Duration of classicality in degenerate quantum scalar fields

Elisa Todarello (KIT)

15th Patras Workshop on Axions, WIMPs and WISPs
Freiburg, June 6th 2019

A. Arza, S. Chakrabarty, S. Enomoto, Y. Han, P. Sikivie

Outline

- Motivation
- Duration of classicality
- Method to calculate duration of classicality in quantum field theory
- Application of method
  - Self-gravitating homogeneous field
  - Inhomogeneous field with repulsive contact interactions
- Conclusions

Axion Dark Matter

- Axions from vacuum realignment

$$\mathcal{N} \sim 10^{60} \overset{?}{\Rightarrow}$$ classical field theory

- Is this true in the presence of interactions? For how long?

- Axion interactions
  - Quartic self-interactions $V_a \simeq \lambda \, a^4$
  - Gravitational self-interactions
  - Number changing interactions

- Are these important? Relaxation rate vs. Hubble rate

$$\Gamma(t) \overset{?}{>} H(t)$$

Bose-Einstein Condensation

Sikivie, Yang, Phys. Rev. Lett. 103 (2009) 111301

- Thermalizing interaction: gravity

$$\Gamma(t) > H(t) \qquad \text{for} \qquad T \lesssim 500 \text{ eV} \left(\frac{f_a}{10^{12} \text{ GeV}} \right)^{1/2}$$

- Necessary and sufficient conditions for Bose-Einstein condensation
  1. Large number of identical Bosons
  2. Conservation of the number of particles
  3. High degeneracy $\mathcal{N} > \mathcal{O}(1)$
  4. Effective thermalizing interaction

- Quantum mechanics $\Rightarrow$ Bose-Einstein distribution $\Rightarrow$ BEC

- Can we use classical field theory instead if $\mathcal{N} \gg 1$?

Duration of Classicality

$$H = \sum_j \omega_j a_j^\dagger a_j + \sum_{jklm} \frac{1}{4} \Lambda_{jk}^{lm} \; a_j^\dagger a_k^\dagger a_l a_m$$

Quantum theory

$$[a_j, a_k^\dagger] = \delta_{jk}$$

$$i\dot{a}_j = \omega_j a_j + \sum_{klm} \frac{1}{2} \Lambda_{jk}^{lm} \; a_k^\dagger a_l a_m$$

$$\mathcal{N}_j(t) = a_j^\dagger(t) a_j(t)$$

$$\langle \mathcal{N}_j(t) \rangle = \langle \Psi | \mathcal{N}_j(t) | \Psi \rangle$$

Classical analogue

$$i\dot{A}_j = \omega_j A_j + \sum_{klm} \frac{1}{2} \Lambda_{jk}^{lm} \; A_k^* A_l A_m$$

$$N_j(t) = A_j^*(t) A_j(t)$$

Duration of Classicality

Quantum theory

$$\langle \mathcal{N}_j(0) \rangle$$

$$\downarrow$$

$$\langle \mathcal{N}_j(t) \rangle$$

$$\downarrow \tau$$

$$\langle \mathcal{N}_j \rangle = \frac{1}{\mathrm{e}^{\frac{1}{T}(\varepsilon_j - \mu)} - 1}$$

$$\downarrow t_{\mathrm{cl}}?$$

Classical theory

$$N_j(0) \sim \langle \mathcal{N}_j(0) \rangle$$

$$\downarrow$$

$$N_j(t)$$

$$\downarrow$$

$$N_j = \frac{T}{\varepsilon_j - \mu}$$

Gravitational Self-Interactions

- Non-relativistic limit

$$H_G = -\frac{G}{2}\int d^3r\ d^3r'\ \frac{:\psi^\dagger(\vec{r},t)\psi(\vec{r},t)\psi^\dagger(\vec{r}\,',t)\psi(\vec{r}\,',t):}{|\vec{r}-\vec{r}\,'|}$$

- Heisenberg equations of motion

$$\begin{aligned} i\partial_t\psi(\vec{r},t) &= -\tfrac{1}{2m}\nabla^2\psi(\vec{r},t) + m\varphi(\vec{r},t)\psi(\vec{r},t) \\ \nabla^2\varphi(\vec{r},t) &= 4\pi G m\psi^\dagger(\vec{r},t)\psi(\vec{r},t) \end{aligned}$$

