

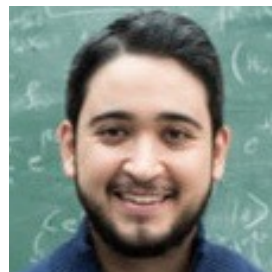
COMPLEXITY, QUANTUM FIELDS AND GEOMETRY

Michal P. Heller

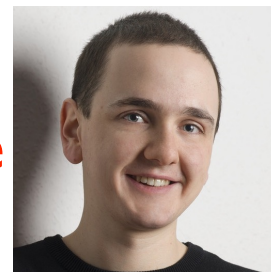
aei.mpg.de/GQFI

based on 1707.08582, 1807.07075, 1810.05151 with a focus on

1904.02713 with Hugo Camargo



, Johannes Knaute



, Ro Jefferson

and 191x.xxxxx with Mario Flory and Volker Schomerus

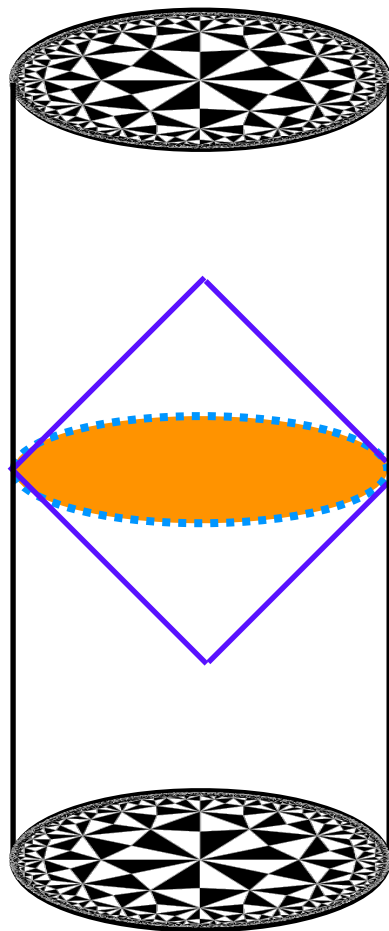
see also Hugo Camargo's TODAY 18:40 talk @ String & Math Physics

Quantum field theory meets gravity

Holography (AdS/CFT) provides perhaps the deepest realization of this slogan

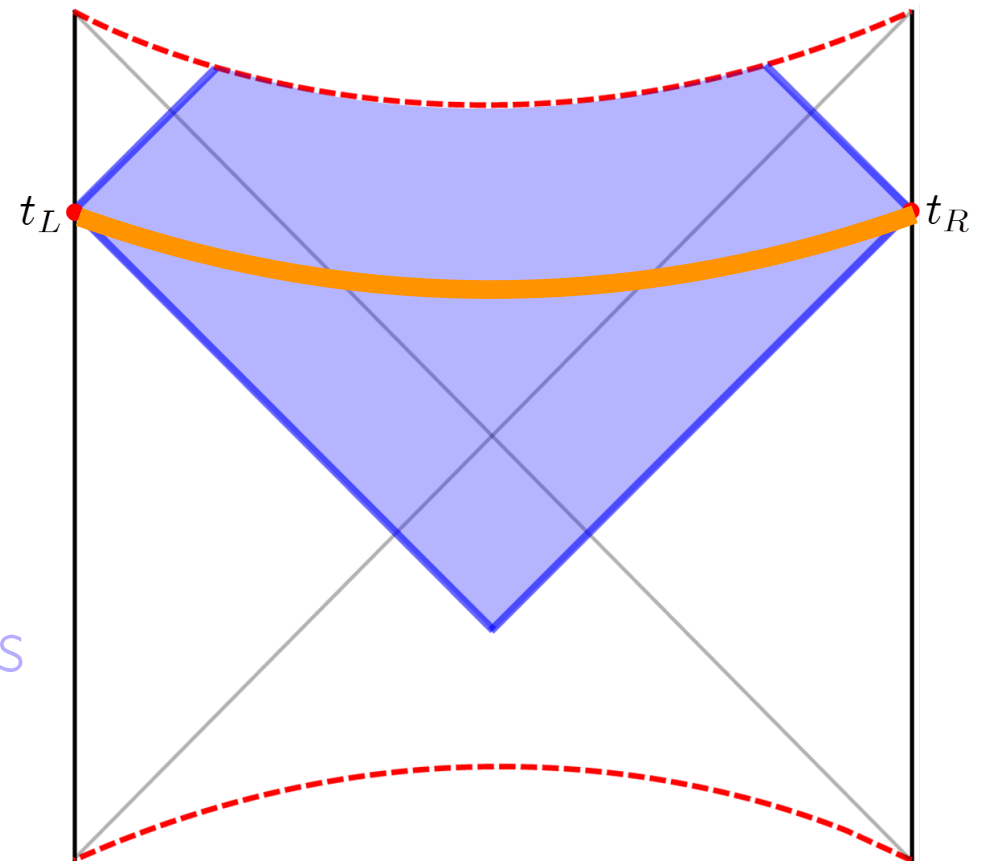
New and interesting: holographic complexity proposals

1402.5674 by Susskind, 1509.07876 by Brown et al., ...



