Generalized Wilson lines and next-to-soft emissions

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based on arXiv:1905.13710, 1610.06842, ...
Bahjat-Abbas, DB, Sinnighe-Damste, Laenen, Magnea, Vernazza, White

Outline

Motivations from Phenomenology: Logs

Soft and collinear gluons generate $\log(\xi)$ that spoil perturbation theory when $\xi \to 0$ (threshold)

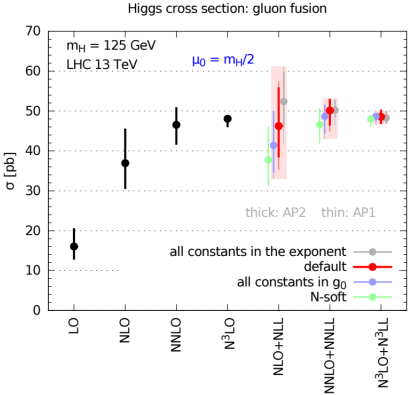
$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \sum_{m=0}^{2n-1} \alpha_s^n \left(a_{nm} \left(\frac{\log^m(\xi)}{\xi} \right)_+ + a_{nm}^{(\delta)} \delta(\xi) + b_{nm} \log^m(\xi) + \mathcal{O}(\xi) \right)$$

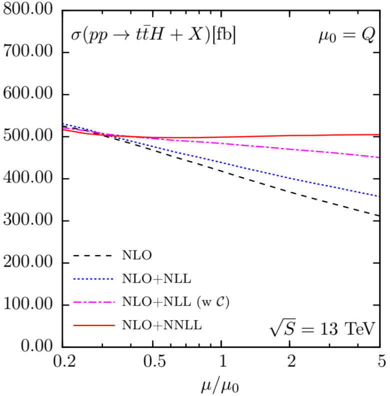
- a_{nm}: **LP** Logs $\mathcal{D}^i$ $\to$ leading soft limit
- a_{nm}^{δ}: delta's $\to$ purely virtual contribution
- b_{nm}: **NLP** Logs L^i $\to$ extend to "next-to-soft"

DY	$\xi = 1 - z$	$z = Q^2/s$
Higgs	$\xi = 1 - z$	$z = m_H^2/s$
DIS	$\xi = 1 - x$	$x = -q^2/(2p \cdot q)$
$t\bar{t}$	$\xi = 1 - z$	$z = 4m_t^2/s$

Motivations from Phenomenology: Logs

Soft gluon **resummation** restores the predictive power of perturbation theory

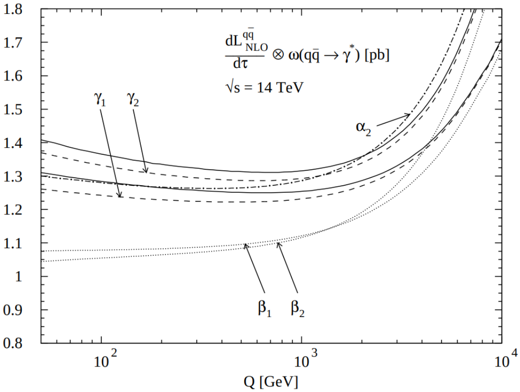




- Higgs [Bonvini, Marzani, Muselli, Rottoli, 2016]
- $t\bar{t}H$ [Kulesza, Motyka, Stebel, Theeuwes, 2017]

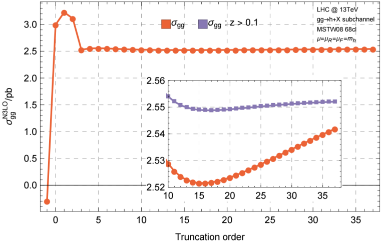
Motivations from Phenomenology

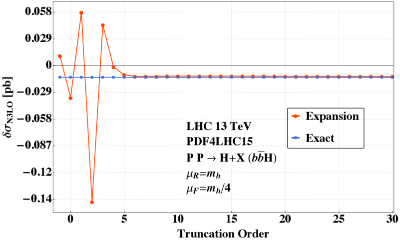
Subleading terms in the soft expansion might be important
[Krämer, Laenen, Spira 1998]



Motivations from Phenomenology

Fixed order calculation as a threshold expansion: Higgs production at N^3LO





- Gluon fusion [Anastasiou, Duhr, Dulat, Herzog, Mistlberger 2015]
- Bottom-quark fusion [Duhr, Dulat, Mistlberger 2019]

An old story

Is there a theoretical framework to organize these subleading terms?