- Schrödinger-Poisson equations (classical)

$$\begin{aligned} i\partial_t\Psi(\vec{r},t) &= -\tfrac{1}{2m}\nabla^2\Psi(\vec{r},t) + m\Phi(\vec{r},t)\Psi(\vec{r},t) \\ \nabla^2\Phi(\vec{r},t) &= 4\pi G m|\Psi(\vec{r},t)|^2 \end{aligned}$$

S. Chackrabarty, S. Enomoto, Y. Han, P. Sikivie, E. T., PR D97 (2018) no. 4, 043531

Method to get t_{cl} in QFT

Step 1: Expand field over ONC mode functions

$$\psi(\vec{r},t) = \sum_{\vec{\alpha}} u^{\vec{\alpha}}(\vec{r},t) a_{\vec{\alpha}}(t) \qquad \Psi(\vec{r},t) = \sum_{\vec{\alpha}} u^{\vec{\alpha}}(\vec{r},t) A_{\vec{\alpha}}(t)$$

$$u^{\vec{\alpha}}(\vec{r},t) = \frac{1}{\sqrt{N}} \Psi(\vec{r},t)\ \mathrm{e}^{i\vec{\alpha}\cdot\vec{\chi}(\vec{r},t)}$$

where $\Psi(\vec{r},t)$ solves the classical field equations

- $\Psi(\vec{r},t)$ defines a fluid with number density and velocity

$$\begin{aligned} n(\vec{r},t) &= \Psi^*(\vec{r},t)\Psi(\vec{r},t) \\ \vec{v}(\vec{r},t) &= \frac{1}{2imn}(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*) \end{aligned}$$

- Choose $\vec{\chi}$ to be comoving coordinates, such that the density in $\vec{\chi}$ space is constant

- The classical evolution is trivial $\qquad A_{\vec{\alpha}}(t) = \sqrt{N}\delta^{\vec{0}}_{\vec{\alpha}}$

Method to get $t_{\rm cl}$ in QFT

- Deviation from classical theory as $\vec{\alpha} \neq \vec{0}$ modes get occupied quantum mechanically

Step 2: calculate coefficients of equations of motion

$$i\partial_t a_{\vec{\alpha}} = \sum_{\vec{\beta}} M_{\vec{\alpha}}^{\vec{\beta}}\ a_{\vec{\beta}} + \sum_{\vec{\beta}\vec{\gamma}\vec{\delta}} \frac{1}{2}\Lambda_{\vec{\alpha}\vec{\beta}}^{\vec{\gamma}\vec{\delta}}\ a_{\vec{\beta}}^\dagger a_{\vec{\gamma}} a_{\vec{\delta}}$$

$$\begin{aligned} M_{\vec{\alpha}\vec{\beta}}(t) &= \int_V d^3r\ u^{\vec{\alpha}}(\vec{r},t)^* (-i\partial_t - \frac{1}{2m}\nabla^2) u^{\vec{\beta}}(\vec{r},t) \\ \Lambda_{\vec{\alpha}\vec{\beta}}^{\vec{\gamma}\vec{\delta}}(t) &= -Gm^2 \int_V d^3r \int_V d^3r' \frac{1}{|\vec{r}-\vec{r}\,'|} \times \\ &\quad \times\ u^{\vec{\alpha}}(\vec{r},t)^* u^{\vec{\beta}}(\vec{r}\,',t)^*\ (u^{\vec{\gamma}}(\vec{r},t) u^{\vec{\delta}}(\vec{r}\,',t) + u^{\vec{\gamma}}(\vec{r}\,',t) u^{\vec{\delta}}(\vec{x},t)) \end{aligned}$$

Step 3: Bogoliubov ansatz

- Large occupation of $\vec{\alpha} = \vec{0}$

$$\begin{aligned} a_{\vec{0}} &\to \sqrt{N} \\ a_{\vec{\alpha}} &\to b_{\vec{\alpha}}, \quad \text{for} \quad \vec{\alpha} \neq \vec{0}\ . \end{aligned}$$

