Modularity from Monodromy

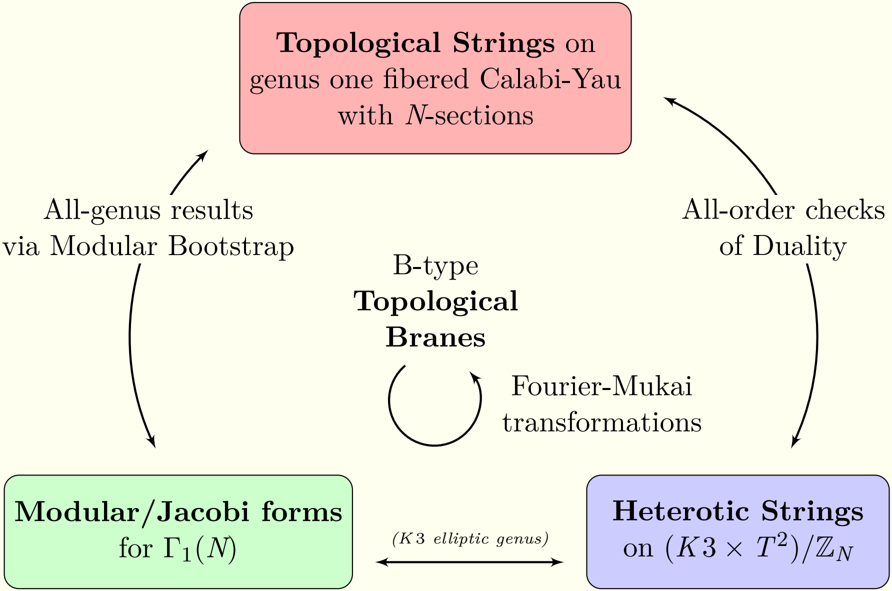
DESY Theory Workshop 2019

Thorsten Schimannek

based on [1902.08215], T.S.

and [190x.xxx], C. F. Cota, A. Klemm, T.S.

25.9.2019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genus one fibered Calabi-Yau
with N-sections
All-genus results
via Modular Bootstrap
All-order checks
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B-type
Topological
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for $\Gamma_1(N)$
($K3$ elliptic genus)
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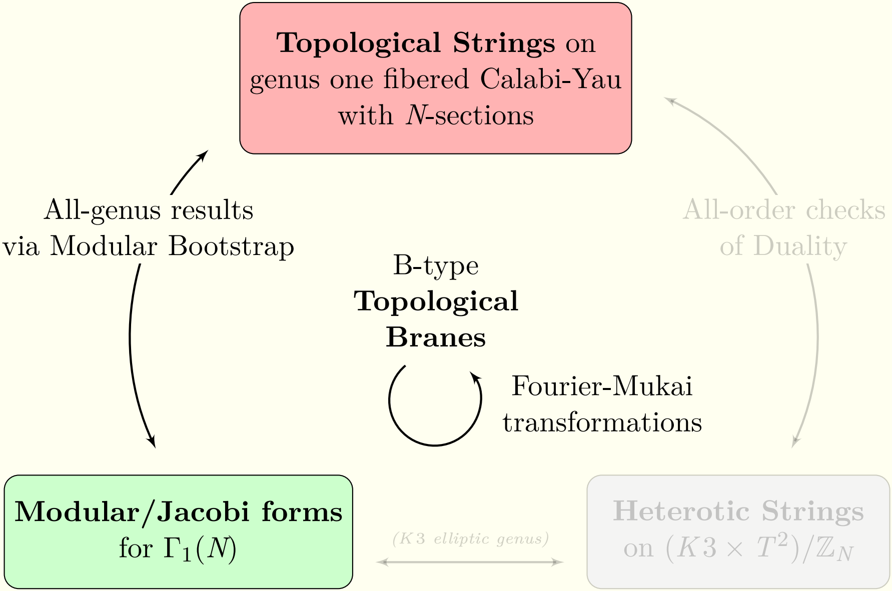
on $(K3 \times T^2)/\mathbb{Z}_N$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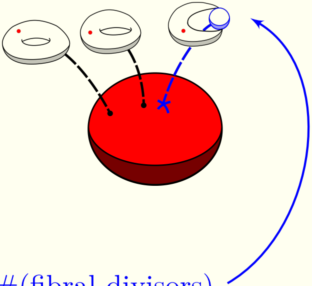
Consider *elliptically fibered* Calabi-Yau 3-fold $\pi : X \to B$

- Generic fibers of X are tori
- Fibers degenerate over *discrimininant locus* $\subset B$

e.g. I_2 singularity



Theorem (Shioda-Tate-Wazir):

$$h^{1,1}(X) = h^{1,1}(B) + \#(\text{sections}) + \#(\text{fibral divisors})$$

Analogous relation if X has no section but only N-sections. It is then called *genus one fibered*.