- Story begins in 1958: Low's theorem
  Dressing an amplitude with a next-to-soft emission
- Generalize to spinors [Burnett-Kroll 1968]
- Collinear effects [Del Duca 1990]
- More recently: preliminary attempts to systematize [Moch, Vogt 2009, Eynck, Laenen, Magnea 2003, Laenen, Magnea, Stavenga 2008]

A LIVELY FIELD NOWADAYS

Both at fixed order:

- Drell-Yan [DB, Laenen, Magnea, Vernazza, White 2016, Bahjat-Abbas, Sinninghe-Damste, Vernazza, White 2018]
- Higgs [Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger 2013-17]
- Final state jets (DIS, e^+e^-, prompt photon) [vanBeekveld, Beenakker, Laenen, White 2019]
- N-jettines subtraction [Ebert, Moult, Stewart, Tackmann, Vita, Zhu 2017, Boughezal, Isgro, Liu, Petriello 2017, Beneke, Gamy, Szafron, Wang 2018]

and to all-orders:

- Resummation in event shapes [Moult, Stewart, Vita, Zhu 2018]
- Resummation in DY via SCET [Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, 2018]
- Resummation in colourless final states [Bahjat-Abbas, DB, Sinnighe-Damste, Laenen, Magnea, Vernazza, White, 2019]

Parallel study: Next-to-soft theorems

Soft factorization theorems as Ward identities:

generalization of Weinberg soft thereom for gravitons to next-to-soft level, by studying particular transformations (BMS) on past and future null infinity of the S matrix. [Cachazo, Strominger 2014, Casali 2014, Bern, Davis, Nohle 2014, Larkosky, Neill, Stewart 2014, Sen 2016-18, ...]

$$\begin{aligned} \mathcal{A}_{n+1}(\{p_i\},k) &= \mathcal{S}_n^0 \, \mathcal{A}_n(\{p_i\}) \, , \\ \mathcal{A}_{n+1}(\{p_i\},k) &= \mathcal{S}_n^1 \, \mathcal{A}_n(\{p_i\}) \, , \end{aligned} \tag{1}$$

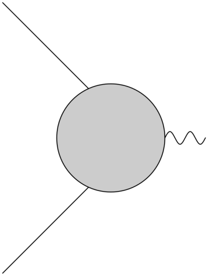
$$\begin{aligned} \mathcal{S}_n^0 &= \sum_{i=1}^{n} \frac{\epsilon_\mu(k)\, p_i^\mu}{p_i \cdot k} \, , \\ \mathcal{S}_n^1 &= \sum_{i=1}^{n} \frac{\epsilon_\mu(k)\, k_\alpha J^{(i)\mu\alpha}}{p_i \cdot k} \, , \end{aligned} \tag{2}$$

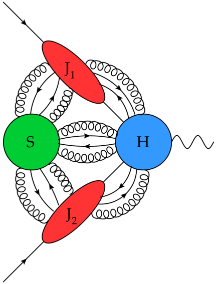
$J^{(i)}_{\mu\alpha}$ total angular momentum of the i-th leg

Factorization at Leading Power

Soft Collinear Factorization

Consider the singular regions of the electroweak form factor at a generic loop-order:





Goal: factorize these regions. Two strategies:

- effective field theory (SCET) [Bauer, Fleming, Pirjol, Stewart, Beneke, Neubert, Becher,... 1410.1892]
- diagrammatic analysis [Collins, Soper, Sterman hep-ph0409313]

Soft Collinear Factorization

Generic diagram $G(p_1, \ldots, p_E)$:

$$G = \left(\prod_{i=1}^{I} \int_0^1 dx_i \right) \delta(1 - \sum_{k=1}^{I} x_k) \left(\prod_{j=1}^{L} \int \frac{d^d k_j}{(2\pi)^d} \right) \frac{\widetilde{\mathcal{N}}(p_r, k_j, x_i)}{[\mathcal{D}(p_r, k_j, x_i)]^I} ,$$

$$\mathcal{D}(p_r, k_j, x_i) = \sum_{i=1}^{I} x_i \left(\ell_i^2 - m_i^2 \right) + i\eta \, .$$

Strategy is well-known:

- **Identify** potential IR singularities via **Landau equations**
- Power counting for each singular region
- Diagrammatic analysis and Wilson lines

Soft Collinear Factorization

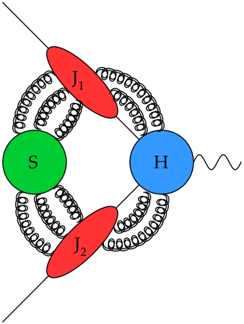
Power counting is performed via an **expansion by regions**, a systematic procedure for expanding loop integrals about their singular regions. [Beneke, Smirnov 1998, Jantzen 2011]

