Dilatation operator and spin chains

Renormalisation and anomalous dimensions

Typical local gauge-invariant operator

$$\mathcal{O}(x) = \underbrace{\mathrm{Tr}\left[\phi^{i_1}(x)\phi^{i_2}(x)\ldots\phi^{i_n}(x)\right]}_{\text{product of fields at the same space-time point}}$$

Renormalized operators and mixing matrix (dilatation operator)

$$\underbrace{\mathcal{O}^I_{\text{ren}}}_{\text{convergent}} = Z^I{}_J \underbrace{\mathcal{O}^J_{\text{bare}}}_{\text{divergent}} \implies D = \frac{dZ}{d\log\Lambda} Z^{-1}$$

mixing matrix

$$D\mathcal{O}_n = \Delta_n \mathcal{O}_n$$

$$\Delta_n \equiv \Delta_n\Big(\underbrace{g^2_{YM}N}_{\lambda}, \frac{1}{N}\Big)$$

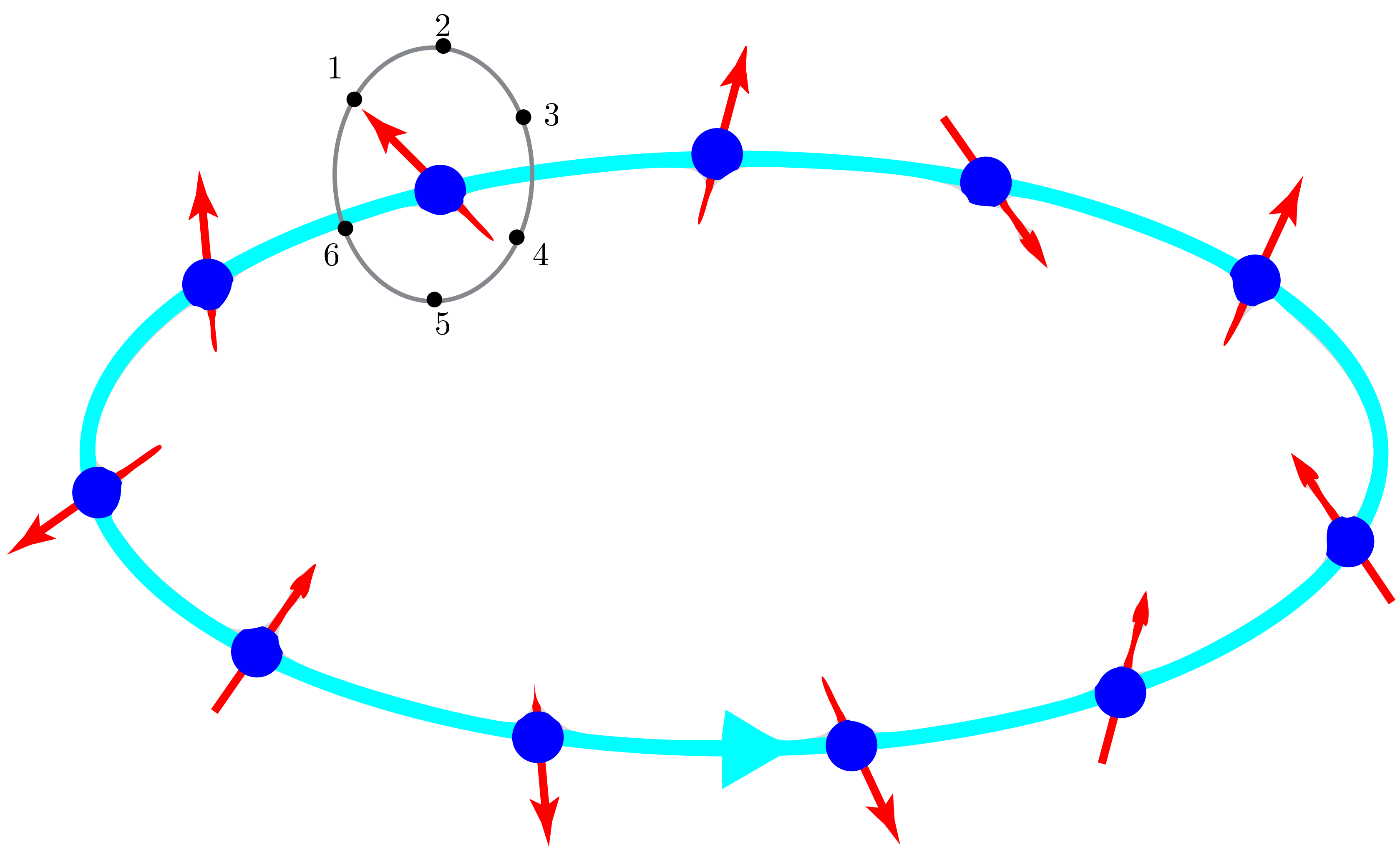
Eigenvalues of D are anomalous dimensions.

