Holography and Integrability

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These lectures are about the spectral problem

$\mathcal{N}=4$ super Yang-Mills $\leftrightarrow$ Strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Plan

0. Integrable systems

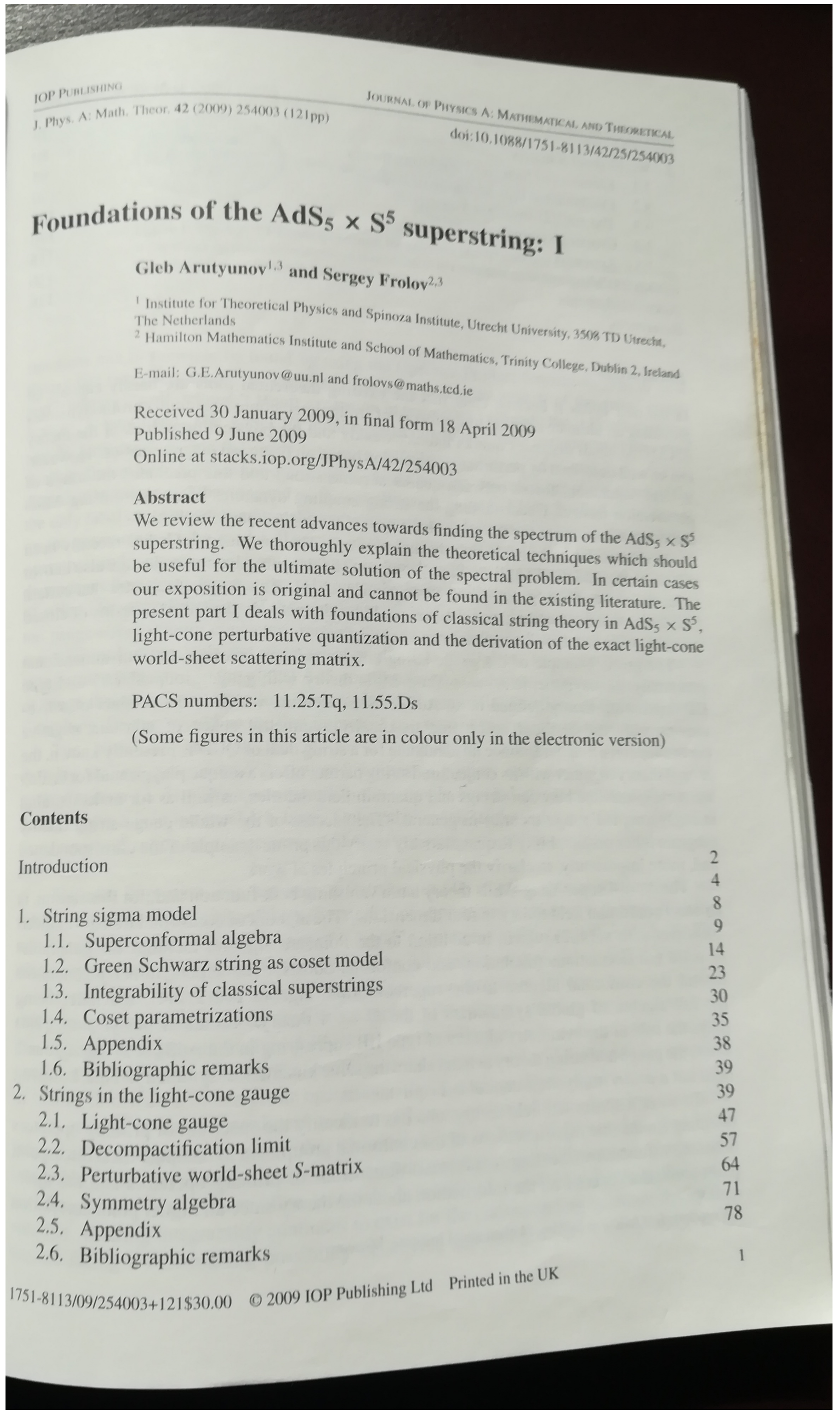
Liouville

I. Integrable structure of gauge theory

Elements of superconformal representation theory, dilatation operator, long – range spin chains

II. Integrable structure of string theory

Sigma – model, Lax representation, classical solutions, S – matrix, Bethe – Ansatz, TBA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Foundations of the $AdS_5 \times S^5$ superstring: I

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Abstract
We review the recent advances towards finding the spectrum of the $AdS_5 \times S^5$ superstring. We thoroughly explain the theoretical techniques which should be useful for the ultimate solution of the spectral problem. In certain cases our exposition is original and cannot be found in the existing literature. The present part I deals with foundations of classical string theory in $AdS_5 \times S^5$, light-cone perturbative quantization and the derivation of the exact light-cone world-sheet scattering matrix.