We distinguish different regions for the momentum l^μ by the different scalings of its components

$$\begin{aligned} \text{Hard}: &\quad l \sim \sqrt{\hat{s}}\,(1,1,1) \\ \text{Soft}: &\quad l \sim \sqrt{\hat{s}}\,(\lambda^2,\lambda^2,\lambda^2) \\ \text{Collinear}: &\quad l \sim \sqrt{\hat{s}}\,(1,\lambda,\lambda^2) \\ \text{Anticollinear}: &\quad l \sim \sqrt{\hat{s}}\,(\lambda^2,\lambda,1) \end{aligned} \tag{3}$$

Soft Collinear Factorization

After Power counting: only longitudinally polarized gluons still connect the regions

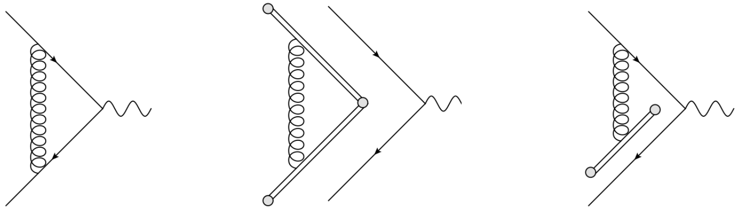


Note: in physical gauge we achieved our goal. Gauge independent factorization?

Soft Collinear Factorization

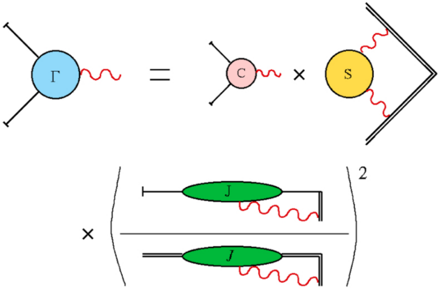
Introduce **Wilson lines**:

$$\Phi_n(\lambda_2, \lambda_1) \equiv \mathcal{P} \exp\left(ig\mu^{\epsilon} \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A(\lambda n) \right)$$



Soft Collinear Factorization

Taking into account soft/collinear double counting [Dixon, Magnea, Sterman 2008]



$$\textbf{Soft} \quad \mathcal{S}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon\right) = \langle 0|\Phi_{\beta_2}(\infty,0)\Phi_{\beta_1}(0,-\infty)|0\rangle$$

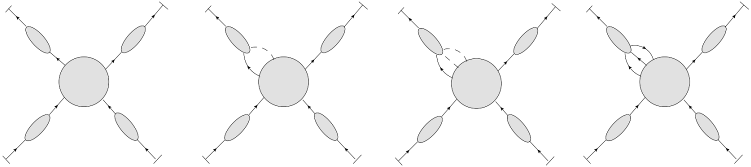
$$\textbf{Jet} \quad J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)u(p) = \langle 0|\Phi_n(\infty,0)\psi(0)|p\rangle$$

$$\textbf{Eik Jet} \quad \mathcal{J}\left(\frac{(\beta_1\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right) = \langle 0|\Phi_n(\infty,0)\Phi_{\beta_1}(0,-\infty)|0\rangle$$

Factorization at Next-to-Leading Power

Revisit power counting at NLP

Gauge and gravity emissions from **scalar** lines [Gervais 2017]



(a) (b) (c) (d)

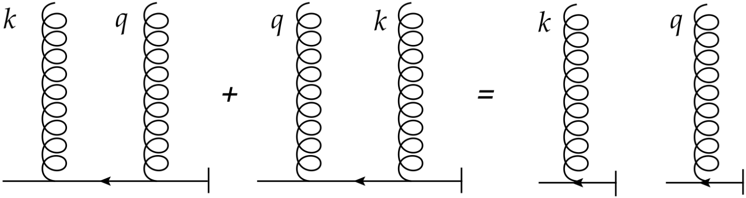
$$
\begin{aligned}
M_{el} = & \left(\prod_{i=1}^{n} J_i{}^{f}\right) \otimes H \\
& + \sum_{i=1}^{n}\left(\prod_{j\neq i} J_j{}^{f}\right) J_i{}^{fs} \otimes H_i{}^{fs} + \sum_{i=1}^{n}\left(\prod_{j\neq i} J_j{}^{f}\right) J_i{}^{f\partial s} \otimes H_i{}^{f\partial s} \\
& + \sum_{i=1}^{n}\left(\prod_{j\neq i} J_j{}^{f}\right) J_i{}^{fss} \otimes H_i{}^{fss} + \sum_{i=1}^{n}\left(\prod_{j\neq i} J_j{}^{f}\right) J_i{}^{fff} \otimes H_i{}^{fff} \\
& + \sum_{1\leq i<j\leq n}\left(\prod_{l\neq i,j} J_l{}^{f}\right) J_i{}^{fs} J_j{}^{fs} \otimes H_{ij}^{fsfs} + \sum_{1\leq i<j\leq n}\left(\prod_{l\neq i,j} J_l{}^{f}\right) J_i{}^{f\underline{s}} J_j{}^{f\underline{s}} S \otimes H_{ij}^{f\underline{s}f\underline{s}} \\
& + O(\lambda^3)\,.
\end{aligned}
$$