Spin chain interpretation

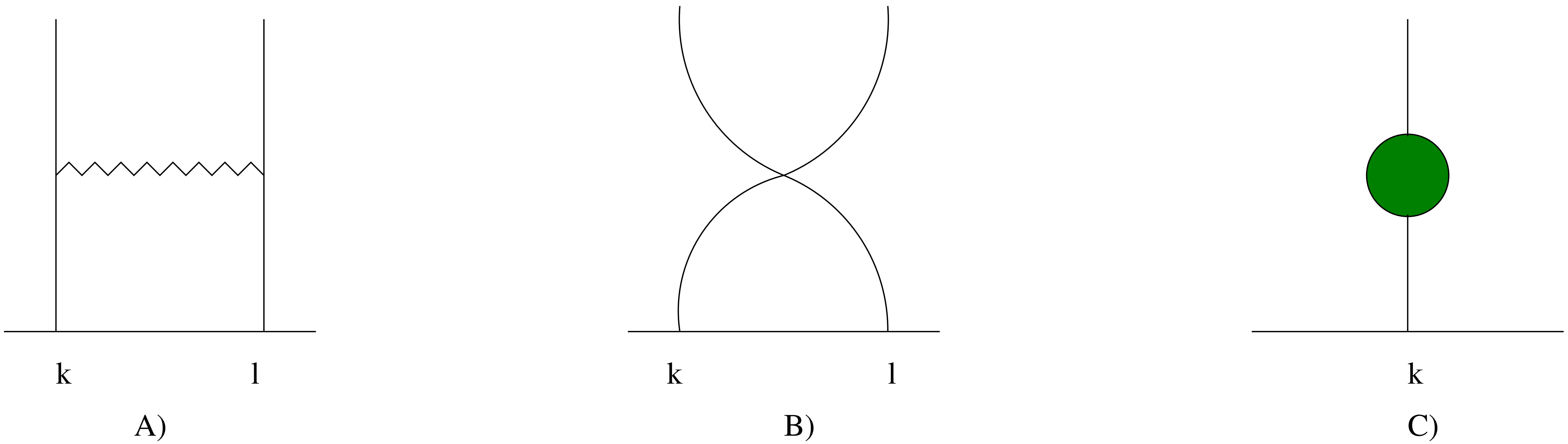
Consider a scalar operator made of L elementary scalars

$$\mathrm{Tr}\big(\phi^{i_1}\phi^{i_2}\ldots\phi^{i_L}\big)$$

View it as the spin chain of length L where spin at every site can have 6 polarizations



Spin chain interpretation



One-loop graphs

$$
\begin{aligned}
Z_{A\ldots i_k i_{k+1}\ldots}^{\ \ \ldots j_k j_{k+1}\ldots} &= I - \frac{\lambda}{16\pi^2}\ln\Lambda\ \delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}} \\
Z_{B\ldots i_k i_{k+1}\ldots}^{\ \ \ldots j_k j_{k+1}\ldots} &= I - \frac{\lambda}{16\pi^2}\ln\Lambda\ \left(2\delta_{i_k}^{j_{k+1}}\delta_{i_{k+1}}^{j_k} - \delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}} - \delta_{i_k i_{k+1}}\delta^{j_k j_{k+1}}\right) \\
Z_{C\ldots i_k i_{k+1}\ldots}^{\ \ \ldots j_k j_{k+1}\ldots} &= I + \frac{\lambda}{8\pi^2}\ln\Lambda\ \delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}}
\end{aligned}
$$

