Gravity, Information and Fundamental Symmetries

@MPI, Garching, Nov.4-6, 2019

Entanglement Wedges and AdS/CFT (in the light of Information Metric)

Tadashi Takayanagi

Yukawa Institute for Theoretical Physics (YITP), Kyoto Univ.

It from Qubit
Simons Collaboration

YITP
YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Contents

Refs for ④

arXiv: 1908.09939 with Yuki Suzuki (Kyoto U.)
and Koji Umemoto (YITP, Kyoto U.)
arXiv: 19XX.YYYYY with Yuya Kusuki (YITP, Kyoto),
Yuki Suzuki and Koji Umemoto

① Introduction

Microscopes are basic devices for experiments in science, such as accelerators in particle physics.

What are "microscopes" in gravity ?

⇒ **One answer is Holography !**
(AdS/CFT, gauge/gravity duality)

Holography is not a real experiment but provides a useful device in thought experiments of quantum gravity.

Indeed, holography magnifies a gravitational spacetime into tiny bits of quantum information.

Bekentein-Hawking Formula of BH Entropy

$$S_{BH} = \frac{k_B \cdot c^3}{4G_N \cdot \hbar} \cdot \mathrm{Area(Horizon)}.$$

➡ BH thermodynamics !

A= Surface Area of Black hole $\Rightarrow$ **Geometry**
G$_N$=Newton constant $\Rightarrow$ **Gravity**
$\hbar$=Planck constant $\Rightarrow$ **Quantum Mechanics**
k$_B$=Boltzmann constant $\Rightarrow$**Stat.Mech./Information**

Quantum Gravity!

BH Entropy is proportional to the area, not the volume!

➡ Degrees of Freedom in Gravity $\propto$ Surface Area !

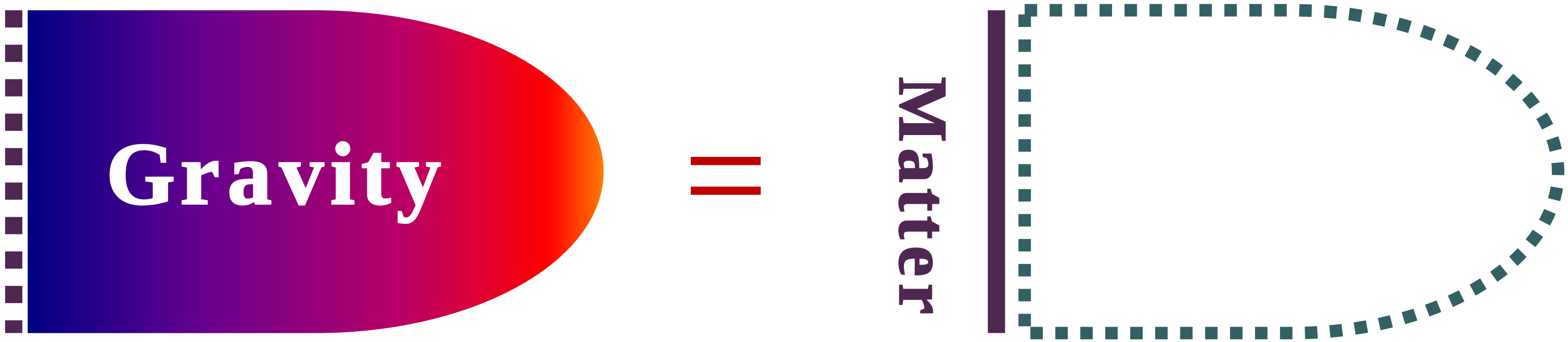
The idea of holographic principle !

['t Hooft 1993, Susskind 1994]

Bdy of M

Holography

Gravity on M = Quantum Matter on ∂M



BH entropy($\propto$Area)= Thermal Entropy of Matter ($\propto$Volume)

[String theory derivation of BH entropy: Strominger-Vafa 1996]

The best example of holography in string theory:

AdS/CFT Correspondence [Maldacena 1997]

AdS/CFT

Gravity (String theory) on D+1 dim. AdS (anti de-Sitter space)

=

Conformal Field Theory (CFT) on D dim. spacetime

CFT=Massless QFT

⬇ Classical limit

General relativity with $\Lambda < 0$

⬇ Large N + Strong coupling

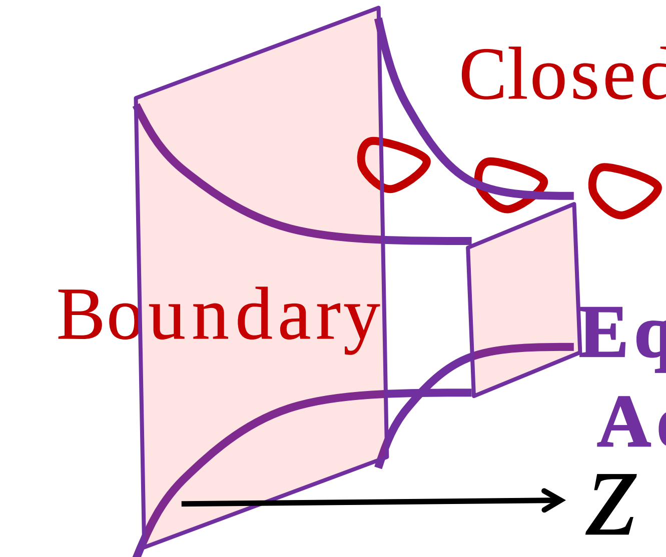
Strongly interacting Quantum Field Theories

Basic Principle
(Bulk-Boundary relation) :