Analogous results in SCET [Larkoski, Neill, Stewart 2014]

Things are simpler for specific processes at given order.

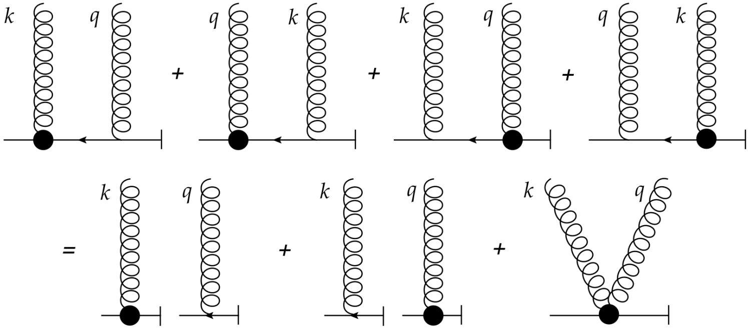
EIKONAL IDENTITY

Let us consider only soft gluon emissions, with no collinear effect.



Factorization (and exponentiation) works very easily at eikonal level.

Next-to-Eikonal identity



There are correlations between pairs of gluons. Naive factorization is broken at NE level (seagull vertex). Do these vertices exponentiate?

Exponentiation in the worldline formalism

Dressed propagators Δ_{ij} in the background of a radiating field A_{ij} can be written as single-particle QM correlators evolving in Schwinger proper time τ

$$\Delta_{ij}(p_1, x_1) = \frac{1}{2} \int_0^\infty d\tau \, \langle p_1 | e^{-i(H_{ij}(\hat{x}_\mu, \hat{p}_\mu) + i\epsilon)\tau} | x_1 \rangle \, .$$

with Hamiltonian

$$H_{ij}^{ab} = -\frac{1}{2} \left((\hat{p}\delta_{ij} - g A_{ij}(\hat{x}))^2 \delta^{ab} + g S^{ab}_{\mu\nu} F^{\mu\nu}_{ij}(\hat{x}) \right) \, .$$

The correlator can be evaluated as a path integral (worldline)

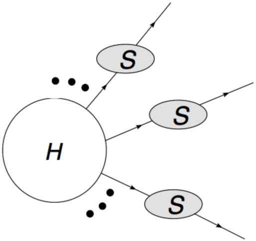
$$(M_{ij}^{ab})^{-1}(p_f, x_i) = \mathcal{N}(p_f) \frac{1}{2} e^{-i p_f x_i} \int_0^\infty d\tau \, e^{-i\frac{1}{2}(p_f^2 - m^2 + i\epsilon)\tau}$$

$$\int \mathcal{D}x \, e^{i \int_0^\tau dt \, (\frac{1}{2}\dot{x}^2)\delta_{ij}\delta_{ab} + (p_f + \dot{x})\cdot A_{ij}(x_i + p_f t + x(t))\delta_{ab} + \frac{i}{2}\partial \cdot A_{ij}(x_i + p_f t + x(t))\delta_{ab} + g S^{ab}_{\mu\nu} F^{\mu\nu}_{ij}}$$

Exponentiation in the worldline formalism

Setting p_f as the hard momentum, we can perform a soft expansion in the exponent. [Laenen, Stavenga, White 2018]

- At LP we obtain the soft limit and the propagator becomes a Wilson line (classical straight trajectory).
- At NLP we get **fluctuations** along the classical trajectory - recoil and spin effects



Exponentiation follows from standard QFT properties of connected diagrams. Both gauge and gravity [White 2011]

We call this new objet a **generalized Wilson line**.