$\mathcal{C}_V \sim$ Volume of codim-1
max volume bulk slice

$\mathcal{C}_A \sim$ Action in codim-0
bulk region with null bdries



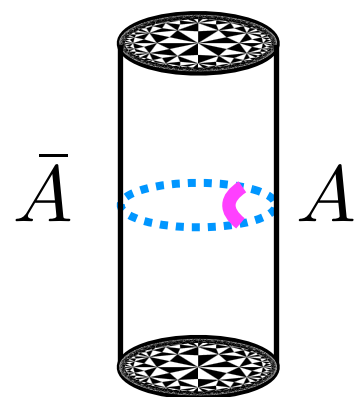
Motivation: what do holography complexity proposals stand for in hQFT?

Gravity, Quantum Fields and Information

Holographic complexity proposals provide a very interesting realization of deep relations between gravity and (quantum) information

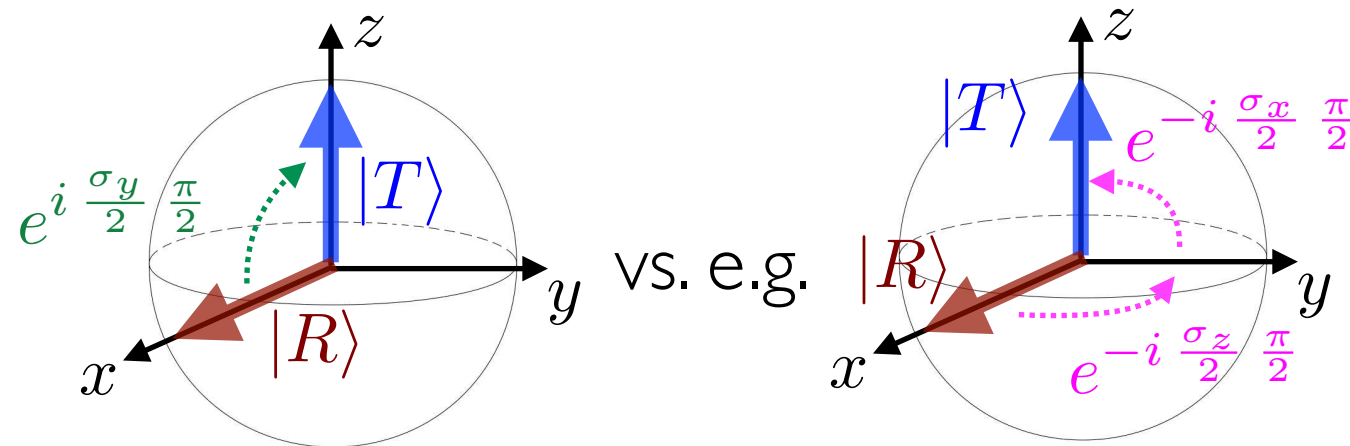
The best known such connection concerns certain codim-2 bulk surfaces and von Neumann entropy of density matrices of spatial regions on the boundary

Ryu-Takayanagi 0603001 [hep-th]



$$\frac{\text{AREA}}{4G_N} = -\text{tr} \rho_A \log \rho_A \text{ with (here) } \rho_A = \text{tr}_{\bar{A}} |0\rangle\langle 0|$$

Complexity is another quant-info notion concerning hardness of defining an operator or a state from another state with restricted resources, e.g.



naively: $\frac{\pi}{2} < \frac{\pi}{2} + \frac{\pi}{2}$

say y-rot is $\mathcal{O}(10)$
harder to do: $10 \times \frac{\pi}{2} > \frac{\pi}{2} + \frac{\pi}{2}$

Complexity for free quantum fields

To my taste, \mathcal{C}_V and \mathcal{C}_A looked a lot like calculating RT surfaces before first works on entanglement entropy in QFT (pioneers: 1980s, explosion > 2004)