A **lattice Jacobi form** $\phi(\tau, \vec{z})$ of weight k and index matrix C

- Is periodic under $\tau \to \tau + 1$ and $z_i \to z_i + 1$
- Satisfies *"Modular Transformation Law" (MTL)*

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{\vec{z}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(\frac{2\pi i c z^t C z}{c\tau + d}\right) \phi(\tau, \vec{z})$$

- MTL **implies** *"Elliptic Transformation Law" (ETL)*

$$\phi\left(\tau, \vec{z} + \vec{\lambda}\tau\right) = \exp\left(-2\pi i \left[\lambda^t C \lambda \tau + \lambda^t C z + z^t C \lambda\right]\right) \phi(\tau, z)$$

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Expand **Topological String partition function** as

$$Z_{\text{top}}(\vec{t},\lambda) = Z_0(\tau,\vec{m},\lambda)\left(1+\sum_{\beta\in H_2(B,\mathbb{Z})} Z_\beta(\tau,\vec{m},\lambda)\,Q^\beta\right)$$

Conjecture: $Z_\beta(\tau,\underline{m},\lambda)$ are *lattice Jacobi forms* of weight 0

[Huang,Katz,Klemm'15], **[DelZotto,Lockhart'16]**

[DelZotto,Gu,Huang,Kashani-Poor,Klemm,Lockhart'17], 2x**[Lee,Lerche,Weigand'18]**

- Index with respect to λ is $\frac{1}{2}\beta\cdot(\beta - c_1(B))$
- Index matrix w.r.t. volumes of fibral curves

$$C^\beta_{ab} = -\frac{1}{2}\beta\cdot\pi(D_a\cdot D_b)$$

Together with a smart ansatz this led to all genus results.

Expand **Topological String** ... as

$\vec{m}$: volumes of fibral curves

$$Z_{\text{top}}(\vec{t}, \lambda) = Z_0(\tau, \vec{m}, \lambda) \left(1 + \sum_{\beta \in H_2(B,\mathbb{Z})} Z_\beta(\tau, \vec{m}, \lambda) \, Q^\beta \right)$$

τ: fiber volume

Conjecture: $Z_\beta(\tau, \underline{m}, \lambda)$ are *lattice Jacobi forms* of weight 0

[Huang,Katz,Klemm'15], **[DelZotto,**

[DelZotto,Gu,Huang,Kashani-Poor,

$$Q^\beta \;=\; \exp\left(2\pi i \beta \cdot t\right)$$

t^j: (shifted) volumes of base curves

- Index with respect to λ is $\frac{1}{2}\beta \cdot (\beta - c_1(B))$
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Expand **Topological String partition function** as

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How does this relate to monodromies?

Idea: *Interpret Kähler moduli as central charges of 2-branes*

1. Identify actions on D-branes that generate modular group
2. Show that those actions arise from monodromies
   They identify dual points in the stringy Kähler moduli space!
3. Use general automorphic properties of $Z_{\mathrm{top.}}$

A brief reminder of topological B-branes

- Representable by finite complexes of vector bundles

$$\mathcal{F}^\bullet = 0 \to V_1 \to \cdots \to V_n \to 0$$
$$(\cdots \to \text{brane} \to \text{anti-brane} \to \text{brane} \to \dots)$$

  Objects of *derived category of quasi-coherent sheaves $D^b(X)$*

- Central charge depends on Kähler class ω **[Iritani'09]**

$$Z(\mathcal{F}^\bullet) = \int_X e^\omega \, \Gamma(X) \, (\mathrm{ch}\, \mathcal{F}^\bullet)^\vee + (\text{instanton corr.})$$

  $\to$ For 2-branes just volume of support (no instantons!)

Homological mirror symmetry:

Monodromies in stringy Kähler moduli space
act as autoequivalences on $D^b(X)$! **[Kontsevich'96]**, **[Horja'99]**

Autoequivalences of $D^b(X)$ are always expressible as *Fourier-Mukai transformations* **[Orlov'96]**

$$\Phi_{\mathcal{E}} : \mathcal{F}^\bullet \mapsto R\pi_{1*}\left(\mathcal{E} \otimes_L L\pi_2^* \mathcal{F}^\bullet\right) , \quad \mathcal{E} \in D^b(X \times X) .$$

The sheaf $\mathcal{E}$ is called the *Fourier-Mukai kernel*.

Physical intuition:

- Kernel $\mathcal{E}$ corresponds to defect between two non-linear sigma models with target space X
- FM-transformation $\approx$ *fusing defect with boundary* e.g. **[Brunner,Jockers,Roggenkamp'08]**

Consider genus one fibration with N-sections and take *ideal sheaf of relative diagonal* $\mathcal{I}_\Delta$ on $X \times_B X$ as kernel:

$$U: \begin{cases} \tau & \mapsto \quad \tau/(1+N\tau) \\ m_i & \mapsto \quad m_i/(1+N\tau)\,, \qquad i=1,...,\mathrm{rk}(G) \\ Q^\beta & \mapsto \quad (-1)^{a_i}\exp\left(\frac{2\pi i}{1+N\tau}\cdot m^a m^b C^\beta_{ab} + \mathcal{O}(Q_i)\right) Q_i \end{cases}$$

With B-field shifts $\tau \to \tau + 1$, this generates action of $\Gamma_1(N)$!