Spin chain interpretation

Total Z is

$$Z^{\dots j_k j_{k+1}\dots}_{\dots i_k i_{k+1}\dots} = I + \frac{\lambda}{16\pi^2}\ln\Lambda\left(2\delta^{j_{k+1}}_{i_k}\delta^{j_k}_{i_{k+1}} - 2\delta^{j_k}_{i_k}\delta^{j_{k+1}}_{i_{k+1}} + \delta_{i_k i_{k+1}}\delta^{j_k j_{k+1}}\right)$$

Introducing the trace operator K and the permutation operator P:

$$K = \delta_{i_k i_{k+1}}\delta^{j_k j_{k+1}}\,, \qquad\qquad P = \delta^{j_{k+1}}_{i_k}\delta^{j_k}_{i_{k+1}}$$

the mixing matrix is

$$\boxed{\mathrm{D}_{1-\mathrm{loop}} = \frac{g^2}{2}\sum_{i=1}^{L}\left(I - P_{i,i+1} + \frac{1}{2}K_{i,i+1}\right)}$$

$$g^2 = \frac{\lambda}{4\pi^2} = \frac{g^2_{\mathrm{YM}}N}{4\pi^2}$$

the Hamiltonian of the SO(6) *integrable* spin chain!

[Minahan and Zarembo, '03]

[Previously observed integrable structures in QCD: Lipatov, '94; Faddeev and Korchemsky '95]

Spin chain interpretation

Reduction to the $\mathfrak{su}(2) \subset \mathfrak{su}(4) \simeq \mathfrak{so}(6)$: $K = 0$

The Hamiltonian of the $\mathfrak{su}(2)$ spin chain is

$$\mathbf{H} = \sum_{i=1}^{L} \Big(I - P_{i,i+1} \Big)$$

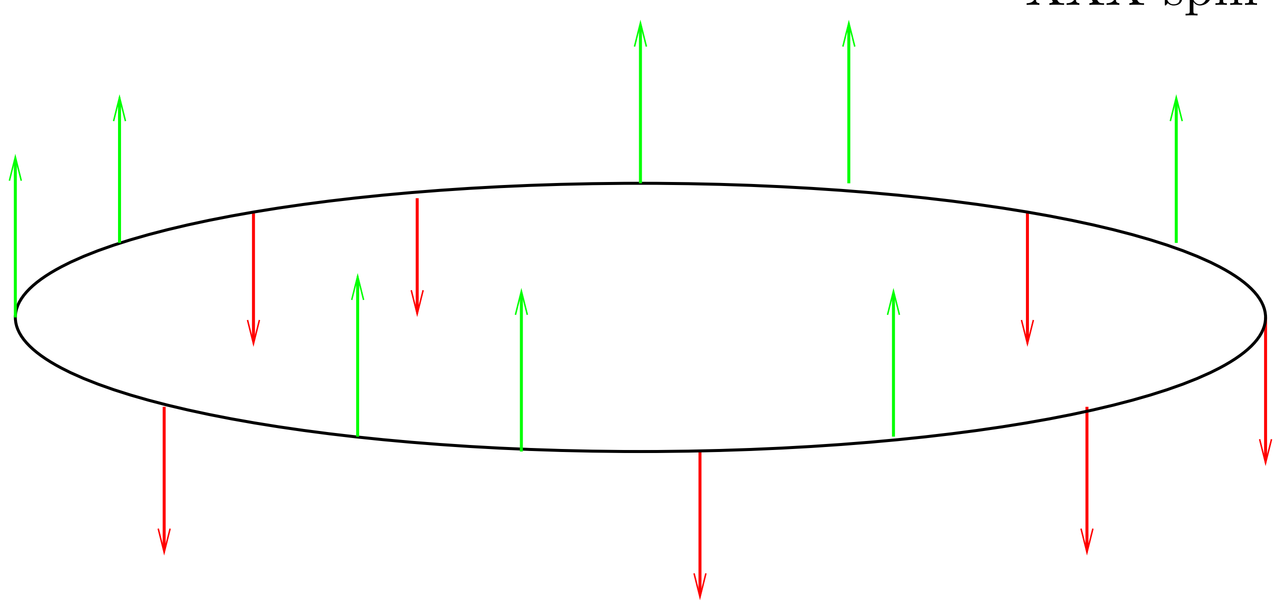
$P(\uparrow\downarrow) = (\downarrow\uparrow)$