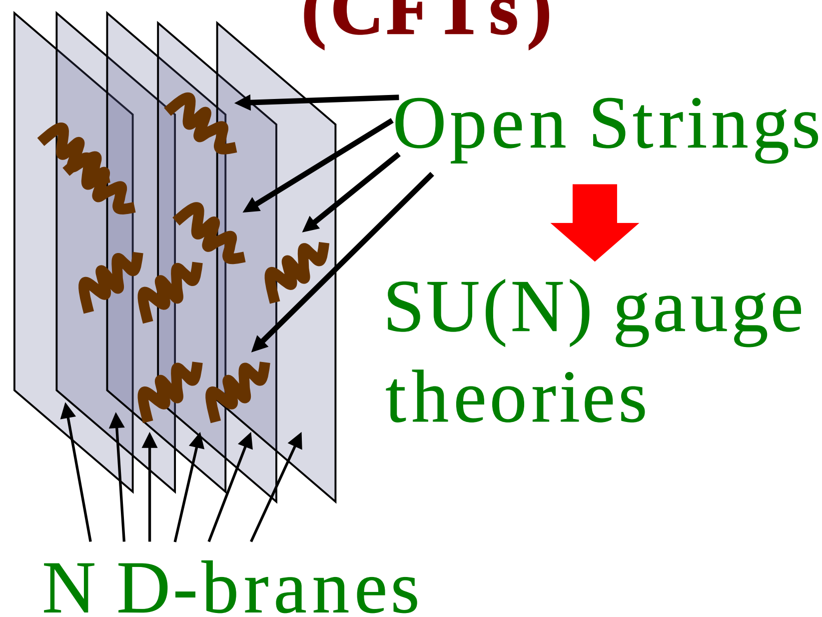
$$\boxed{Z_{Gravity} = Z_{CFT}}$$

Gravity in Anti de-Sitter space

Conformal Field Theories (CFTs)



Equivalent
AdS/CFT



SU(N) gauge theories

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2}{z^2}$$

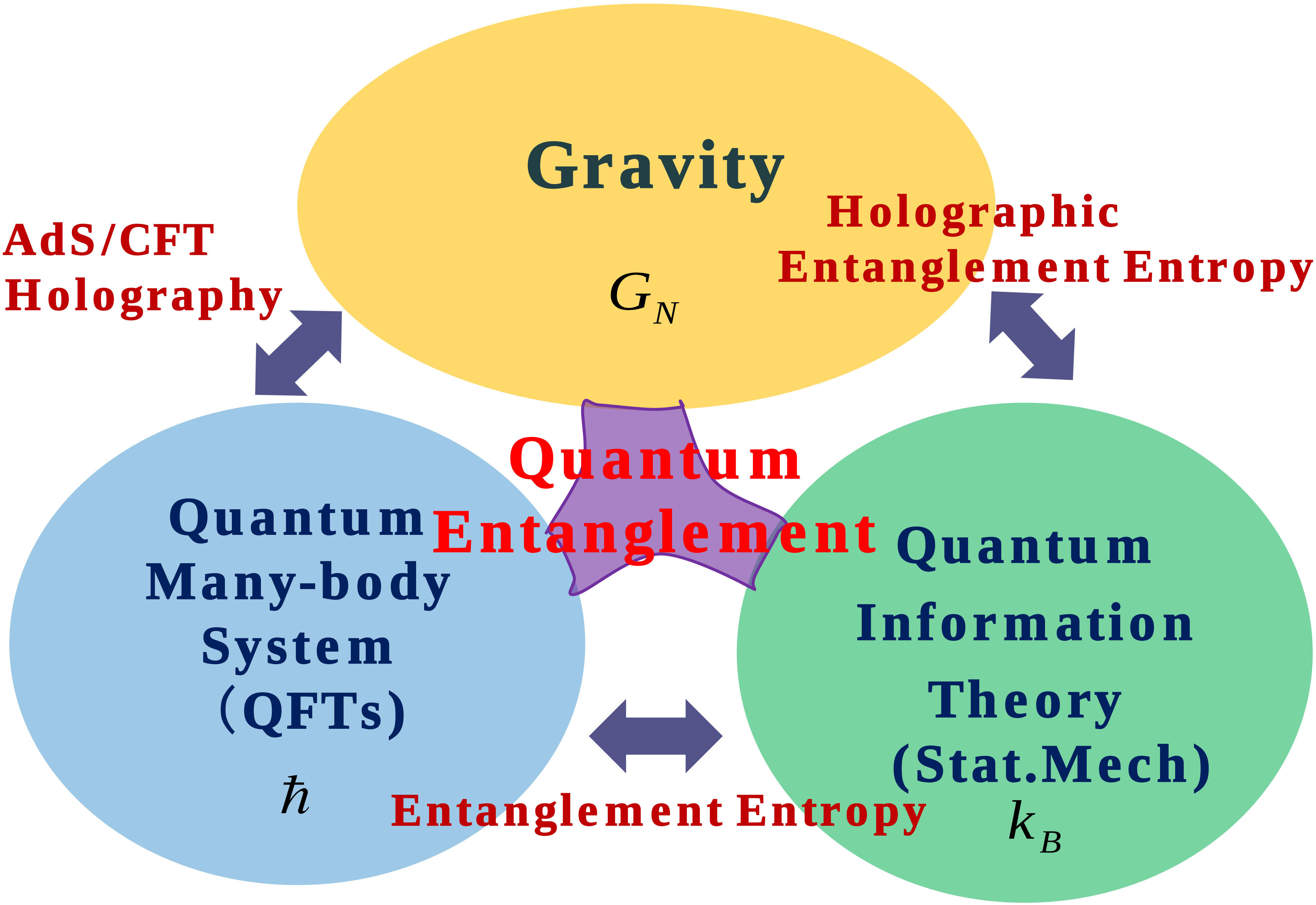
Equivalent

Thermodynamics of Black holes (branes) ⟷ **Thermodynamics of various materials**

However, the AdS/CFT has been a conjecture
for more than 20 years, without any definite proof,
in spite of so many evidences and successful checks.

Recent developments strongly suggest that
in order to understand basic mechanisms how the
AdS/CFT works, quantum information theoretical
ideas play crucial roles.