$$
\begin{aligned}
&\widetilde{W}_p(0,\infty) = \\
&= \mathcal{P}\exp\left[g\int_k A_\mu(k)\left(-\frac{p^\mu}{p\cdot k} + \frac{k^\mu}{2p\cdot k} - k^2\frac{p^\mu}{2(p\cdot k)^2} - \frac{\mathrm{i}k_\nu\Sigma^{\nu\mu}}{p\cdot k}\right)\right. \\
&\quad + \int_k\int_l A_\mu(k)A_\nu(l)\left(\frac{\eta^{\mu\nu}}{2p\cdot(k+l)} - \frac{p^\nu l^\mu p\cdot k + p^\mu k^\nu p\cdot l}{2(p\cdot l)(p\cdot k)\left[p\cdot(k+l)\right]}\right. \\
&\quad \left.\left. + \frac{(k\cdot l)p^\mu p^\nu}{2(p\cdot l)(p\cdot k)\left[p\cdot(k+l)\right]} - \frac{\mathrm{i}\Sigma^{\mu\nu}}{p\cdot(k+l)}\right)\right] .
\end{aligned}
$$

We obtain precisely the effective vertices of the NE identity.
Hence, they exponentiate.

New definitions

New correlators for **virtual** radiation:

- next-to-soft function

$$\widetilde{\mathcal{S}}\left(p_1, p_2\right) = \left\langle 0 \left| \widetilde{W}_{p_2}(\infty, 0) \widetilde{W}_{p_1}(0, -\infty) \right| 0 \right\rangle \Big|_{\mathrm{NLP}}$$

- next-to-soft jet function

$$\widetilde{\mathcal{J}}\Big(p, n\Big) = \left\langle 0 \left| \Phi_n(\infty, 0) \widetilde{W}_p(0, -\infty) \right| 0 \right\rangle \Big|_{\mathrm{NLP}}$$

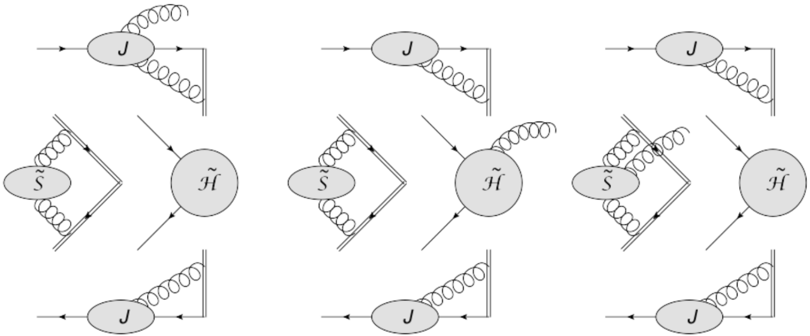
Non-radiative amplitude $\mathcal{A}$:

$$\mathcal{A}\left(p_1, p_2\right) = \widetilde{\mathcal{H}}\left(p_j, n_j\right) \widetilde{\mathcal{S}}\left(p_j\right) \prod_{i=1}^{2} \frac{J\left(p_i, n_i\right)}{\widetilde{\mathcal{J}}\left(p_i, n_i\right)}$$

A factorization formula

Dressing the non-radiative amplitude with a next-to-soft **real** emission:

$$\mathcal{A}_{\mu,a}\left(p_j,k\right) = \sum_{i=1}^{2} \left\{ \left[\frac{1}{2} \frac{\widetilde{\mathcal{S}}_{\mu,a}(p_j,k)}{\widetilde{\mathcal{S}}(p_j)} + g\,\mathbf{T}_{i,a} G^{\nu}_{i,\mu} \frac{\partial}{\partial p_i^{\nu}} + \frac{J_{\mu,a}\left(p_i,n_i,k\right)}{J(p_i,n_i)} \right.\right.$$
$$\left.\left. - \, g\,\mathbf{T}_{i,a}\, G^{\nu}_{i,\mu} \frac{\partial}{\partial p_i^{\nu}} \log\left(\frac{J(p_i,n_i)}{\widetilde{\mathcal{J}}(p_i,n_i)}\right) \right] \mathcal{A}(p_j) - \mathcal{A}^{\widetilde{\mathcal{J}}_i}_{\mu,a}(p_j,k) \right\}$$



The theorem at one-loop (abelian, 2 legs)

$$\mathcal{A}^{\mu,(1)}(p_j,k) = \sum_{i=1}^{2}\Bigg[q_i\left(\frac{p_i^\mu}{p_i\cdot k} - \frac{\not{k}\gamma^\mu}{2p_i\cdot k}\right)\mathcal{A}^{(1)}(p_i;p_j) + q_i\left(G_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu}\right)\mathcal{A}^{(1)}(p_i;p_j) + J^{\mu\,(1)}(p_i,k)\mathcal{A}^{(0)}(p_i;p_j)\Bigg]\,, \tag{4}$$