Q^β **transforms like Jacobi form!** **[TS'19]**, **[Cota,Klemm,TS'19]**

$\to$ can derive elliptic transformation law for $Z_{\text{top.}}$

Does it correspond to a monodromy?

Consider Calabi-Yau with effective divisor J_b such that

$$J_b^3 = 0\,, \quad \text{and} \quad \int_M c_2(M) \cdot J_b = 36\,.$$

Denote by M_C the generic Conifold monodromy and by M_b the action of a shift of the B-field by J_b.

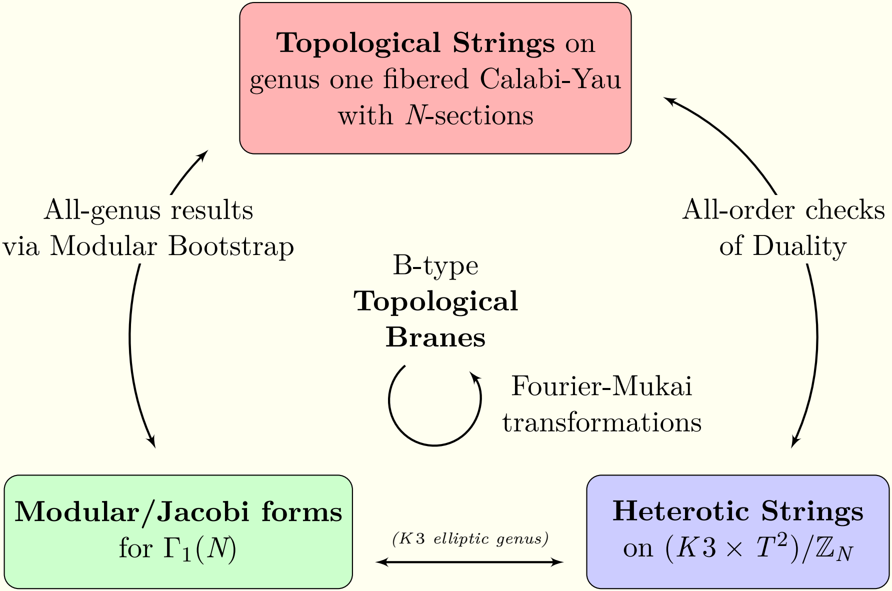
Then

$$M_{\mathrm{W}} = M_b^{-1} \cdot M_C \cdot M_b^{-1} \cdot M_C \cdot M_b^{-1} \cdot M_C \cdot M_b^3\,,$$

reproduces the action of the U-transformation!

[Cota,Klemm,TS'19]

Example: All genus one fibrations over $\mathbb{P}^2$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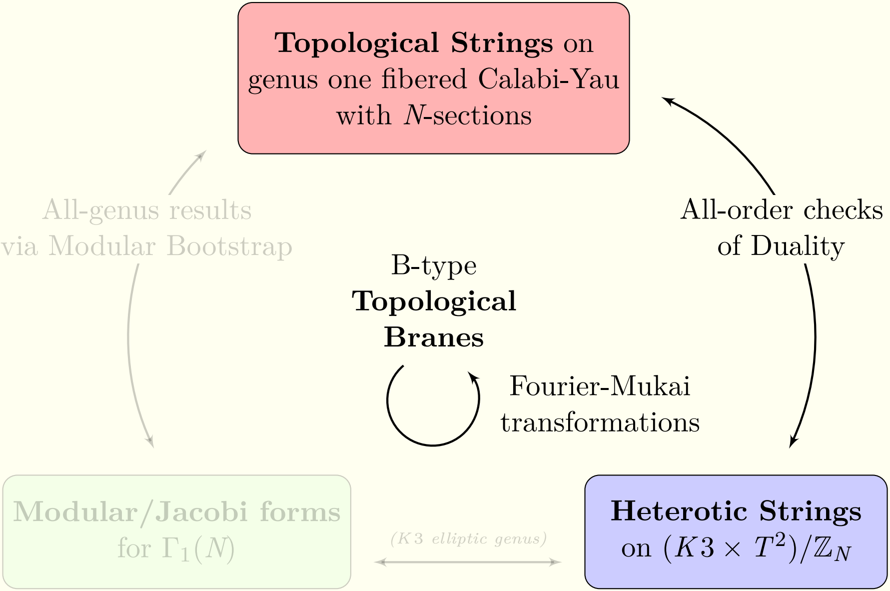

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How does this relate to Heterotic strings?

1. Heterotic on $K3 \times T^2$ dual to Type IIA on CY 3-fold **[Kachru,Vafa'95]**
2. Dualities of Heterotic strings on $K3 \times T^2$ contain $SL(2, \mathbb{Z})$
3. Realized as monodromies of dual Calabi-Yau **[Klemm,Lerche,Mayr'95]**
4. Dualities of Het. strings on $(K3 \times T^2)/\mathbb{Z}_N$ contain $\Gamma_1(N)$

$\rightarrow$ *Dual CY should be genus one fibration with N-sections*

We use modular bootstrap for genus one fibrations to perform arbitrary order checks against Heterotic calculation by **[Chattopadhyaya,David'16]**

Thank you for your attention!