Quantum entanglement in quantum systems (CFT)
$\Rightarrow$ Emergent Geometry of Anti de-Sitter spacetime


Gravity
G_N
AdS/CFT
Holography
Holographic
Entanglement Entropy
Quantum
Entanglement
Quantum
Many-body
System
(QFTs)
$\hbar$
Quantum
Information
Theory
(Stat.Mech)
k_B
Entanglement Entropy

② Quantum Entanglement and AdS/CFT

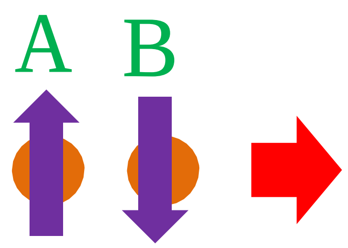
(2-1) Quantum Entanglement (QE)

QE = quantum correlations between two subsystems

Simple example: 2 Qubits system

(1) Direct Product State

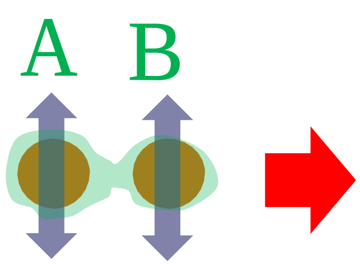
$$\left|\Psi_c\right\rangle = \left|\uparrow\right\rangle_A \otimes \left|\downarrow\right\rangle_B$$



Not Entangled

(2) EPR (Bell) States

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_A \otimes \left|\downarrow\right\rangle_B \pm \left|\downarrow\right\rangle_A \otimes \left|\uparrow\right\rangle_B\right)$$



Entangled

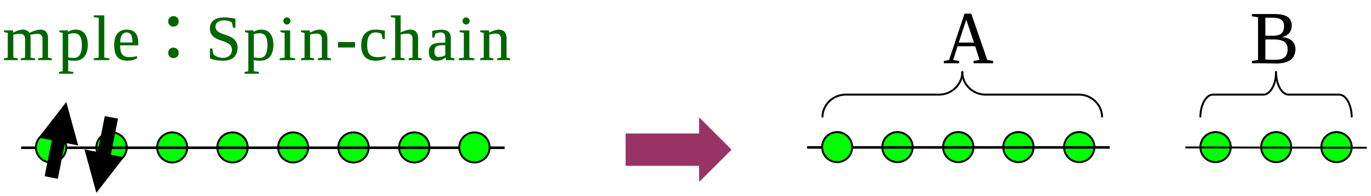
EPR Pair

Entanglement Entropy (EE)

Amount of QE = # of EPR Pairs ➡ **Entanglement Entropy (EE)**

First we decompose the Hilbert space: $H_{tot} = H_A \otimes H_B$.

Example：Spin-chain



We introduce the reduced density matrix ρ_A

by tracing out B $\rho_A = \mathrm{Tr}_B\left[\left|\Psi_{tot}\right\rangle\left\langle\Psi_{tot}\right|\right].$

The entanglement entropy (EE) S_A is defined by

$$\boxed{S_A = -\mathrm{Tr}[\rho_A \log \rho_A]}$$

(2-2) Holographic Entanglement Entropy (HEE)

[Ryu-Takayanagi 2006, Hubeny-Rangamani-Takayanagi 2007]

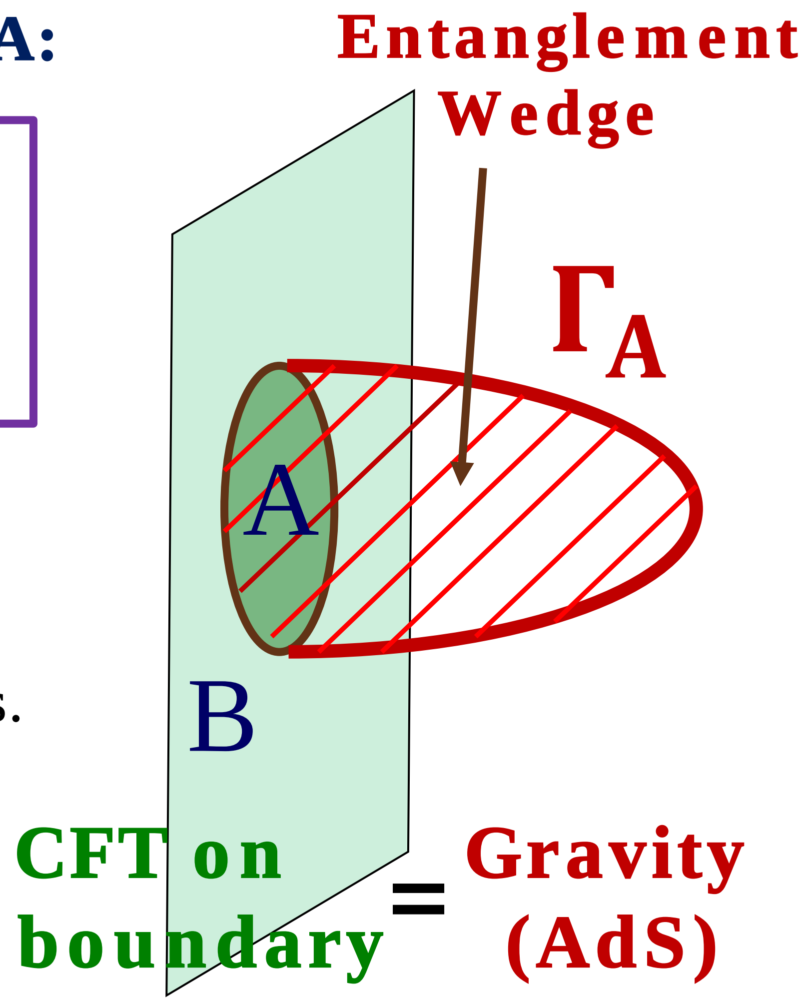
EE in CFT can be computed from the minimal area surface ΓA:

$$\boxed{S_A = \min_{\Gamma_A}\left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N}\right]}$$

Note: The bdy of ΓA =The bdy of A.

Many evidences for this conjecture have been found for more than 10 years.

This formula was proved by Lewkowycz-Maldacena 2013 based on the bulk-bdy relation of AdS/CFT.