1st and 2nd lines:

$$\frac{\epsilon_\mu(k)\,p_i^\mu}{p_i\cdot k}\,\mathcal{A}^{(1)} + \frac{\mathrm{i}\,\epsilon_\mu(k)\,k^\nu}{p_i\cdot k}\,\left[L_{\mu\nu}^{(i)} + \Sigma_{\mu\nu}^{(i)}\right]\mathcal{A}^{(1)}\,, \tag{5}$$

3rd line $\rightarrow$ breakdown of next-to-soft theorems at loop level

It is made out of 3 main ingredients

- external (scalar) emission
- derivative of the non radiative amplitude
- jet emission function J^{μ}

$$K^{(2)}_{\rm ext}(z) + K^{(2)}_{\partial\mathcal{A}}(z) + K^{(2)}_{\rm collinear}(z) = K^{(2)}_{1r1v}$$

Explicit verification for the Drell-Yan NNLO K-factor [DB, Laenen, Magnea, Melville, Vernazza, White 2015]

LL Resummation

Leading Logs

The argument is simple:

- order by order in α_s, LL requires highest pole in ϵ
- Hence, maximum number if singular integrations
- Radiation must be soft **and** collinear.

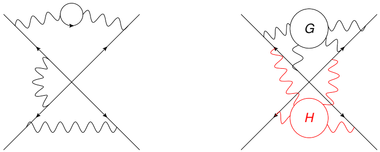
Purely collinear effects contribute at subleading Log.
Everything is captured by the **(next-to-) soft function** S.

$$\sigma = \mathcal{H} \times \mathcal{S}$$

At Leading Log, NLP = next-to-soft
i.e. loop corrections to next-to-soft theorems don't contribute
We can use next-to-soft theorems for exponentiantion

DIAGRAMMATIC EXPONENTIATION OF S

Replica trick: N copies of the original theory



(a) (b)

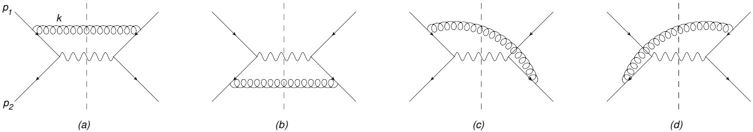
Replicas do not interact! Hence

$$\mathcal{S}_R = \mathcal{S}^N = 1 + N\log(\mathcal{S}) + \mathcal{O}(N^2)$$

Exponentiation means to find W such that $\mathcal{S} = \exp(\sum_W W)$
$\Longrightarrow$ Simply select $\mathcal{O}(N)$ in $\mathcal{S}_\mathcal{R}$!

- QED: W are connected diagrams
- QCD: W are called webs

Resummation at LP (Drell-Yan)



Cross-section (not amplitude) $\Longrightarrow$ **Mellin** space

$$F(N) = \int_0^1 dz\, z^{N-1} f(z)$$

Large N expansion $\leftrightarrow$ Soft expansion

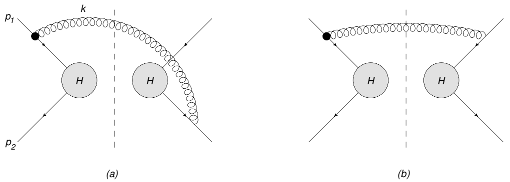
$$\mathcal{S}^{(1)}\Big|_{\mathrm{LL}} = \left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} \frac{2\alpha_s}{\pi} C_F \left[\frac{\log N}{\epsilon} + \log^2 N\right]$$

$$\int_0^1 d\tau\, \tau^{N-1} \frac{d\sigma_{\mathrm{DY}}}{d\tau}\Big|_{\mathrm{LL}} =$$

$$\sigma_0(Q^2)\, q_{\mathrm{LL}}(N, Q^2)\, \bar{q}_{\mathrm{LL}}(N, Q^2) \exp\left[\frac{2\alpha_s}{\pi} C_F \log^2 N\right]$$

(well-known result)

Resummation at NLP (Drell-Yan)



Consider next-to-soft function in terms of generalized Wilson lines. Same argument for exponentiation!

$$\mathcal{S}_{\rm LP+NLP} = \frac{2\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\left[\frac{1}{\epsilon}\left(\log N + \frac{1}{2N}\right) + \log^2 N + \frac{\log N}{N}\right]$$

$$\int_0^1 d\tau\, \tau^{N-1}\, \frac{d\sigma_{\rm DY}}{d\tau}\bigg|_{\rm LL,\,NLP} = \sigma_0(Q^2)\, q_{\rm LL,\,NLP}\big(N, Q^2\big)\ \bar{q}_{\rm LL,\,NLP}\big(N, Q^2\big) \times \exp\left[\frac{2\alpha_s C_F}{\pi}\left(\log^2 N + \frac{\log N}{N}\right)\right]$$

Note simplicity: no need for RG equations!