Method to get t_{cl} in QFT

Step 4: expand equations for $b_{\vec{\alpha}}$ in powers of $1/\sqrt{N}$ and keep first order terms only. Solve.

$$i\partial_t b_{\vec{\alpha}} = \sum_{\vec{\beta}\neq\vec{0}} \left[\left(M_{\vec{\alpha}}^{\vec{\beta}} + \Lambda_{\vec{\alpha}\vec{0}}^{\vec{\beta}\vec{0}} \right) N \; b_{\vec{\beta}} + \frac{1}{2} \Lambda_{\vec{\alpha}\vec{\beta}}^{\vec{0}\vec{0}} \; N \; b_{\vec{\beta}}^{\dagger} \right]$$

Step 5: duration of classicality

- Expectation value $\langle \mathcal{N}_{\vec{\alpha}}(t) \rangle = \langle 0 | b_{\vec{\alpha}}^{\dagger}(t) b_{\vec{\alpha}}(t) | 0 \rangle$ for $\vec{\alpha} \neq \vec{0}$
- Evaporation $$N_{ev}(t) = \sum_{\vec{\alpha}\neq\vec{0}} \langle \mathcal{N}_{\vec{\alpha}}(t) \rangle$$
- Duration of classicality

$$N_{ev}(t_{\mathrm{cl}}) \sim N$$

Homogeneous State

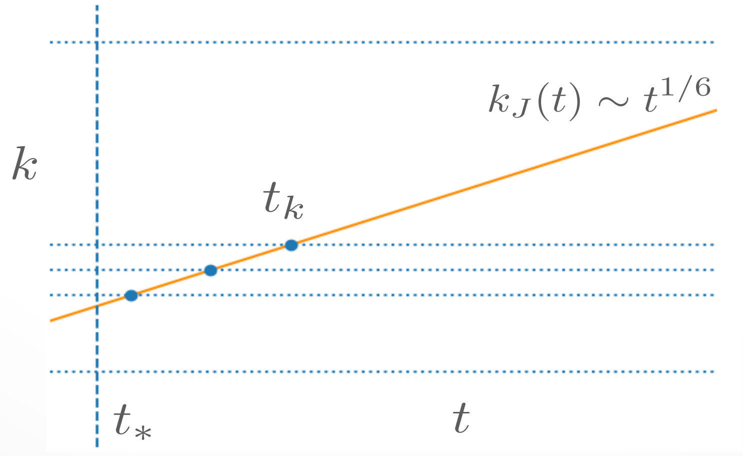
- Classical solution

$$\Psi_0(\vec{r},t) = \sqrt{n_0(t)}\ \mathrm{e}^{\frac{i}{2}mH(t)r^2} \qquad\qquad \Phi_0(\vec{r},t) = \frac{2\pi G m}{3} n_0(t) r^2$$

- Mode functions

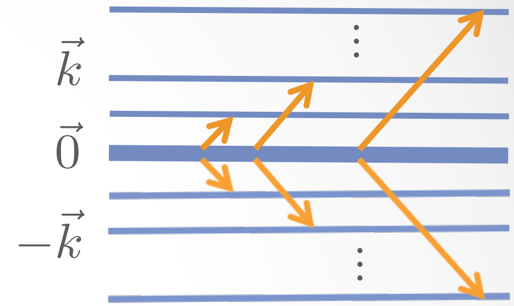
$$u^{\vec{k}}(\vec{r},t) = \sqrt{\frac{n_0(t)}{N}}\ \mathrm{e}^{\frac{i}{2}mH(t)r^2 + i\frac{\vec{k}\cdot\vec{r}}{R(t)}}$$

- Jeans wavenumber $\quad k_J(t) = (16\pi G m^3 n(t))^{1/4} R(t)$



Homogeneous State

- Unstable modes $k < k_J(t)$



$$\langle \mathcal{N}_{\vec{k}}(t) \rangle = \frac{1}{10}\left(\frac{t}{t_*}\right)^2 \left(\frac{t_*}{t_k}\right)^{1/3} \qquad t \gg t_k > t_*$$

$$\langle \mathcal{N}_{\vec{k}}(t) \rangle = \frac{1}{10}\left(\frac{t}{t_k}\right)^2 \qquad t \gg t_* > t_k$$