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(Some figures in this article are in colour only in the electronic version)

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Elements of Classical and Quantum Integrable Systems

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Review of AdS/CFT Integrability: An Overview

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Lett.Math.Phys. 99 (2012)

Part 0

Integrable systems

"When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic theoretic interest."

George D. Birkhoff
Dynamical systems, AMS, 1927

Integrability

Consider a dynamical system which has an (infinite) number of constants of motion. A theory which exhibits non-stochastic behaviour of this sort is said to be exactly integrable

- Classical finite-dimensional systems solved by *the Liouville theorem*
- Classical infinite-dimensional integrable systems (integrable PDE's) solved by the *Classical Inverse Scattering Method* or by the *Finite-Gap Integration Technique*
- Quantum integrable many-body systems solved by means of *"Bethe Ansatz"* It exists in a variety of forms: Coordinate, Algebraic, Functional, Nested, Asymptotic, Thermodynamic. . .

NOTE

Sur l'intégration des équations différentielles de la Dynamique, présentée au Bureau des Longitudes le 29 juin 1853;

PAR M. J. LIOUVILLE.

Dans le Rapport qui précède, sur le Mémoire de M. Edmond Bour, j'ai parlé d'un théorème que j'ai donné il y a deux ans dans mon cours au Collége de France, puis communiqué au Bureau des Longitudes. L'extrait suivant du procès-verbal de la séance du 29 juin 1853, que je dois à l'obligeance du Secrétaire du Bureau, M. Daussy, contient textuellement la Note que j'ai présentée à ce corps savant.

BUREAU DES LONGITUDES.

Paris, le 28 mars 1855.

Le Secrétaire du Bureau des Longitudes certifie que ce qui suit est extrait du procès-verbal de la séance du Bureau du mercredi 29 juin 1853.

« M. Liouville communique la Note ci-jointe sur l'intégration des équations de la » Dynamique. Il y ajoute verbalement tous les développements nécessaires pour en » expliquer l'utilité et les applications.

Note sur les équations de la Dynamique; par M. J. LIOUVILLE.

(Communiquée au Bureau des Longitudes, le mercredi 29 juin 1853.)

» Considérons les équations de la Dynamique, ou plutôt le système plus général » d'équations différentielles que voici :

$$\frac{dx}{dt} = \frac{dV}{dx'},\quad \frac{dy}{dt} = \frac{dV}{dy'},\ldots,\quad \frac{dz}{dt} = \frac{dV}{dz'},$$

$$\frac{dx'}{dt} = -\frac{dV}{dx},\quad \frac{dy'}{dt} = -\frac{dV}{dy},\ldots,\quad \frac{dz'}{dt} = -\frac{dV}{dz},$$

» V étant une fonction donnée quelconque de $t, x, y,\ldots, x', y',\ldots$; admettons » qu'on ait trouvé la moitié des intégrales de ce système

$$\alpha = \varphi(t, x, y,\ldots, x', y',\ldots),$$
$$\beta = \psi(t, x, y,\ldots, x', y',\ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\gamma = \varpi(t, x, y,\ldots, x', y',\ldots),$$

» et supposons que toutes les quantités (β, α), $(\gamma, \beta),\ldots$, $(\alpha, \gamma),\ldots$, où nous » prenons

$$(\alpha, \beta) = \frac{d\alpha}{dx'}\cdot\frac{d\beta}{dx} - \frac{d\alpha}{dx}\cdot\frac{d\beta}{dx'} + \frac{d\alpha}{dy'}\cdot\frac{d\beta}{dy} - \frac{d\alpha}{dy}\cdot\frac{d\beta}{dy'} + \ldots + \frac{d\alpha}{dz'}\cdot\frac{d\beta}{dz} - \frac{d\alpha}{dz}\cdot\frac{d\beta}{dz'},$$

» soient égales à zéro. Je dis que si, cela étant, les équations intégrales trouvées peu- » vent fournir les valeurs de $x', y',\ldots, z'$, en $t, x, y,\ldots, z$, la quantité

$$x'dx + y'dy + \ldots + z'dz - V\,dt$$

» sera une différentielle exacte par rapport à $x, y,\ldots, z, t$. Soit dS cette différen- » tielle; S contiendra les constantes $\alpha, \beta,\ldots, \gamma$; et le système complet des inté- » grales de nos équations différentielles sera

$$\frac{dS}{dx} = x',\quad \frac{dS}{dy} = y',\ldots,\quad \frac{dS}{dz} = z',$$

$$\frac{dS}{d\alpha} = -\alpha',\quad \frac{dS}{d\beta} = -\beta',\ldots,\quad \frac{dS}{d\gamma} = -\gamma',$$

» $\alpha', \beta',\ldots, \gamma'$, étant de nouvelles constantes arbitraires.