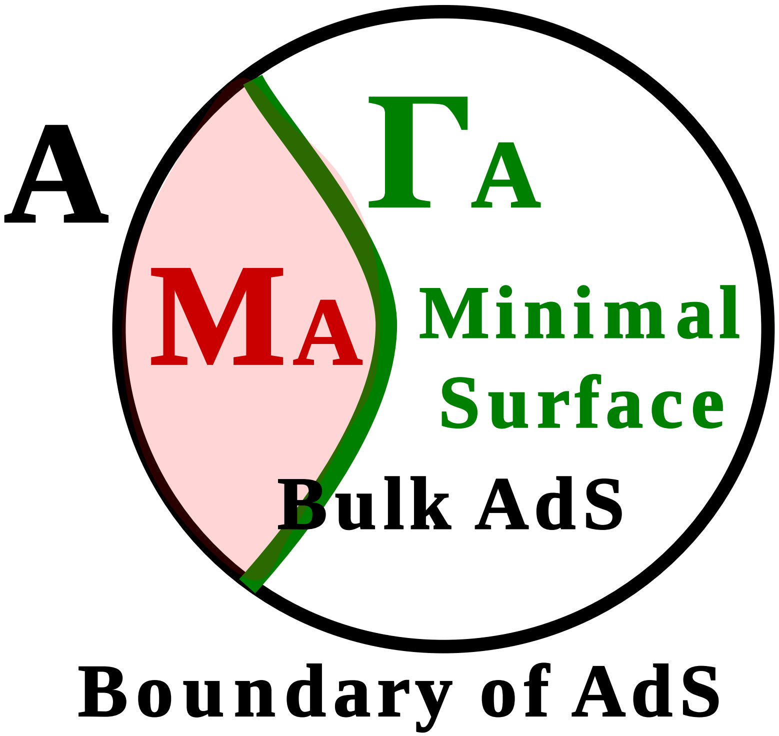
③ Entanglement Wedges in AdS/CFT

Definition of Entanglement Wedges

Which bulk region is dual to a given region A in CFT ?

⇒ Entanglement Wedge (EW) M_A

M_A = A region surrounded by A and Γ_A



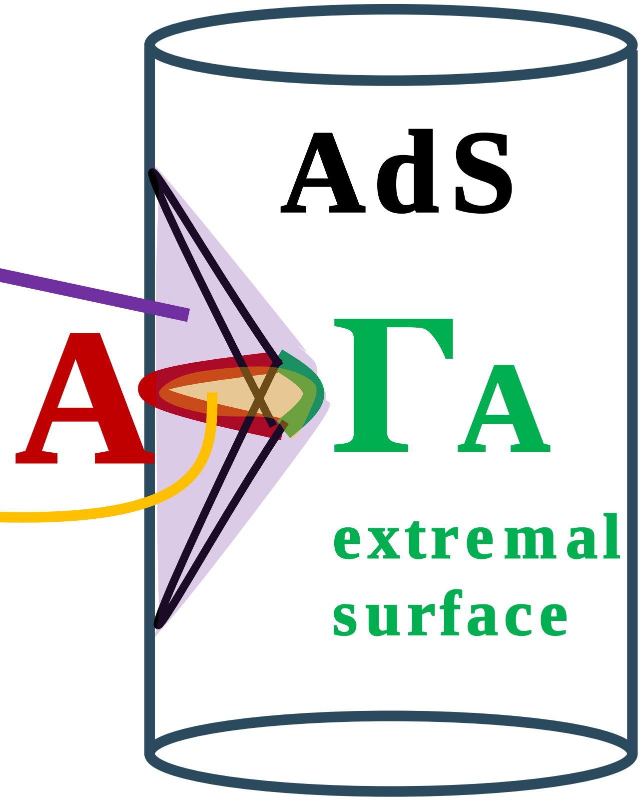
ρ_A in CFT (Low energy info)

$\Leftrightarrow$ ρ_{MA} in AdS gravity

[Hamilton-Kabat-Lifschytz-Lowe 2006, Czech-Karczmarek-Nogueira-Raamsdonk 2012, Wall 2012, Headrick-Hubeny-Lawrence-Rangamani 2014, Jafferis-Lewkowycz-Maldacena-Suh 2015, Dong-Harlow-Wall 2016, . . .]

Covariant Definition of EW

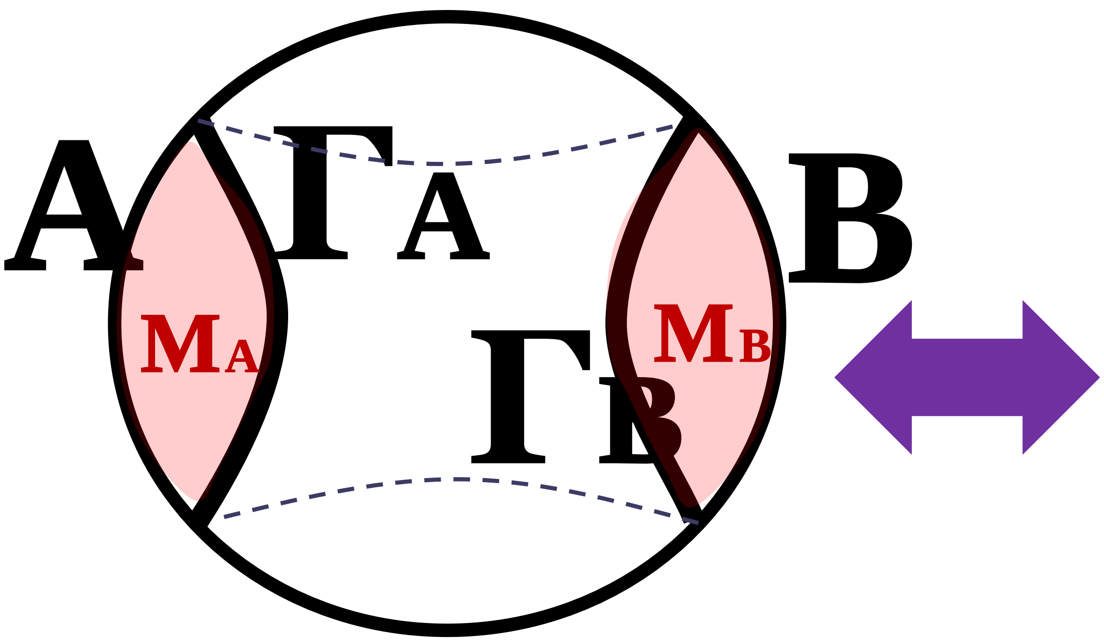
Entanglement Wedge
=Domain of dependence
of M_A