XXX spin chain

$\mathrm{Tr}(W^M Z^{L-M}) + \ldots$

$Z = \phi_1 + i\phi_2$

$W = \phi_3 + i\phi_4$



The planar one-loop mixing matrix is identical to the Heisenberg Hamiltonian H. It acts as $2^L \times 2^L$ matrix on states of the spin chain of length L. M is the number of magnons.

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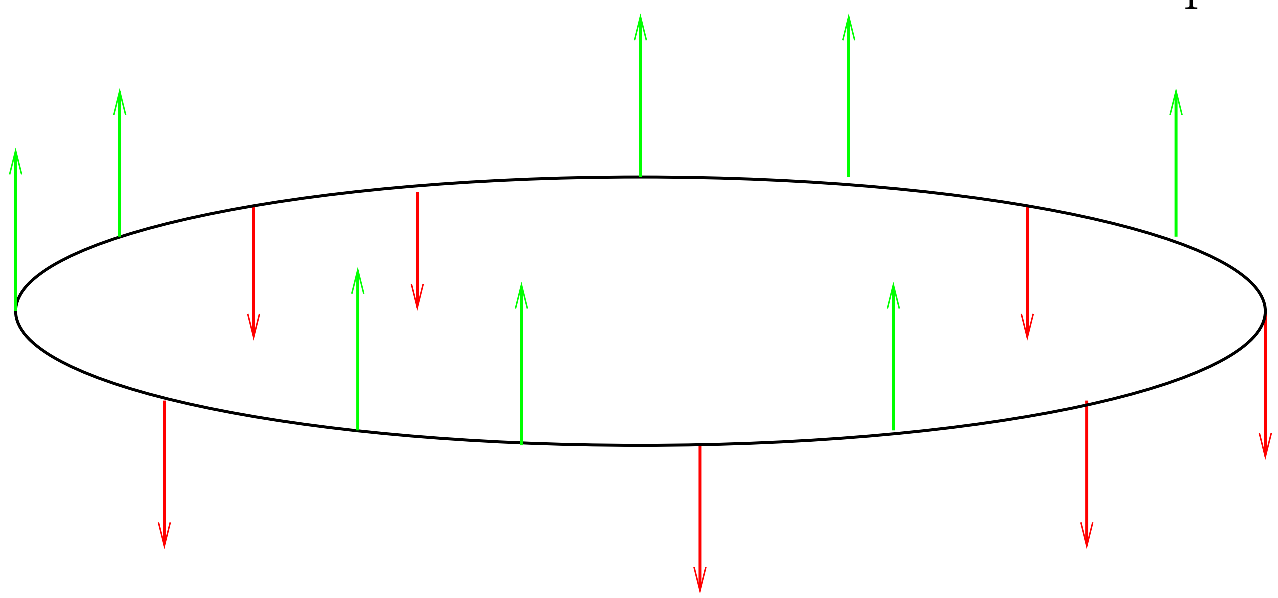