» Il n'y aura à cet énoncé qu'un très-léger changement à faire si les intégrales don- » nées $\alpha = \varphi$, $\beta = \psi,\ldots$, $\gamma = \varpi$, au lieu de fournir $x', y',\ldots, z'$, fournissaient » les valeurs d'une moitié quelconque des quantités $x, y,\ldots, z, x', y',\ldots, z'$, » en fonction de t et des autres, pourvu que dans cette moitié il n'y ait que des lettres » de nom différent comme $x', y',\ldots, z$. J'ai donné de longs développements sur » toutes ces questions dans mes leçons au Collége de France.

» Le 29 juin 1853.

» *Signé* J. LIOUVILLE. »

P. DAUSSY.

Liouville integrable systems (1853)

Let the dimension of the phase space be $2n$. The system is Liouville integrable if it has n independent *globally defined* conserved quantities I_i which mutually Poisson commute

$$\{I_i, I_j\} = 0$$

$$\{f,h\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial p_j} \frac{\partial h}{\partial q^j} - \frac{\partial h}{\partial p_j} \frac{\partial f}{\partial q^j} \right)$$

and H is a function of I_i

A Liouville integrable system can be solved by "quadrature"

- Find *canonical* transformation $(p_i, q_i) \to (I_i, \theta_i)$. Equations of motion in new coordinates

$$\{f,h\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial I_j} \frac{\partial h}{\partial \theta^j} - \frac{\partial h}{\partial I_j} \frac{\partial f}{\partial \theta^j} \right)$$

$$\begin{aligned} \dot{I}_i &= \{H, I_i\} = 0 \\ \dot{\theta}_i &= \{H, \theta_i\} = \frac{\partial H}{\partial I_i} \equiv \underbrace{\omega_i(I_j)}_{\text{time independent}} \end{aligned}$$

- Solution

$$I_i(t) = I_i(0)\,, \qquad \theta_i(t) = \theta_i(0) + t\omega_i$$

Arnold-Liouville theorem

$$\{f_i, f_j\} = 0$$

$$\mathscr{P}_c = \{x \in \mathscr{P} : \ f_i(x) = c_i, \quad i = 1, \ldots, N\}$$

where c_i are constants. Assume that functions f_i are independent on $\mathscr{P}_c$, which means that the 1-forms df_i are linearly independent at each point of $\mathscr{P}_c$. Then

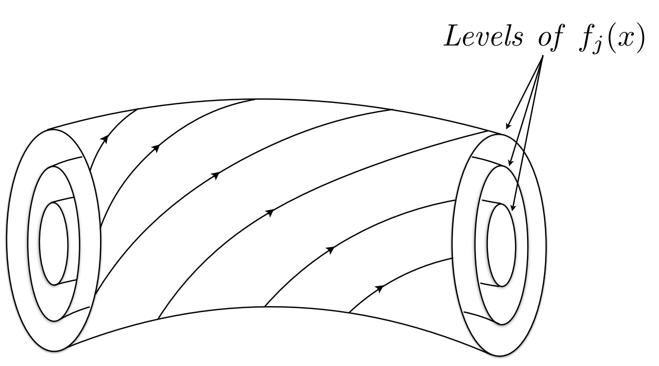
1. $\mathscr{P}_c$ is a smooth manifold invariant under the hamiltonian flow with $H = H(f_i)$.
2. If $\mathscr{P}_c$ is compact and connected then it is diffeomorphic to the N-dimensional torus

$$\mathbb{T}^N = \{(\varphi_1, \ldots, \varphi_N) \bmod 2\pi\}.$$

3. The motion on $\mathscr{P}_c$ under H is conditionally periodic, that is,

$$\frac{d\varphi_i}{dt} = \omega_i(c).$$

4. The equations of motion can be integrated by quadratures.



Arnold-Liouville theorem

$$f_j(p,q) = c_j \quad \longrightarrow \quad p_j = p_j(c,q)$$

Action variables

$$I_j(c) = \frac{1}{2\pi}\oint_{\gamma_j} p_i(q,c)\,dq_i = \frac{1}{2\pi}\oint_{\gamma_j} \alpha \qquad \Longrightarrow \qquad c_j = c_j(I)$$