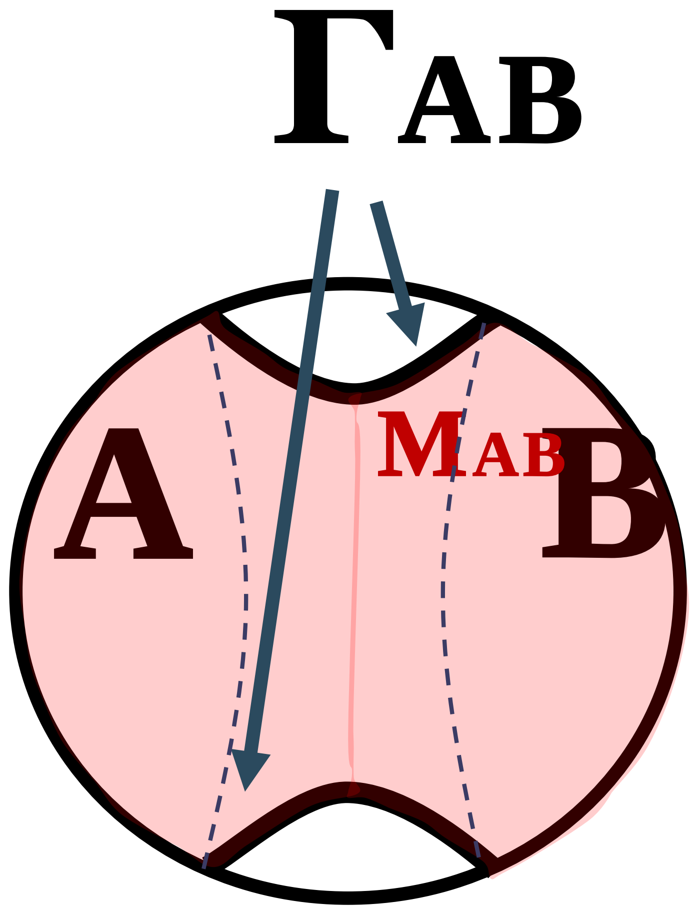


Though we assume static setups in this talk,
we extend to non-static setups in a covariant way.

EW for Disconnected Subregions

$$\mathrm{I}(A:B) = S_A + S_B - S_{AB} = 0$$



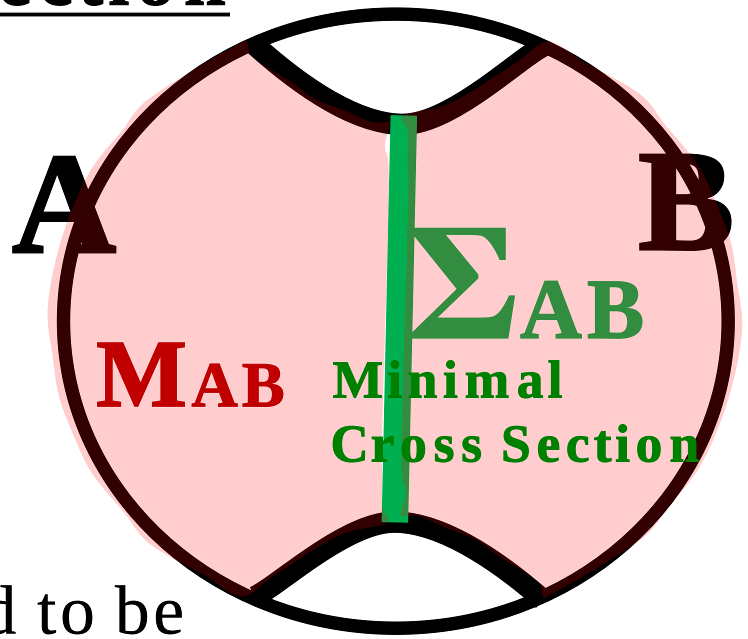


$M_{AB} = M_A \cup M_B$, $\Gamma_{AB} = \Gamma_A \cup \Gamma_B$ $\quad \mathrm{I}(A:B) > 0$

Entanglement Wedge Cross Section

Define the **EW cross section:**

$$E_W(\rho_{AB}) = \frac{\mathrm{Area}(\Sigma_{AB})}{4G_N}$$



Recently, this quantity is conjectured to be dual to a quantity called **entanglement of purification**.

$$E_W(\rho_{AB}) = E_P(\rho_{AB})$$

↑

Minimum of entanglement entropy when we purify the mixed state ρ_{AB}

[Umemoto-TT 2017, Nguyen-Devakul-Halbasch-Zaletel-Swingle 2017,
Explicit Checks from CFT calculations: Caputa-Miyaji-Umemoto-TT 2018]

[Classical correlation enhancement in EoP: Bhattacharyya-Jahn-Umemoto-TT 2019]

④ Derivation of Entanglement Wedges from CFTs

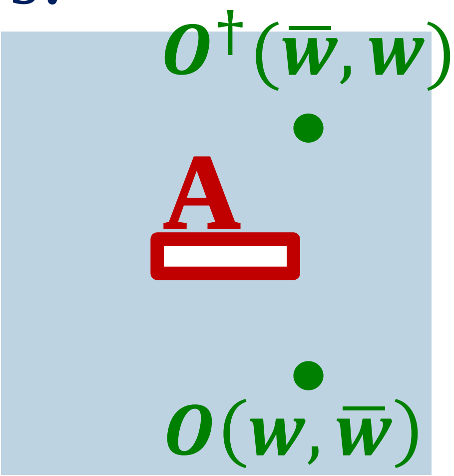
(4-1) Our Setup

Consider locally excited states in 2d CFTs:

$$\boxed{|\psi(w,\bar{w})\rangle = O(w,\bar{w})|0\rangle}$$

$O(w,\bar{w})$: **A primary in 2d CFT with conformal dim. h**

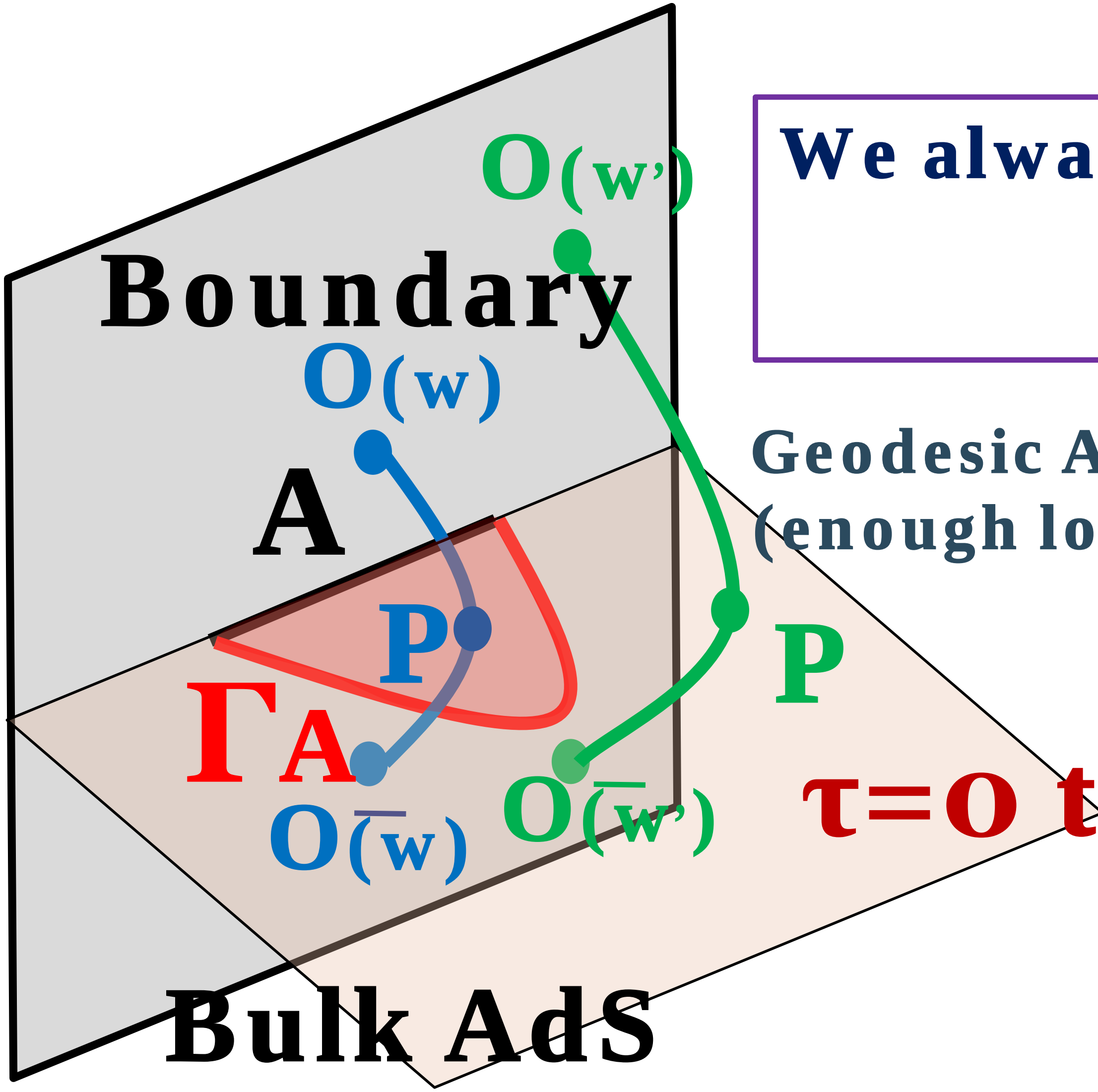
w$= x + i\tau \in$ complex plane R^2



By tracing out the subsystem B, we obtain the reduced density matrix: $\boxed{\rho_A(w,\bar{w}) = \mathrm{Tr}_B\left[O(w,\bar{w})|0\rangle\langle 0|O^\dagger(\bar{w},w)\right]}$

We will study the w-dependence of $\rho_A(w,\bar{w})$.

Gravity Dual of $\rho_A(w,\bar{w}) = \mathrm{Tr}_B\left[O(w)|0\rangle\langle 0|O^\dagger(w)\right]$



We always assume

$$1 << h << c \quad .$$

Geodesic Approximation (enough localization)

Negligible Back-reactions

ρ_A may be sensitive to O(w) but not to O(w').

(4-2) Information Metric

ρA is sensitive to O(w)
⇔ We can distinguish $\rho_A(w)$ and $\rho_A(w')$ if $w \neq w'$.

To study distinguishability of quantum states, consider the Bures distance between density matrices ρ and ρ'.

Bures distance: $$D_B(\rho,\rho')^2 = 2 - 2\underbrace{\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]}_{\textbf{Fidelity}}$$

For pure states, this is simplified as

$$D_B(|\Psi\rangle\langle\Psi|, |\Psi'\rangle\langle\Psi'|)^2 = 2(1 - |\langle\Psi|\Psi'\rangle|).$$

Assume density matrices depend on parameters λi, denoted by ρ(λ). **The Bures metric** is defined as follows:

$$\boxed{ds^2 \equiv D_B(\rho(\lambda), \rho(\lambda + d\lambda))^2 \cong G_{ij} d\lambda^i d\lambda^j}$$

We can calculate Bures distance for a locally excited pure state : $|\Psi(w)\rangle = \boldsymbol{O}(\boldsymbol{w}, \overline{\boldsymbol{w}})|\boldsymbol{0}\rangle$.