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Closed sectors

- Sector $\mathfrak{su}(2)$. Operators with $[M, J-M, M]$ and $\Delta_0 = J + M$. *Classically they are $\frac{1}{4}$-BPS states.* Operators:

$$\mathrm{Tr}(Z_1^J Z_2^M) + \ldots.$$

- Sector $\mathfrak{sl}(2)$. The Dynkin content $[0, J, 0; M, M]$ and $\Delta_0 = J + M$, Lorentz spin $S = M$. Operators

$$\mathrm{Tr}(\mathcal{D}^M Z^J) + \ldots$$

where $\mathcal{D} = \mathcal{D}_1 + i\mathcal{D}_2$

- Sector $\mathfrak{su}(1|1)$. The Dynkin content $[0, J - \frac{1}{2}M, M; 0, \frac{1}{2}M]$ and $\Delta_0 = J + M$. Operators:

$$\mathrm{Tr}(Z^{J-\frac{M}{2}} \Psi^M) + \ldots.$$

$\dim\Psi = 3/2$

$\vdots$

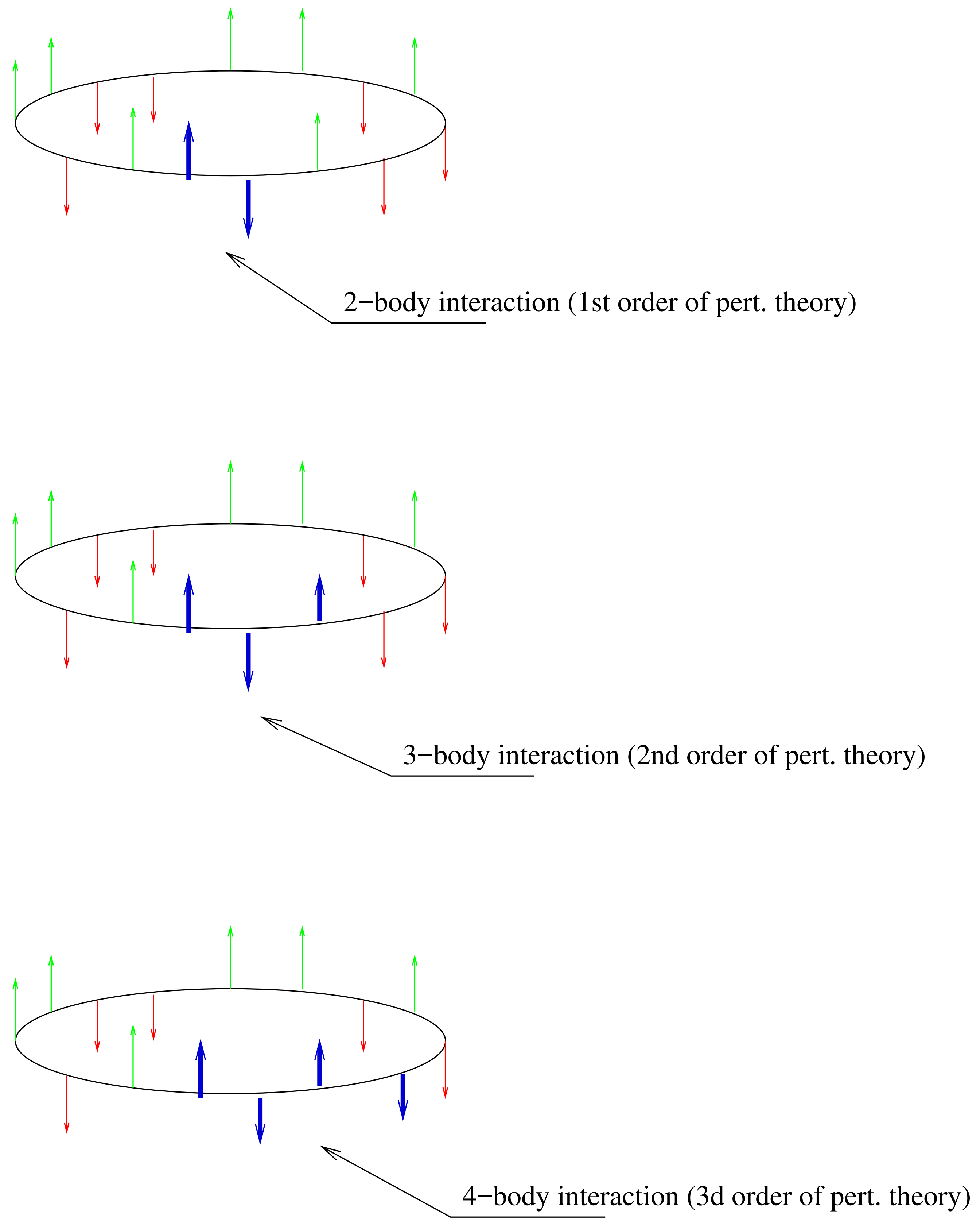
- Sector $\mathfrak{su}(2|3)$. Three complex scalars, two complex fermions. *Classically they are $\frac{1}{8}$-BPS states.* Operators:

$$\mathrm{Tr}(Z_1^{n_1} Z_2^{n_2} Z_3^{n_3} \Psi_1^{n_4} \Psi_2^{n_5}) + \ldots.$$

$\mathfrak{psu}(2,2|4)$ integrable spin chain

can be solved by the Bethe Ansatz

Dilatation operator at higher loops



Dilatation operator acts as the Hamiltonian $\mathbb{H}$ of an integrable long-range spin chain

Dilatation operator at higher loops

On a chain of length L made of two complex scalars the dilop acts as

$$
\begin{aligned}
\mathbf{H}_{1\ell} &= \sum_{i=1}^{L} \Big(I - P_{i,i+1} \Big) \quad \Leftarrow \quad \text{Heisenberg Hamiltonian} \\
\mathbf{H}_{2\ell} &= \sum_{i=1}^{L} \Big(-\frac{3}{2} I + 2 P_{i,i+1} - \frac{1}{2} P_{i,i+2} \Big) \\
\mathbf{H}_{3\ell} &= \sum_{i=1}^{L} \Big(5 I - 7 P_{i,i+1} + 2 P_{i,i+2} \\
&- \frac{1}{2} \left(P_{i,i+3} P_{i+1,i+2} - P_{i,i+2} P_{i+1,i+3} \right) \Big)
\end{aligned}
$$

One inevitably runs into the wrapping problem! At this point the interpretation in terms of long-range spin chains does not work anymore and one has to resort to a sigma-model description coming from string on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Part II

Integrable structure of string theory

Sigma model

$$\text{non}-\text{linear sigma model with target} \qquad \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$$

$$\mathrm{SO}(4,2)\times\mathrm{SO}(6) \Big/ \mathrm{SO}(4,1)\times\mathrm{SO}(5) \simeq \mathrm{AdS}_5\times\mathrm{S}^5$$

$$\underbrace{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}_{\text{Local Lorentz group}} \subset \mathrm{SU}(2,2)\times\mathrm{SO}(6)$$

Sigma model action

$$\mathfrak{sl}(4|4): \qquad M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}_{8\times 8} \qquad \mathrm{str}M \equiv \mathrm{tr}\, m - \mathrm{tr}\, n = 0.$$

4×4

θ, η – grassmann (fermionic)

$$\mathfrak{su}(2,2|4): \qquad M^\dagger H + HM = 0$$

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}_{8\times 8} \qquad \Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}_{4\times 4}$$

$$\underbrace{m^\dagger = -\Sigma\, m\, \Sigma}_{\mathfrak{u}(2,2)}\,, \qquad \underbrace{n^\dagger = -n}_{\mathfrak{u}(4)}\,, \qquad \eta^\dagger = -\Sigma\,\theta \qquad \mathfrak{u}(1)\text{-generator } i\mathbb{1}$$

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(2,2|4)$$

$$\mathfrak{psu}(2,2|4) = \mathfrak{su}(2,2|4)/\mathfrak{u}(1)$$

Sigma model action

An external automorphism – supertransposition.

$$M \to -M^{st}$$

where the supertransponse M^{st} is defined as

$$M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix} \longrightarrow (M^{st})^{st} = \begin{pmatrix} m & -\theta \\ -\eta & n \end{pmatrix}$$

$\mathbb{Z}_4$ – grading $\qquad \mathscr{G} = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)} \oplus \mathscr{G}^{(2)} \oplus \mathscr{G}^{(3)} \qquad [\mathscr{G}^{(k)}, \mathscr{G}^{(m)}] \subset \mathscr{G}^{(k+m)}$ modulo $\mathbb{Z}_4$.