fundamental cycles

Generating function of the canonical transformation

new momenta

old coordinates

$$S(I,q) = \int_{q_0}^{q} p_i(\tilde{q},I)\,d\tilde{q}_i$$

$$p_j = \frac{\partial S}{\partial q_j} \quad \rightarrow \quad p_j = p_j(I,q)\,, \qquad\qquad \theta_j = \frac{\partial S}{\partial I_j} \quad \rightarrow \quad \theta_j = \theta_j(I,q)$$

angle coordinates

Exercise

Show that $(p_j, q_j) \rightarrow (I_j, \theta_j)$ is a canonical transformation

Exercise

Find the action-angle variables for the harmonic oscillator

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

Part I

Integrable structure of gauge theory

Conformal symmetry

d-dimensional Minkowski space with the metric $\quad \eta_{ab} = (-1, 1, ..., 1)\,, \qquad a, b = 0, ..., d-1\,.$

The conformal group is generated by a set of generators

$$
\begin{array}{rcl}
\mathrm{M}_{ab} & - & \text{Lorentz rotations} \\
\mathrm{P}_a & - & \text{translations} \\
\mathrm{K}_a & - & \text{conformal boosts} \\
\mathrm{D} & - & \text{dilatation}
\end{array}
$$

subject to the following non-trivial commutation relations

$$
\begin{aligned}
[\mathrm{M}_{ab}, \mathrm{M}_{cd}] &= i(\eta_{ac}\mathrm{M}_{bd} - \eta_{bc}\mathrm{M}_{ad} - \eta_{ad}\mathrm{M}_{bc} + \eta_{bd}\mathrm{M}_{ac})\,, \\
[\mathrm{M}_{ab}, \mathrm{P}_c] &= i(\eta_{ac}\mathrm{P}_b - \eta_{bc}\mathrm{P}_a)\,, \qquad [\mathrm{M}_{ab}, \mathrm{K}_c] = i(\eta_{ac}\mathrm{K}_b - \eta_{bc}\mathrm{K}_a)\,, \\
[\mathrm{D}, \mathrm{P}_a] &= i\mathrm{P}_a\,, \qquad [\mathrm{D}, \mathrm{K}_a] = -i\mathrm{K}_a\,, \\
[\mathrm{K}_a, \mathrm{P}_b] &= -2i\mathrm{M}_{ab} - 2i\eta_{ab}\mathrm{D}\,.
\end{aligned}
$$

The Lorentz generators form a Lie algebra $\mathfrak{so}(d-1,1)$. In unitary representations of the conformal algebra the generators are realized by hermitian operators

$$
\mathrm{M}^{\dagger}_{ab} = \mathrm{M}_{ab}\,, \quad \mathrm{P}^{\dagger}_{a} = \mathrm{P}_{a}\,, \quad \mathrm{K}^{\dagger}_{a} = \mathrm{K}_{a}\,, \quad \mathrm{D}^{\dagger} = \mathrm{D}\,.
$$

In $d = 4$ the conformal algebra is $\mathfrak{so}(4,2) \simeq \mathfrak{su}(2,2)$

Field representations of the conformal algebra

$$|\Phi_I\rangle = \lim_{x\to 0} \Phi_I(x)|0\rangle \qquad \leftarrow \qquad \text{conformal state}$$

The action of algebra generators on a conformal state

$$\mathrm{K}_a|\Phi_I\rangle = 0, \quad \mathrm{D}|\Phi_I\rangle = i\Delta|\Phi_I\rangle, \quad \mathrm{M}_{ab}|\Phi_I\rangle = |\Phi_J\rangle(\Lambda_{ab})^J{}_I$$

$$\mathrm{Coset} = \mathrm{SO(d,2)}/\{\mathrm{K}_a, \mathrm{D}, \mathrm{M}_{ab}\}$$