(dim=h primary)

$$D_B(|\Psi(w)\rangle\langle\Psi(w)|, |\Psi(w')\rangle\langle\Psi(w')|)^2 = 2(1 - |\langle\Psi(w)|\Psi(w')\rangle|).$$

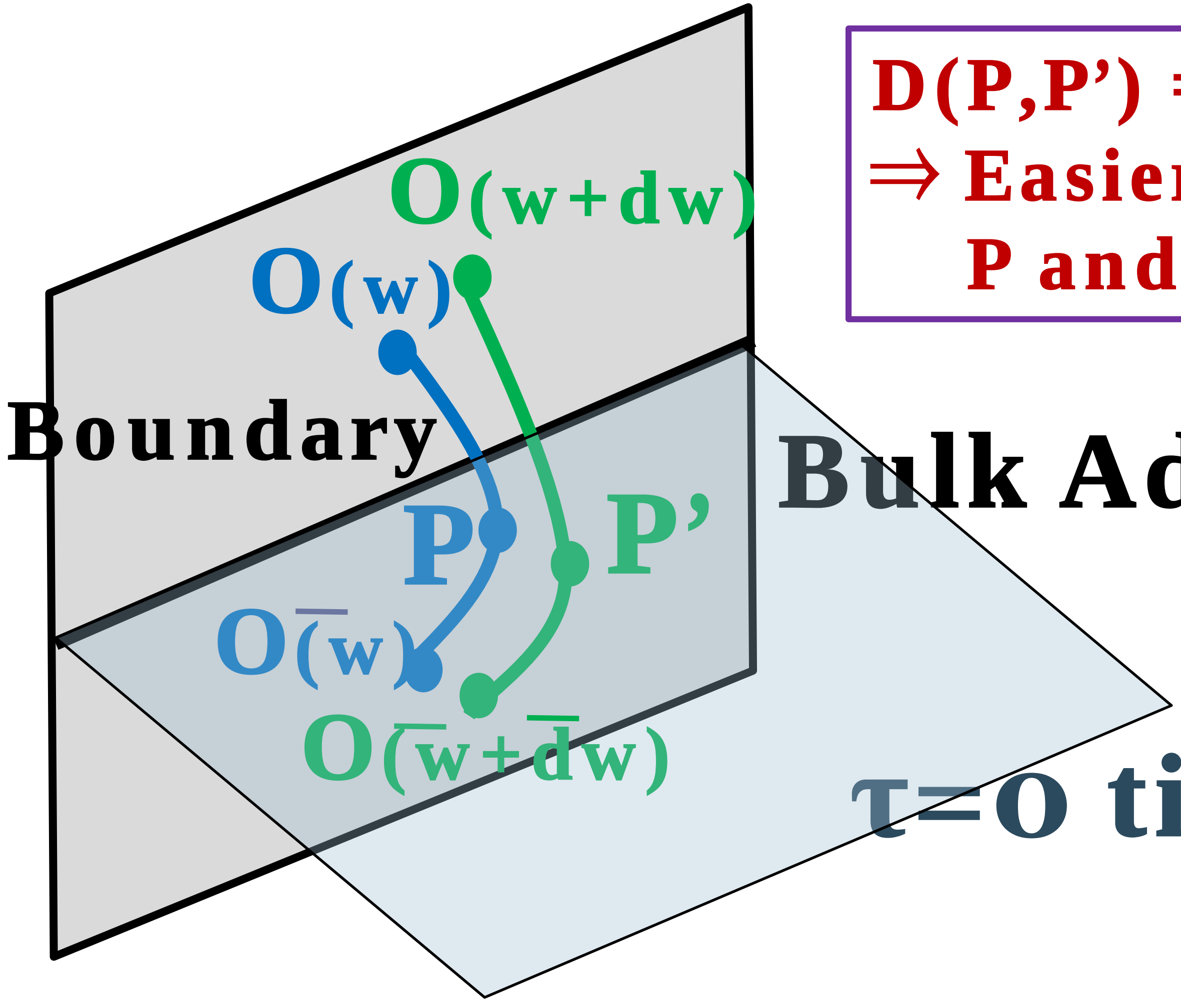
$$|\langle\Psi(w)|\Psi(w')\rangle| = |w - \overline{w}|^{2h} |w' - \overline{w'}|^{2h} |w - \overline{w'}|^{-4h}.$$

Finally, we get Bures metric (note: w=x+iτ):

$$\boxed{ds_B^2 = \frac{h}{\tau^2}(\mathrm{d}x^2 + d\tau^2)}$$

⇒ **The metric on a time slice of AdS !**

Bures Info. Metric ∝ Metric in Gravity Dual



D(P,P') = larger
⇒ Easier to distinguish P and P'

Bures metric for pure states is universal for any CFT !

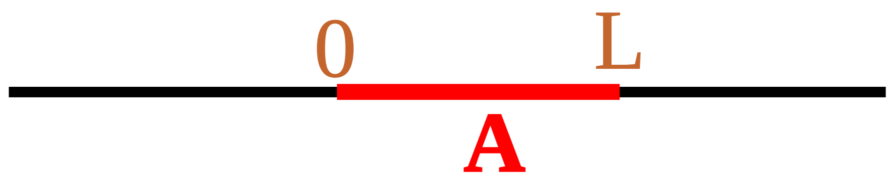
⇒ The situation largely changes for ρA as we will see.

(4-3) Single Interval Reduced Density Matrices

Consider the reduced density matrix

$$\boldsymbol{\rho_A(w,\bar{w}) = \mathrm{Tr}_B\left[O(w,\bar{w})|0\rangle\langle 0|O^\dagger(\bar{w},w)\right]}.$$

We choose A to be an interval A=[0,L] on R^2.



As a first step, let us start with the computation of a simpler version of Fidelity (like `2nd Renyi'):

$$\boldsymbol{I(\rho,\rho') = \frac{\mathrm{Tr}[\rho\rho']}{\sqrt{\mathrm{Tr}[\rho^2]\mathrm{Tr}[\rho'^2]}}}$$