- Evaporation of the homogenous state

$$N_{ev}(t) = \sum_{\vec{k}} \langle \mathcal{N}_{\vec{k}}(t) \rangle \sim 0.221 N G m^2 \sqrt{m t_*} \left(\frac{t}{t_*}\right)^2$$

- Duration of classicality

$$\boxed{t_{\mathrm{cl}} \sim t_* \frac{1}{(G m^2 \sqrt{m t_*})^{1/2}}}$$

Contact Self-Interactions

- Gross-Pitaevskii equation

$$i\partial_t \Psi(\vec{x},t) = -\frac{1}{2m}\nabla^2\Psi(\vec{x},t) + \frac{\lambda}{8m^2}\Psi^*(\vec{x},t)\Psi(\vec{x},t)\Psi(\vec{x},t)$$

- Homogenous case

$$\Psi_0(\vec{x},t) = \sqrt{n_0}\,\mathrm{e}^{-i\delta\omega t} \qquad \delta\omega = \frac{\lambda n_0}{8m^2}$$

- Duration of classicality

$$t_{\mathrm{cl}} = \infty$$

A. Arza, S. Chackrabarty, S. Enomoto, Y. Han, E. T., in preparation

One-mode perturbation

- Linearize

$$\Psi(\vec{x},t) = \Psi_0(t) + \Psi_1(\vec{x},t)$$

- Number density and velocity

$$\begin{aligned} n(\vec{x},t) &= n_0 + n_1(\vec{x},t) \\ \vec{v}(\vec{x},t) &= \vec{v}_1(\vec{x},t) \end{aligned}$$

- Classical solution: one plane wave

$$\begin{aligned} n_1(\vec{x},0) &= \delta n \ \cos(\omega_p t - \vec{p}\cdot\vec{x}) \\ \vec{v}_1(\vec{x},0) &= \delta\vec{v} \ \cos(\omega_p t - \vec{p}\cdot\vec{x}) \end{aligned}$$

$$\omega_p = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2\delta\omega\right)}$$

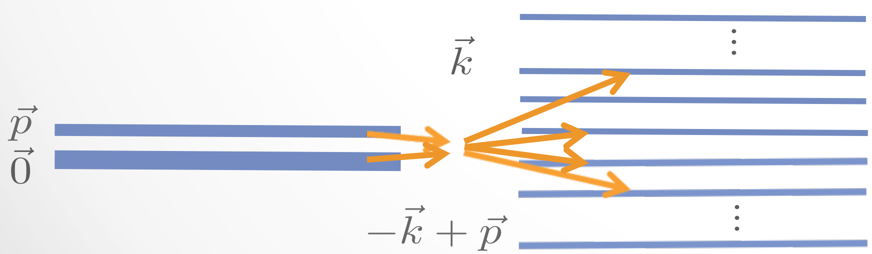
One-mode perturbation

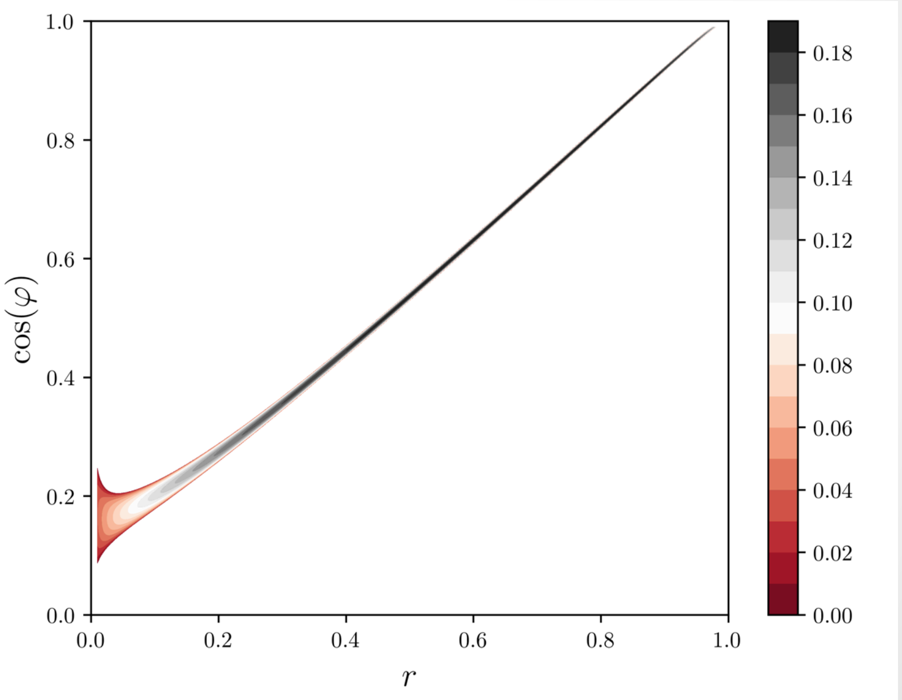
- Quantum behavior: occupation of modes grows exponentially within a resonance region