Space of the conformal irrep span by P_a (they act like usual creation operators in the quantum mechanics):

$$\mathrm{P}_{a_1}\dots\mathrm{P}_{a_n}|\Phi_I\rangle$$

Generating function of conformal states (physical field on space-time)

$$\Phi_I(x)|0\rangle \quad = \quad \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}x^{a_1}\dots x^{a_n}\mathrm{P}_{a_1}\dots\mathrm{P}_{a_n}|\Phi_I\rangle = e^{-ix^a\mathrm{P}_a}|\Phi_I\rangle$$

not normalizable

Relation to positive energy irreps

The standard treatment of unitary irreps is based on another coset:

$$\mathrm{SO(d,2)/SO(d) \times SO(2)}$$

Set of generators

$$\begin{aligned}
&\mathrm{M}_{rs} \Leftarrow \text{generators of } \mathrm{SO(d)} \\
&\mathrm{E}_r^{\pm} = \mathrm{M}_{d+1,r} \pm i\mathrm{M}_{0r}\,, \qquad r = 1, \ldots, d \\
&\mathrm{H} = -\tfrac{1}{2}(\mathrm{P}_0 + \mathrm{K}_0) \Leftarrow \text{conformal Hamiltonian}
\end{aligned}$$

Algebra

$$\begin{aligned}
[\mathrm{E}_r^-, \mathrm{E}_s^+] &= 2\delta_{rs}\mathrm{H} - 2i\mathrm{M}_{rs}\,, \\
[\mathrm{H}, \mathrm{E}_r^{\pm}] &= \pm\mathrm{E}_r^{\pm}\,, \qquad [\mathrm{E}_r^+, \mathrm{E}_s^+] = 0
\end{aligned}$$

Relation to positive energy irreps

Conformal states are related

$$|\Psi_I\rangle = e^{-\frac{\pi}{2}\mathrm{M}_{0d}}|\Phi_I\rangle\,, \qquad \mathrm{H}|\Psi_I\rangle = \Delta|\Psi_I\rangle$$

$$e^{\frac{\pi}{4}(\mathrm{P}_0-\mathrm{K}_0)}(-i\mathrm{D})e^{-\frac{\pi}{4}(\mathrm{P}_0-\mathrm{K}_0)} = \mathrm{H}$$

Exercise
Prove!

Conformal primary state is defined

$$\mathrm{E}^-_r|\Psi_I\rangle = 0\,, \qquad \mathrm{H}|\Psi_I\rangle = \Delta|\Psi_I\rangle$$

and all its descendants span a basis of positive energy irrep:

$$\mathrm{E}^+_{r_1}\ldots\mathrm{E}^+_{r_n}|\Psi_I\rangle$$

Conformal states (normalizable) form a positive definite matrix:

$$\langle\Psi_I|\Psi_J\rangle$$

Superconformal symmetry

Add new generators

$$
\begin{array}{ccl}
\mathrm{Q}^i_\alpha, \bar{\mathrm{Q}}_{i\dot\alpha}, \mathrm{S}^\alpha_i, \bar{\mathrm{S}}^{i\dot\alpha} & - & \text{Supercharges} \quad i = 1, \ldots, \mathcal{N} \\
\mathrm{R}^i{}_j & - & \text{Internal} \quad (\mathrm{R}-) \text{ symmetry} \quad \mathrm{U}(\mathcal{N})
\end{array}
$$

$$
\mathrm{P}_{\alpha\dot\alpha} = \sigma^a_{\alpha\dot\alpha}\mathrm{P}_a \qquad\qquad \mathrm{K}^{\dot\alpha\alpha} = (\bar\sigma^a)^{\dot\alpha\alpha}\mathrm{K}_a
$$

$$
\begin{aligned}
\{\mathrm{Q}^i_\alpha, \bar{\mathrm{Q}}_{j\dot\alpha}\} &= 2\delta^i{}_j \mathrm{P}_{\alpha\dot\alpha}\,, \\
\{\bar{\mathrm{S}}^{i\dot\alpha}, \mathrm{S}^\alpha_j\} &= 2\delta^i{}_j \mathrm{K}^{\dot\alpha\alpha}\,,
\end{aligned}
\qquad
\begin{aligned}
\{\mathrm{Q}^i_\alpha, \mathrm{S}^\beta_j\} &= 4\delta^i{}_j(\mathrm{M}_\alpha{}^\beta - \frac{i}{2}\delta_\alpha{}^\beta \mathrm{D}) - 4\delta_\alpha{}^\beta \mathrm{R}^i{}_j \\
[\mathrm{R}^i{}_j, \mathrm{R}^k{}_l] &= \delta^k{}_j \mathrm{R}^i{}_l - \delta^i{}_l \mathrm{R}^k{}_j
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathrm{R}^i{}_j, \mathrm{Q}^k_\alpha] &= \delta^k{}_j \mathrm{Q}^i_\alpha - \frac{1}{4}\delta^i{}_j \mathrm{Q}^k_\alpha \\
[\mathrm{R}^i{}_j, \mathrm{S}^\alpha_k] &= -\delta^i{}_k \mathrm{S}^\alpha_j + \frac{1}{4}\delta^i{}_j \mathrm{S}^\alpha_k
\end{aligned}
$$