Resummation at NLP (Drell-Yan)

Clearly, at LP no need to limit at LL!

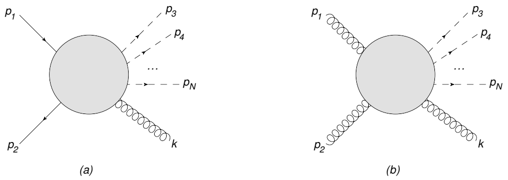
$$\begin{aligned}
\ln\left[\Delta(N,Q^2)\right] &= F_{\mathrm{DY}}\left[\alpha_s(Q^2)\right] \\
&+ \int_0^1 dz\, z^{N-1}\left\{\frac{1}{1-z}\, D\left[\alpha_s\left(\frac{(1-z)^2Q^2}{z}\right)\right]\right. \\
&+ \left. 2\int_{Q^2}^{(1-z)^2Q^2/z}\frac{dq^2}{q^2}\, P_{qq}^{\mathrm{LP+NLP}}\left[z,\alpha_s(q^2)\right]\right\}_+
\end{aligned}$$

- F resums constants
- D is soft-wide angle function at LP
- $P_{qq}^{\mathrm{LP+NLP}}$ is the soft expansion of the DGLAP splitting function. It generates NLP LL.

Part of the subleading NLP Logs are exponentiated [Laenen, Magnea, Stavenga 2008]

Resummation for generic colourless final states

DY, Higgs, 2 Higgs, etc..



$$M_{\rm NLP} \;=\; M_{\rm scal.} + M_{\rm spin} + M_{\rm orb.}$$

but separation between interal/external emission not gauge invariant!

$$\mathcal{S}_{\rm NLP} = \exp\left[\sum_i W_{\rm LP}^{(i)} + \sum_j W_{\rm NLP}^{(j)}\right] = \exp\left[\sum_i W_{\rm LP}^{(i)}\right]\left(1 + \sum_j W_{\rm NLP}^{(j)}\right)$$

Exponentiation seems to be controlled by the LP Soft function

Resummation for generic colourless final states

NLP radiation generated by **shifted kinematics** [DelDuca, Laenen, Magnea, Vernazza, White 2017]

$$\left|\mathcal{M}^{(q\bar{q}g)}_{\mathrm{NLP}}(p_1,p_2,k)\right|^2 = g_s^2 C_F \frac{\hat{s}}{p_1\cdot k\, p_2\cdot k}\left|\mathcal{M}^{(q\bar{q})}_{\mathrm{LO}}(p_1+\delta p_1, p_2+\delta p_2)\right|^2$$

$$\begin{aligned}
\delta p_1 &= -\frac{1}{2}\left(\frac{p_2\cdot k}{p_1\cdot p_2}p_1^\alpha - \frac{p_1\cdot k}{p_1\cdot p_2}p_2^\alpha + k^\alpha\right)\\
\delta p_2 &= -\frac{1}{2}\left(\frac{p_1\cdot k}{p_1\cdot p_2}p_2^\alpha - \frac{p_2\cdot k}{p_1\cdot p_2}p_1^\alpha + k^\alpha\right) .
\end{aligned}$$

$$\widehat{\Delta}^{(q\bar{q})}_{\mathrm{NLP}}(z,\epsilon) = z\,\mathcal{S}_{\mathrm{LP}}(z,\epsilon)\,\hat{\sigma}^{(q\bar{q})}_{\mathrm{LO}}(z\hat{s})$$

Resummation of NLP LL by just shifting!