Let $\mathfrak{g}$ be an element of the supergroup $\mathrm{SU}(2,2|4)$.

Introduce the following one-form with values in $\mathfrak{su}(2,2|4)$

$$A = -\mathfrak{g}^{-1}\mathrm{d}\mathfrak{g} = A^{(0)} + A^{(2)} + A^{(1)} + A^{(3)}$$

$$F = \mathrm{d}A - A \wedge A = 0 \longrightarrow \partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0$$

Lagrangian for strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$\mathscr{L} = -\frac{g}{2}\Big[\underbrace{\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big)}_{\text{kinetic}} + \underbrace{\kappa\,\epsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)}_{\text{Wess – Zumino}}\Big]$$

$g = \dfrac{R^2}{2\pi\alpha'}$ $\qquad \gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$

Symmetries

Local $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ **symmetry** $\qquad \mathfrak{g}\to\mathfrak{g}\mathfrak{h} \qquad$ $\mathfrak{h}$ belongs to $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$

$$A\to\mathfrak{h}^{-1}A\mathfrak{h}-\mathfrak{h}^{-1}\mathrm{d}\mathfrak{h} \qquad\longrightarrow\qquad A^{(1,2,3)}\to\mathfrak{h}^{-1}A^{(1,2,3)}\mathfrak{h}\,,\qquad A^{(0)}\to\mathfrak{h}^{-1}A^{(0)}\mathfrak{h}-\mathfrak{h}^{-1}\mathrm{d}\mathfrak{h}$$

Global $\mathrm{PSU}(2,2|4)$ **symmetry**

$$G:\ \mathfrak{g}\to\mathfrak{g}' \qquad\qquad G\cdot\mathfrak{g}=\mathfrak{g}'\,\mathfrak{h}\,.$$

Diffeomorphisms

Equations of motion for the world-sheet metric $h_{\alpha\beta}$ are equivalent to vanishing the world-sheet stress-tensor

$$\mathrm{str}(A^{(2)}_\alpha A^{(2)}_\beta)-\tfrac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\mathrm{str}(A^{(2)}_\rho A^{(2)}_\delta)=0 \qquad\longleftarrow\qquad \text{Virasoro constraints}$$

κ-symmetry

$$\left(\begin{array}{cccc|cccc}
0&0&0&0&0&0&\bullet&\bullet\\
0&0&0&0&0&0&\bullet&\bullet\\
0&0&0&0&\bullet&\bullet&0&0\\
0&0&0&0&\bullet&\bullet&0&0\\
\hline
0&0&\bullet&\bullet&0&0&0&0\\
0&0&\bullet&\bullet&0&0&0&0\\
\bullet&\bullet&0&0&0&0&0&0\\
\bullet&\bullet&0&0&0&0&0&0
\end{array}\right)$$

bullets stand for odd elements which cannot be gauged away by κ-symmetry transformations

Equations of motion

$$\Lambda^\alpha = g\Big[\gamma^{\alpha\beta}A^{(2)}_\beta - \tfrac{1}{2}\kappa\,\epsilon^{\alpha\beta}(A^{(1)}_\beta - A^{(3)}_\beta)\Big]$$

Eoms $$\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = 0$$

Define $$J^\alpha = \mathfrak{g}\,\Lambda^\alpha\mathfrak{g}^{-1} \quad\longrightarrow\quad \partial_\alpha J^\alpha = 0$$