For $\mathcal{N} = 4$ we can impose $\mathrm{R}^i{}_i \equiv 0$. Therefore R-symmetry is $\mathfrak{su}(4)$

The superconformal algebra is the $\mathfrak{psu}(2,2|4)$ Lie superalgebra

Superconformal primary state

$$|\text{hws}\rangle = |\Delta; s_1, s_2; a_1, a_2, a_3\rangle$$

$$\begin{array}{rcl}
\Delta & \longrightarrow & \text{Conformal dimension} \\
s_1, s_2 & \longrightarrow & \text{Lorentz spins} \\
a_1, a_2, a_3 & \longrightarrow & \text{Dynkin labels of the } \mathfrak{su}(4) \text{ irrep}
\end{array}$$

$$\mathrm{K}_a|\text{hws}\rangle = 0\,, \qquad \mathrm{S}_i^{\alpha}|\text{hws}\rangle = \bar{\mathrm{S}}^{i\dot{\alpha}}|\text{hws}\rangle = 0\,, \qquad \mathrm{D}|\text{hws}\rangle = i\Delta|\text{hws}\rangle$$

Complete basis is generated from the ordered span

$$\prod_{i,j,\alpha,\dot{\alpha}} (\mathrm{Q}_{\alpha}^{i})^{n_{i\alpha}} (\bar{\mathrm{Q}}_{j\dot{\alpha}})^{\bar{n}_{j\dot{\alpha}}} |\text{hws}\rangle$$

Dimension of a generic long multiplet

$$\dim = 2^{16} \dim(a_1, a_2, a_3)(2s_1 + 1)(2s_2 + 1)$$

Superconformal primary state

$$|\mathrm{hws}\rangle = |\Delta; s_1, s_2; a_1, a_2, a_3\rangle$$

$$\begin{array}{rcl} \Delta & \longrightarrow & \text{Conformal dimension} \\ s_1, s_2 & \longrightarrow & \text{Lorentz spins} \\ a_1, a_2, a_3 & \longrightarrow & \text{Dynkin labels of the } \mathfrak{su}(4) \text{ irrep} \end{array}$$

$$\mathrm{K}_a|\mathrm{hws}\rangle = 0\,, \qquad \mathrm{S}_i^{\alpha}|\mathrm{hws}\rangle = \bar{\mathrm{S}}^{i\dot{\alpha}}|\mathrm{hws}\rangle = 0\,, \qquad \mathrm{D}|\mathrm{hws}\rangle = i\Delta|\mathrm{hws}\rangle$$

Complete basis is generated from the ordered span

$$\prod_{i,j,\alpha,\dot{\alpha}} (\mathrm{Q}^i_\alpha)^{n_{i\alpha}} (\bar{\mathrm{Q}}_{j\dot{\alpha}})^{\bar{n}_{j\dot{\alpha}}} |\mathrm{hws}\rangle$$

Dimension of a generic long multiplet

$$\dim = 2^{16} \dim(a_1, a_2, a_3)(2s_1 + 1)(2s_2 + 1)$$

Relation to positive energy irreps

$\mathfrak{su}(2,2|4)$ has a realization in terms of 8×8 supermatrices with zero supertrace.

H_i with $i=1,\ldots,7$ $\longrightarrow$ Cartan generators

$\mathrm{E}_i^{\pm}$ with $i=1,\ldots,7$ $\longrightarrow$ simple positive and negative root vectors

$$[\mathrm{H}_i,\mathrm{H}_j]=0\,,\qquad [\mathrm{E}_i^{+},\mathrm{E}_j^{-}]=\delta_{ij}\mathrm{H}_j\,,\qquad [\mathrm{H}_i,\mathrm{E}_j^{\pm}]=\pm a_{ij}\mathrm{E}_j^{\pm}$$

Chevalley basis

$$a_{ij}=-\mathrm{str}(\mathrm{H}_i\mathrm{H}_j)\quad\Longrightarrow\quad a=\begin{pmatrix} -2 & +1 & 0 & 0 & 0 & 0 & 0\\ +1 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & -1 & +2 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & +2 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & +2 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & +1\\ 0 & 0 & 0 & 0 & 0 & +1 & -2 \end{pmatrix}$$

