2^3 + E_4^2\otimes E_2^4\right)
\end{aligned}$$

The relations between the z-variable, the momentum and the rescaled rapidity $u \to gu$ transform in the limit $g \to 0$ into

$$p = 2z\,, \qquad u = \cot z = \cot\frac{p}{2}$$

S-matrix cannot be written in the difference form !

$$S_{23}(z_2,z_3)S_{13}(z_1,z_3)S_{12}(z_1,z_2) = S_{12}(z_1,z_2)S_{13}(z_1,z_3)S_{23}(z_2,z_3)$$

String S-matrix at one loop

$$S_{12}^{\rm can} = \frac{u_1-u_2}{u_1-u_2-2i} I_{12}^g - \frac{2i}{u_1-u_2-2i} P_{12}$$

graded identity

$$I_{12}^g = (-1)^{\epsilon_i \epsilon_j} E_i^i \otimes E_j^j$$

obeys the same YBE

$$S^{\rm can}(z_1,z_2) = U_2(z_1)\Big[V_1(z_1)V_2(z_2)S_{12}(z_1,z_2)V_1^{-1}(z_1)V_2^{-1}(z_2)\Big]U_1^{-1}(z_2)\,,$$

where

$$\begin{aligned} U(z) &= \mathrm{diag}(1,1,e^{iz},e^{iz})\,, \\ V(z) &= \mathrm{diag}(e^{i\frac{z}{4}},e^{i\frac{z}{4}},e^{-i\frac{z}{4}},e^{-i\frac{z}{4}}) \end{aligned}$$

twist

gauge transformation

YBE for $S^{\rm can}$

$$\begin{aligned} &U_3(z_2)S_{23}U_2(z_1)U_3(z_1)S_{13}U_1^{-1}(z_3)U_2^{-1}(z_3)S_{12}U_1(z_2) = \\ &= U_2(z_1)U_3(z_1)S_{12}U_1^{-1}(z_2)S_{13}U_3(z_2)S_{23}U_1^{-1}(z_3)U_2^{-1}(z_3) \end{aligned}$$

1 – loop

$$[S, U\otimes U] = 0$$

arbitrary diagonal

all – loop

$$[S, G\otimes G] = 0\,, \qquad G \in \mathrm{SU}(2)\times\mathrm{SU}(2)$$

YBE is preserved by twist at one loop only!