Extending the scope

Dressed propagators

- A crucial ingredient for the exponentiation has been the introduction of the generalized Wilson line
- the (soft) radiated field is set as a classical background and the (hard) emitter is set as a quantum fluctuating field
- The procedure is very clear at cross section level, where all Wilson lines close at infinity $\leftrightarrow$ **effective action** via worldline formalism
- More subtle at amplitude level: **spin** complicates things (different representation for an open dressed propagator)
- Quite generally, worldline rep. for dressed propagators relatively unexplored (see e.g. [Ahmadiniaz, Bastianelli, Corradini 2015])

Gravity and the double copy

- NE Feynman rules have been derived for gravity, for a **scalar** emitter [White, 2011]

$$f(\infty) = \exp\left\{\int \frac{d^dk}{(2\pi)^d} h^{\mu\nu}(k)\left[\frac{\kappa}{2}\frac{p_\mu p_\nu}{p\cdot k} - \frac{\kappa}{4}\frac{p_{(\mu}k_{\nu)}}{p\cdot k} + \frac{m^2}{2}\frac{\kappa}{(d-2)}\frac{\eta_{\mu\nu}}{p\cdot k} + \frac{\kappa}{4}\frac{k^2}{(p\cdot k)^2}p_\mu p_\nu \right.\right.$$
$$\left. + \frac{m^2}{4}\frac{\kappa}{d-2}\frac{k^2}{(p\cdot k)^2}\eta_{\mu\nu}\right] + \int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}h^{\mu\nu}(k)\,h^{\alpha\beta}(l)\left[-\frac{\kappa^2}{16}\left(\frac{p_\alpha\,p_\beta\,p_{(\mu}k_{\nu)}}{p\cdot k\,p\cdot(k+l)}\right.\right.$$
$$\left. + \frac{p_\alpha\,p_\beta\,p_{(\mu}\,l_{\nu)}}{p\cdot l\,p\cdot(k+l)} + (\mu\nu\leftrightarrow\alpha\beta)\right) + \frac{\kappa^2}{8}\frac{p_\mu\,p_\nu\,p_\alpha\,p_\beta\,(k\cdot l)}{p\cdot k\,p\cdot l\,p\cdot(k+l)} - \frac{m^2\kappa^2}{16(d-2)}\left(\frac{\eta_{\mu\nu}\,p_{(\alpha}\,k_{\beta)}}{p\cdot k\,p\cdot(k+l)}\right.$$
$$\left. + \frac{\eta_{\mu\nu}\,p_{(\alpha}\,l_{\beta)}}{p\cdot l\,p\cdot(k+l)} + (\mu\nu\leftrightarrow\alpha\beta)\right) + \frac{m^2\kappa^2}{8(d-2)}\left(\frac{\eta_{\mu\nu}\,p_\alpha\,p_\beta\,(k\cdot l)}{p\cdot k\,p\cdot l\,p\cdot(k+l)} + (\mu\nu\leftrightarrow\alpha\beta)\right)$$
$$\left.\left. + \frac{m^4\kappa^2}{8(d-2)^2}\frac{\eta_{\mu\nu}\,\eta_{\alpha\beta}\,(k\cdot l)}{p\cdot k\,p\cdot l\,p\cdot(k+l)} + \frac{m^2\kappa^2}{4p\cdot(k+l)}\left(\frac{2\eta_{\mu\nu}\,\eta_{\alpha\beta}}{(d-2)^2} - \frac{1}{d-2}(\eta_{\mu\beta}\,\eta_{\nu\alpha} + \eta_{\nu\beta}\,\eta_{\mu\alpha})\right)\right]\right\}$$

- $\mathcal{O}(\kappa^n)$ we get n-graviton vertices
- Possible tool to test the **double copy** in the soft limit. Need to go to next-to-next-to-soft. [Plefka, Shi, Steinhoff, Wang, 2019]

Conclusions

- Next-to-soft physics is interesting for both phenomenology and theory
- Factorization at subleading power is (generally) very intricated
- Simplification at Leading Log for color-singlet production: only soft function
- Many problems still open: NLL, resummation with colored final states, LHC physics
- Many connections with more formal applications in gravity

Thanks for your attention!

THE LOW-BURNETT-KROLL THEOREM

How does a soft emission affect an amplitude? Low theorem: express a radiative amplitude in terms of the non radiative one.

- Low: scalar particles
- Burnett and Kroll: spinor particles
- Del Duca: theorem extended to the region $\frac{m^2}{Q^2} < E < m^2$ (collinear region). In Low original analysis gluon energy E is the smallest scale of the problem.

Del Duca modifications (NuclPhysB345 '90)

Del Duca generalized Low's original analysis attaching an extra soft photon to the factorized amplitude (via Ward Identity)
Introduce 2 polarization tensors:

$$K^{\nu\mu}(p,k) = \frac{k^\nu (2p+k)^\mu}{k^2 + 2p \cdot k}$$

$$G^{\nu\mu}(p,k) = g^{\mu\nu} - K^{\nu\mu}$$

Emission contributes with terms got via Ward identity

- $K + G$ emission from $\mathcal{H}$
- $K + G$ emission from $\mathcal{S}$
- K emission from J

and with a contribution excluded in Low original analysis

- G emission from J