Cartan matrix

Matrix realization

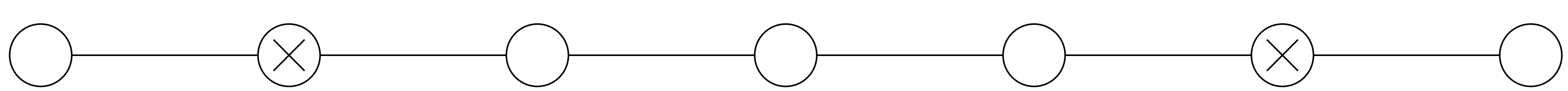
$$H_3 = \begin{pmatrix} 0_{4\times4} & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0_{4\times4} & & & & \\ & 0 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}, \quad H_5 = \begin{pmatrix} 0_{4\times4} & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}$$

$$H_1 = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0_{4\times4} \end{pmatrix}, \quad H_7 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0_{4\times4} \end{pmatrix}.$$

$H_{3,4,5}$ form the Cartan subalgebra of $\mathfrak{su}(4)$

H_1 and H_7 are the Cartan generators of $\mathfrak{su}(2) \times \mathfrak{su}(2) \subset \mathfrak{su}(2,2)$.

$$H_2 = \begin{pmatrix} 0 & & & & & & & \\ & 1 & & & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 1 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}, \qquad H_6 = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & -1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & -1 \end{pmatrix}$$



The Dynkin graph. The hatched roots are fermionic and null.

Matrix realization

For matrix generators $\longrightarrow$ $\mathrm{H}_1 + 2\mathrm{H}_2 + \mathrm{H}_3 - \mathrm{H}_5 - 2\mathrm{H}_6 - \mathrm{H}_7 = \mathbb{1}_{8\times 8}$

$$\mathrm{C} = \mathrm{H}_1 + 2\mathrm{H}_2 + \mathrm{H}_3 - \mathrm{H}_5 - 2\mathrm{H}_6 - \mathrm{H}_7$$

central element

Representations of $\mathfrak{psu}(2,2|4)$ are those of $\mathfrak{su}(2,2|4)$ for which the central charge vanishes

$$\mathrm{H}_1 + 2\mathrm{H}_2 + \mathrm{H}_3 = \mathrm{H}_5 + 2\mathrm{H}_6 + \mathrm{H}_7$$

Matrix realization

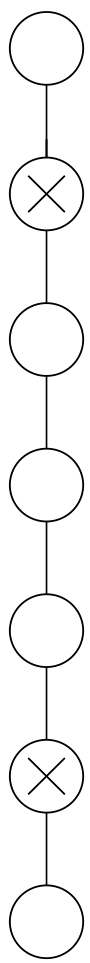
$$\mathrm{D} = \sum_{i=2}^{6} \mathrm{H}_i + \frac{1}{2}(\mathrm{H}_1 + \mathrm{H}_7) \qquad \longrightarrow \qquad \mathrm{D} = \tfrac{1}{2}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 0_{4\times 4} \end{pmatrix}$$

$$\mathrm{H}_2 = \frac{1}{2}\mathrm{D} + \frac{1}{2}\mathrm{C} - \frac{3}{4}\mathrm{H}_3 - \frac{1}{2}\mathrm{H}_4 - \frac{1}{4}\mathrm{H}_5 - \frac{1}{2}\mathrm{H}_1$$

$$\mathrm{H}_6 = \frac{1}{2}\mathrm{D} - \frac{1}{2}\mathrm{C} - \frac{1}{4}\mathrm{H}_3 - \frac{1}{2}\mathrm{H}_4 - \frac{3}{4}\mathrm{H}_5 - \frac{1}{2}\mathrm{H}_7$$

$$\mathrm{E}_i^{+}|\mathrm{hws}\rangle = 0 \qquad \Longleftrightarrow \qquad \mathrm{K}_\mu|\mathrm{hws}\rangle = \mathrm{S}_\alpha^i|\mathrm{hws}\rangle = \bar{\mathrm{S}}_{i\dot{\alpha}}|\mathrm{hws}\rangle = 0$$