What I just described is basically the geometric approach to complexity:

quant-ph/0502070 by Nielsen, ...

$$|T\rangle = \mathcal{P}e^{-\int_{\kappa_1}^{\kappa_2} d\kappa \sum_I \mathcal{O}_I Y^I(\kappa)} |R\rangle \longrightarrow \mathcal{C}_{L_1} \sim \min \left[\underbrace{\int_{\kappa_1}^{\kappa_2} d\kappa \sum_I \eta_I |Y^I(\kappa)|}_{\text{cost}} \right]$$

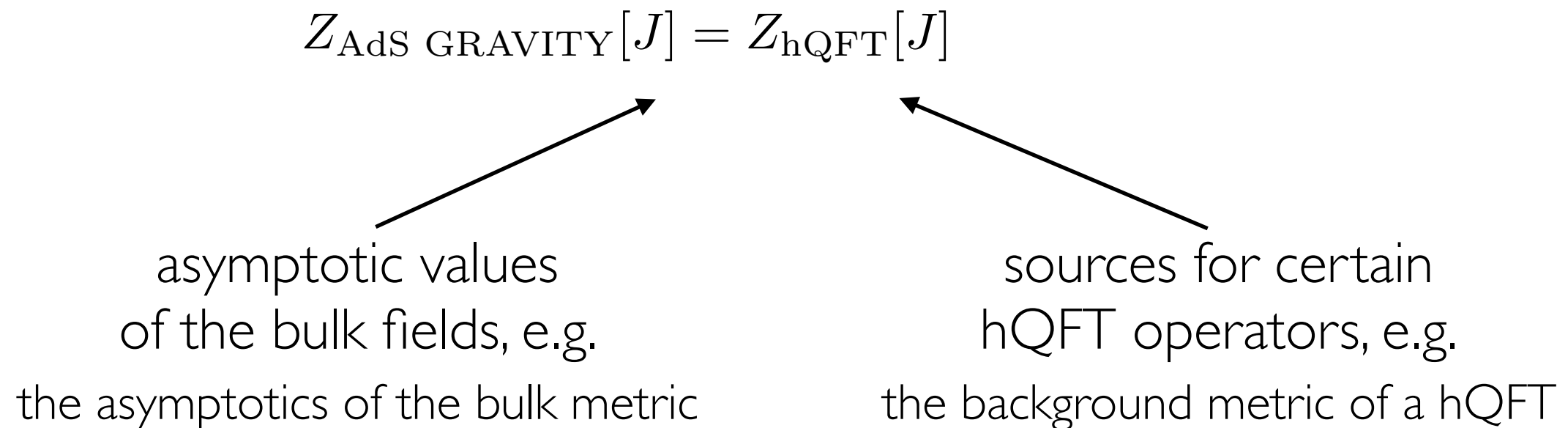
First applications to free QFTs on a lattice / cMERA regularization:

1707.08582, 1707.08570 by Jefferson & Myers, 1807.07075, 1810.05151, ...

- unitary gates (for bosons): $\mathcal{O}_I \sim i \phi_j \phi_l, \quad i \pi_j \pi_l \quad \text{and} \quad i \phi_{(j} \pi_{l)}$
- for very fine-tuned cost function one can get exact results ($\mathcal{C}_{\text{certain } L_2}$)
- circuits then also use very non-local gates, e.g. $\mathcal{O}_I = i \phi_{\text{here}} \phi_{\text{other galaxy}}$
- the good: similar divergence structures to holography (universality?)
- the bad (but expected): very different time dependence than holography

Vision: towards a holographic complexity

If one wants to derive something in holography from first principles (for example the RT proposal), it all eventually boils down to:



The studies I described so far are encouraging tips that holographic complexity proposals might have something to do with complexity

However, in order to derive the gravity dual to complexity, one should phrase things in terms of J 's and this requires some new insights

From Euclidean path-integrals...