$$\gamma = \sqrt{\left(\delta\omega \frac{\delta n}{n} Q^{\vec{p}}_{\vec{k}}\right)^2 - \frac{1}{4}\epsilon^2_{\vec{k}\vec{p}}}$$

$$\epsilon_{\vec{k}\vec{p}} = \omega_k - \omega_p + \omega_{|\vec{k}-\vec{p}|}$$

- γ imaginary: stable modes
- γ real: unstable modes





One-mode perturbation

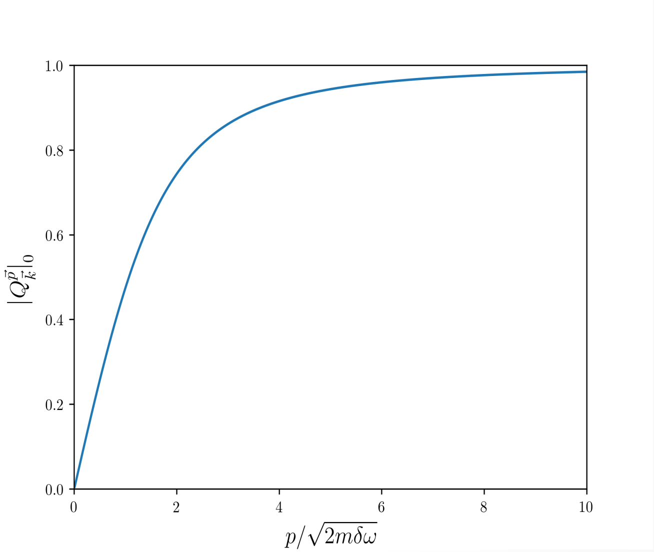
- Evaporation of the classical state

$$N_{ev}(t) = \sum_{\vec{k}} \langle \mathcal{N}_{\vec{k}}(t) \rangle \simeq V \int_{\gamma>0} \frac{d^3k}{(2\pi)^3} \langle \mathcal{N}_{\vec{k}}(t) \rangle$$

- Take integral

$$N_{ev} \approx \mathrm{e}^{2\gamma_0 t}$$

$$\gamma_0 = \delta\omega \frac{\delta n}{n} \left| Q^{\vec{p}}_{\vec{k}} \right|_0$$



- Duration of classicality

$$\boxed{t_{\mathrm{cl}} \approx \frac{1}{\delta\omega \frac{\delta n}{n} \left| Q^{\vec{p}}_{\vec{k}} \right|_0}}$$

Conclusions

- Quantum mechanics is needed to describe interacting quantum fields over timescales of order the relaxation time, even if the occupation numbers are high
- Axion dark matter can relax through gravitational self-interaction
- Quantify timescale of departure from classical field theory for a self-gravitating homogeneous field
- Duration of classicality becomes shorter when there are inhomogeneities? Eg. Repulsive contact self-interactions:

Homogeneous

$$t_{\rm cl} = \infty$$

Homogeneous + perturbation

$$t_{\rm cl} \propto \frac{1}{\delta\omega \frac{\delta n}{n_0}}$$

- Self gravitating inhomogeneous field? Growth of perturbation in axion dark matter?