Weights of a superconformal primary state



$$
\begin{aligned}
\mathrm{H}_1|\mathrm{hws}\rangle &= -s_1|\mathrm{hws}\rangle\,,\\
\mathrm{H}_2|\mathrm{hws}\rangle &= \frac{1}{2}\Big(\Delta-\frac{3}{2}q_1-p-\frac{1}{2}q_2+s_1\Big)|\mathrm{hws}\rangle\,,\\
\mathrm{H}_3|\mathrm{hws}\rangle &= q_1|\mathrm{hws}\rangle\,,\\
\mathrm{H}_4|\mathrm{hws}\rangle &= p\;|\mathrm{hws}\rangle\,,\\
\mathrm{H}_5|\mathrm{hws}\rangle &= q_2|\mathrm{hws}\rangle\,,\\
\mathrm{H}_6|\mathrm{hws}\rangle &= \frac{1}{2}\Big(\Delta-\frac{1}{2}q_1-p-\frac{3}{2}q_2+s_2\Big)|\mathrm{hws}\rangle\,,\\
\mathrm{H}_7|\mathrm{hws}\rangle &= -s_2|\mathrm{hws}\rangle\,.
\end{aligned}
$$

Verma module

$$|\{n_i\}\rangle = (\mathrm{E}_1^-)^{n_1} \dots (\mathrm{E}_7^-)^{n_7} |\mathrm{hws}\rangle$$

$$\begin{aligned}
\mathrm{H}_1|\{n_i\}\rangle &= (2n_1 - n_2 - s_1)|\{n_i\}\rangle\,, \\
\mathrm{H}_2|\{n_i\}\rangle &= \frac{1}{2}\Big(- 2n_1 + 2n_3 + \Delta - \frac{3}{2}q_1 - p - \frac{1}{2}q_2 + s_1\Big)|\{n_i\}\rangle\,, \\
\mathrm{H}_3|\{n_i\}\rangle &= (n_2 - 2n_3 + n_4 + q_1)|\{n_i\}\rangle\,, \\
\mathrm{H}_4|\{n_i\}\rangle &= (n_3 - 2n_4 + n_5 + p)|\{n_i\}\rangle\,, \\
\mathrm{H}_5|\{n_i\}\rangle &= (n_4 - 2n_5 + n_6 + q_2)|\{n_i\}\rangle\,, \\
\mathrm{H}_6|\{n_i\}\rangle &= \frac{1}{2}\Big(2n_5 - 2n_7 + \Delta - \frac{1}{2}q_1 - p - \frac{3}{2}q_2 + s_2\Big)|\{n_i\}\rangle\,, \\
\mathrm{H}_7|\{n_i\}\rangle &= (-n_6 + 2n_7 - s_2)|\{n_i\}\rangle\,.
\end{aligned}$$

Note that only the second and the six roots increase the dimension:

$$\mathrm{D}|\{n_i\}\rangle = \tfrac{1}{2}(2\Delta + n_2 + n_6)|\{n_i\}\rangle$$

There are special representations that contain less states than representations of generic type. The highest weight of such representations obey additional constraints.

Unitarity

There are three series of unitary irreducible representations of the superconfrormal algebra which are usually called A), B) and C). They are formulated as follows

- Series A)

$$\begin{aligned}\Delta &\geq 2 + s_1 + \frac{3}{2}q_1 + p + \frac{1}{2}q_2\,,\\ \Delta &\geq 2 + s_2 + \frac{1}{2}q_1 + p + \frac{3}{2}q_2\end{aligned}$$

These two conditions must be simulteniously satisfied which implies

$$\Delta \geq 2 + s_1 + s_2 + q_1 + p + q_2\,.$$

- Series B)

$$\begin{aligned}\Delta &= \frac{3}{2}q_1 + p + \frac{1}{2}q_2\,, \quad s_1 = 0\\ \Delta &= \frac{1}{2}q_1 + p + \frac{3}{2}q_2\,, \quad s_2 = 0\end{aligned}$$

- Series C)

$$\Delta = p + 2q\,, \quad q \equiv q_1 = q_2, \quad s_1 = 0 = s_2\,.$$

The series A) provides a bound for the conformal dimension Δ. The series B) contains $\frac{1}{8}$-BPS excitations and the series C) comprises $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS operators. Note that the series C) contains only spinless states.

SO(6) charges and Dynkin labels

$$\underbrace{[q_1, p, q_2]}_{\mathrm{SU}(4)} \quad \Longleftrightarrow \quad \underbrace{\begin{aligned} J_1 &= \tfrac{1}{2}(q_1 + 2p + q_2) \\ J_2 &= \tfrac{1}{2}(q_1 + q_2) \\ J_3 &= \tfrac{1}{2}(q_2 - q_3) \end{aligned}}_{\mathrm{SO}(6)}$$

$$\mathrm{Tr}\big(Z_1^{J_1} Z_2^{J_2} Z_3^{J_3}\big) \quad \longrightarrow \quad [J_2 - J_3, J_1 - J_2, J_2 + J_3]$$