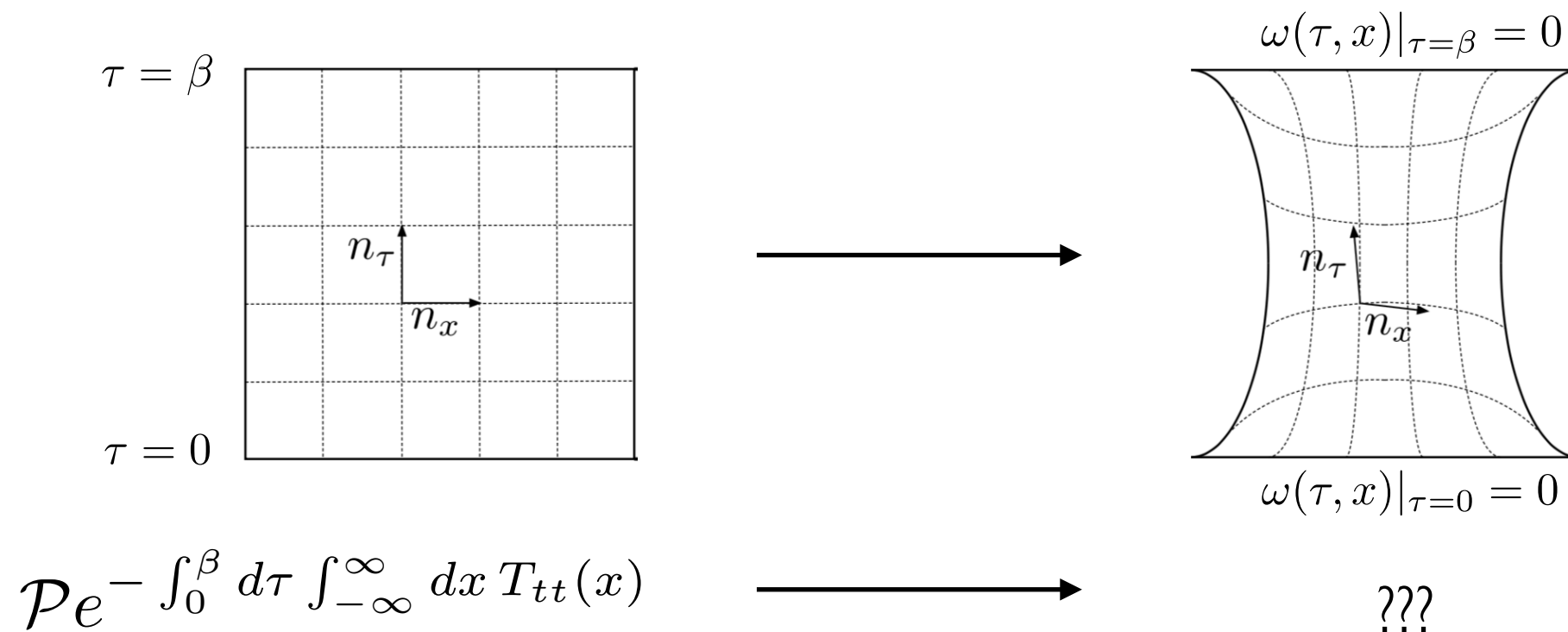
strongly motivated by the ideas from I703.00456 by the Kyoto group, ...

In the following we focus on CFT_2 (no assumptions on c) for now on a line

The object of interest will be $\rho_\beta = e^{-\beta H}$ and we will ignore normalization

Matrix elements of ρ_β are computed by Euclidean path integral on $[0, \beta] \times \mathbb{R}$

We preserve the operator when deforming the metric $e^{2\omega(\tau,x)}(d\tau^2 + dx^2)$:



... to non-unitary circuits...

using the results of [807.0250] by Milsted & Vidal

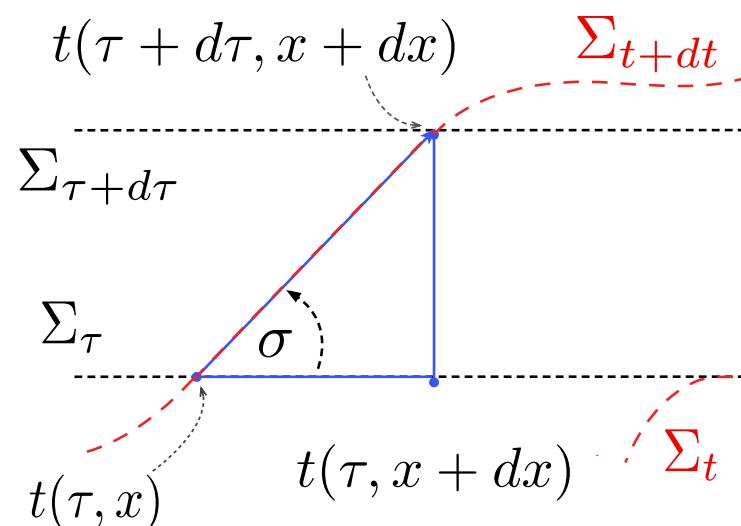
There is a very simple prescription from path integrals on

$$(a^2 + b^2) dt^2 + 2b dt dy + dy^2$$

to circuits involving components $T_{t_M t_M}$ and $T_{t_M y}$ on $-dt_M^2 + dy^2$:

$$\mathcal{P} \exp \left[- \int_{t_i}^{t_f} dt \int_{-\infty}^{\infty} dy \{ a(t, y) T_{t_M t_M}(y) + i b(t, y) T_{t_M y}(y) \} \right]$$