Noether current corresponding to global PSU(2,2|4)-symmetry

Fundamental linear problem

vector of rank $\mathfrak{r}$

$\Psi \equiv \Psi(\sigma, \tau, z)$

$$\frac{\partial \Psi}{\partial \sigma} = L_\sigma(\sigma, \tau, z)\Psi\,,$$
$$\frac{\partial \Psi}{\partial \tau} = L_\tau(\sigma, \tau, z)\Psi\,,$$

$\mathfrak{r} \times \mathfrak{r}$ matrices

$L_\tau \equiv L_\tau(\sigma, \tau, z)$

$L_\sigma \equiv L_\sigma(\sigma, \tau, z)$

spectral parameter

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial \tau \partial \sigma} &= \partial_\tau L_\sigma \Psi + L_\sigma \partial_\tau \Psi = (\partial_\tau L_\sigma + L_\sigma L_\tau)\Psi\,,\\ \frac{\partial^2 \Psi}{\partial \sigma \partial \tau} &= \partial_\sigma L_\tau \Psi + L_\tau \partial_\sigma \Psi = (\partial_\sigma L_\tau + L_\tau L_\sigma)\Psi\,,\end{aligned} \qquad \longrightarrow \qquad \partial_\tau L_\sigma - \partial_\sigma L_\tau + [L_\sigma, L_\tau] = 0$$

Zero-curvature (Lax) representation of an integrable PDE

$$\partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0$$

Example: Sine-Gordon model

Sine-Gordon equation

$$\phi_{\tau\tau} - \phi_{\sigma\sigma} + \frac{m^2}{\beta} \sin \beta\phi = 0$$

Introduce the following 2×2 matrices

$$\begin{aligned} L_\sigma &= \frac{\beta}{4i}\phi_\tau \sigma_3 + \frac{k_0}{i} \sin \frac{\beta\phi}{2}\sigma_1 + \frac{k_1}{i} \cos \frac{\beta\phi}{2}\sigma_2 \\ L_\tau &= \frac{\beta}{4i}\phi_\sigma \sigma_3 + \frac{k_1}{i} \sin \frac{\beta\phi}{2}\sigma_1 + \frac{k_0}{i} \cos \frac{\beta\phi}{2}\sigma_2 \,, \end{aligned}$$

where σ_i are the Pauli matrices and

$$k_0 = \frac{m}{4}\left(z + \frac{1}{z}\right), \qquad k_1 = \frac{m}{4}\left(z - \frac{1}{z}\right)$$

$$\partial_\tau L_\sigma - \partial_\sigma L_\tau + [L_\sigma, L_\tau] = 0 \qquad \text{for any } z \text{ is due to Sine} - \text{Gordon}$$

Integrability through Lax representation

$$\partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0$$

Monodromy

$$\mathrm{T}(z) = \overleftarrow{\exp} \int_0^{2\pi} \mathrm{d}\sigma \, L_\sigma(z) \, .$$

$$\partial_\tau \mathrm{T}(z) \quad = \quad [L_\tau(0,\tau,z), \mathrm{T}(z)]$$

$$\Gamma(z,\mu) \equiv \det(\mathrm{T}(z) - \mu \mathbb{1}) = 0$$

algebraic curve in $\mathbb{C}^2$

spectral curve

Integrability through Lax representation

Ansatz
$$L_\alpha = \ell_0 A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} A_\rho^{(2)} + \ell_3 A_\alpha^{(1)} + \ell_4 A_\alpha^{(3)}$$

must have zero curvature as a consequence of equations of motion and the flatness of A_α

$$\ell_0 = 1\,, \quad \ell_1 = \frac{1}{2}\Big(z^2 + \frac{1}{z^2}\Big)\,, \quad \ell_2 = -\frac{1}{2\kappa}\Big(z^2 - \frac{1}{z^2}\Big)\,, \quad \ell_3 = z\,, \quad \ell_4 = \frac{1}{z}\,.$$

For a given $\kappa = \pm 1$, there is a unique Lax connection which is a meromorphic matrix-valued function on the Riemann sphere.