To use the idea about source configurations preserving the operator we have to transform between (τ, x) and (t, y) . We can express then. $a, b[\omega, \sigma]$:



... and their costs (complexity)

I 904.02713 with Hugo Camargo, Johannes Knaute and Ro Jefferson

We can now write various cost functions, but we cannot use t-derivatives, e.g.

$$\text{cost}_{L_1} \sim \int dt dy \frac{1}{\epsilon^2} (|a| + \epsilon \eta_{\partial a} |\partial_y a| + \epsilon \eta_{\partial b} |\partial_y b| + \dots)$$

UV cut-off in real space

$$\begin{aligned} & \int_0^\beta d\tau \int_{-\infty}^\infty dx \frac{e^\omega}{\epsilon^2} \left\{ e^\omega \right. \\ & \quad + \epsilon \eta_{\partial a} |(\dot{\omega} - \sigma') \sin \sigma + (\omega' + \dot{\sigma}) \cos \sigma| \\ & \quad \left. + \epsilon \eta_{\partial a} |(\omega' + \dot{\sigma}) \sin \sigma - (\dot{\omega} - \sigma') \cos \sigma| \right\} \end{aligned}$$

or

$$\text{cost} "L_1 - 2" \sim \int dt dy \frac{1}{\epsilon^2} \sqrt{a^2 + \epsilon^2 \eta_{(\partial a)^2} (\partial_y a)^2 + \epsilon^2 \eta_{(\partial b)^2} (\partial_y b)^2 + \dots}$$

$$\simeq \int d\tau dx \left\{ \frac{e^{2\omega}}{\epsilon^2} + \frac{1}{2} \eta_{(\partial a)^2} (\dot{\omega}^2 + \omega'^2) \right\} \leftarrow \text{the Liouville action!!!}$$

$$\text{total derivative} \quad + \frac{1}{2} \eta_{(\partial a)^2} (\dot{\sigma}^2 + \sigma'^2) + \eta_{(\partial a)^2} (\omega' \dot{\sigma} - \dot{\omega} \sigma') + \dots \}$$

Comments on complexity in CFT₂

The Liouville action was proposed in the aforementioned [1703.00456](#) as a cost function for “path integral optimizations” and [1904.02713](#) made it rather precise

An ongoing work with Mario Flory and Volker Schomerus:

- the story as described above generalizes straightforwardly to a cylinder (this is interesting since now there is gap $\longrightarrow \lim_{\beta \rightarrow \infty} e^{-\beta H} \sim |0\rangle\langle 0|$)

- the Liouville action was interesting because it is geometric:

$$\sim \int d^2\xi \sqrt{g} \Lambda + \int d^2\xi \sqrt{g} \int d^2\tilde{\xi} \sqrt{\tilde{g}} R \square^{-1} \tilde{R}$$

the cost functions treat t and y differently and are not necessarily geometric

- we like diffeomorphism-invariant objects and, building on [1807.04422](#) by [Caputa and Maghan](#), we want to see if one can construct geometric costs
- also, in the considerations we neglected normalization; can we interpret it?

Outlook

Since 2017 complexity has become a rather lively topic in QFT

We still know very little, but there are reasons for excitement

[1707.08582](#), [1807.07075](#), [1810.05151](#) and many other works by other authors

My own vision for this new field has always been “holographic”:



to prove holographic
complexity proposals

or

to find a gravity dual of
some notion of complexity

Holographic proofs require phrasing complexity in terms of J's



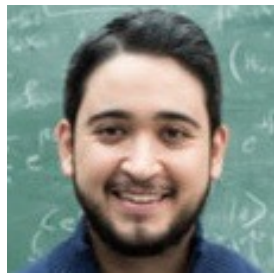
[1904.02713](#) with Hugo Camargo, Johannes Knaute, Ro Jefferson + work-in-progress



$\text{AdS}_3/\text{CFT}_2$ provides the best fully controlled case of holography

THANK YOU
AND PLEASE STAY TUNED,
IN PARTICULAR:

see Hugo Camargo



's TODAY 18:40 talk @ String & Math Physics

have a look at [191x.xxxxxx](#) with Mario Flory and Volker Schomerus when
it appears later this fall