$$\boldsymbol{I(\rho,\rho') = 1 \leftrightarrow \rho = \rho'}.$$

$$\boldsymbol{0 \leq I(\rho,\rho') \leq 1}.$$

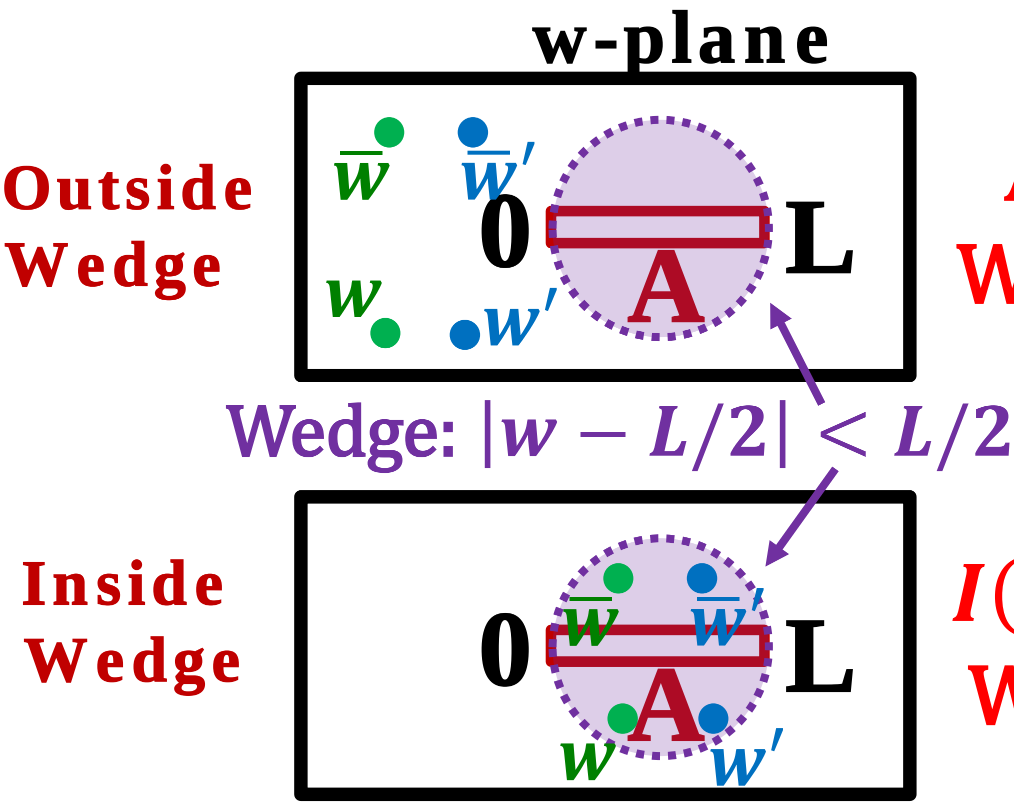
[Introduced by Cardy 2014]

Cf. $\boldsymbol{2 - D_B(\rho,\rho')^2 = 2\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]}$

Fidelity

The quantity $I(\rho(w),\rho(w'))$ is given by a 4-pt function.

For holographic CFTs (=strongly coupled large c CFT), the large c (or large N) factorization property, shows an **emergence of sharp entanglement wedge !**

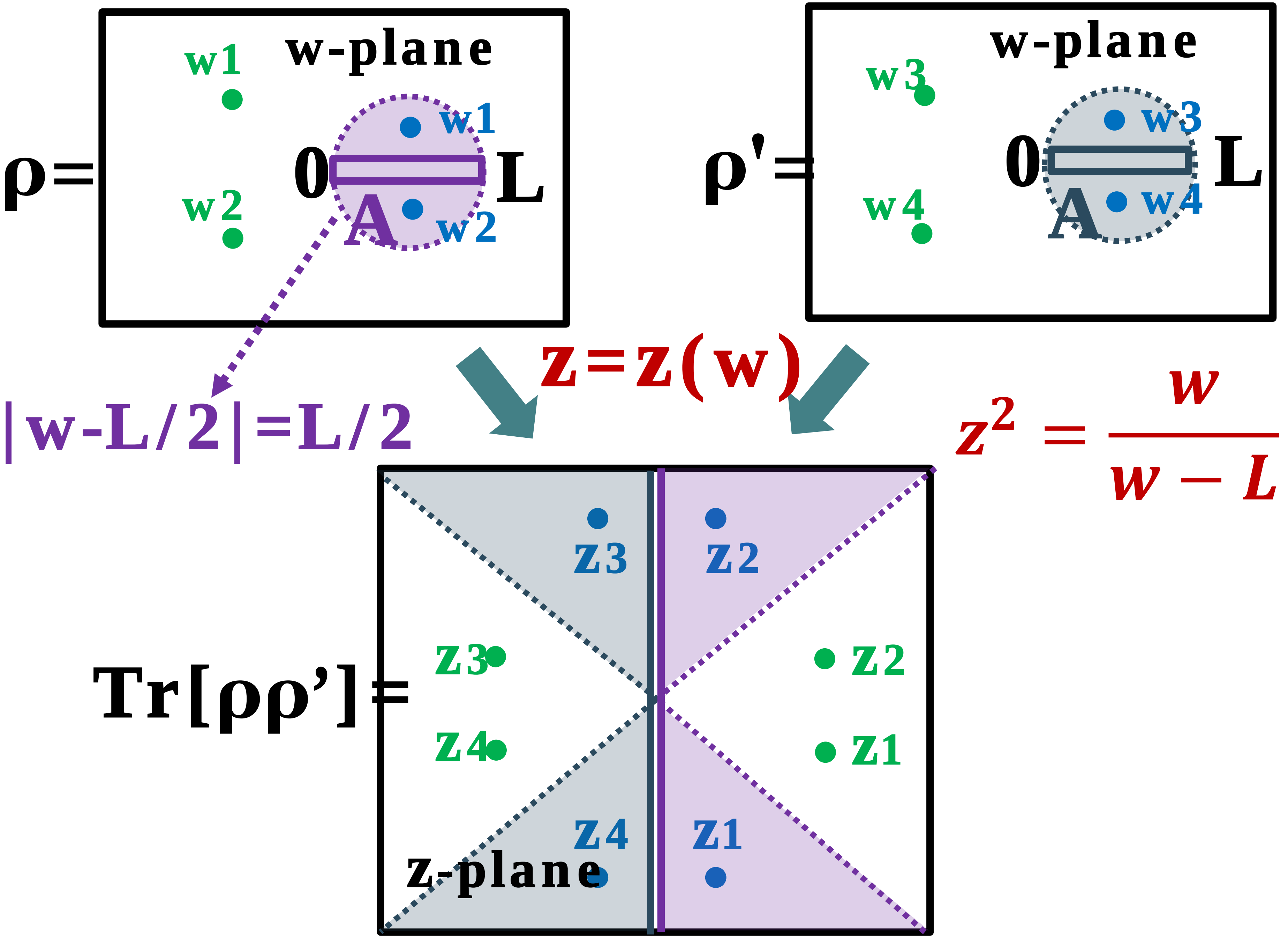


Always $I(\rho,\rho') = 1$

We cannot distinguish !

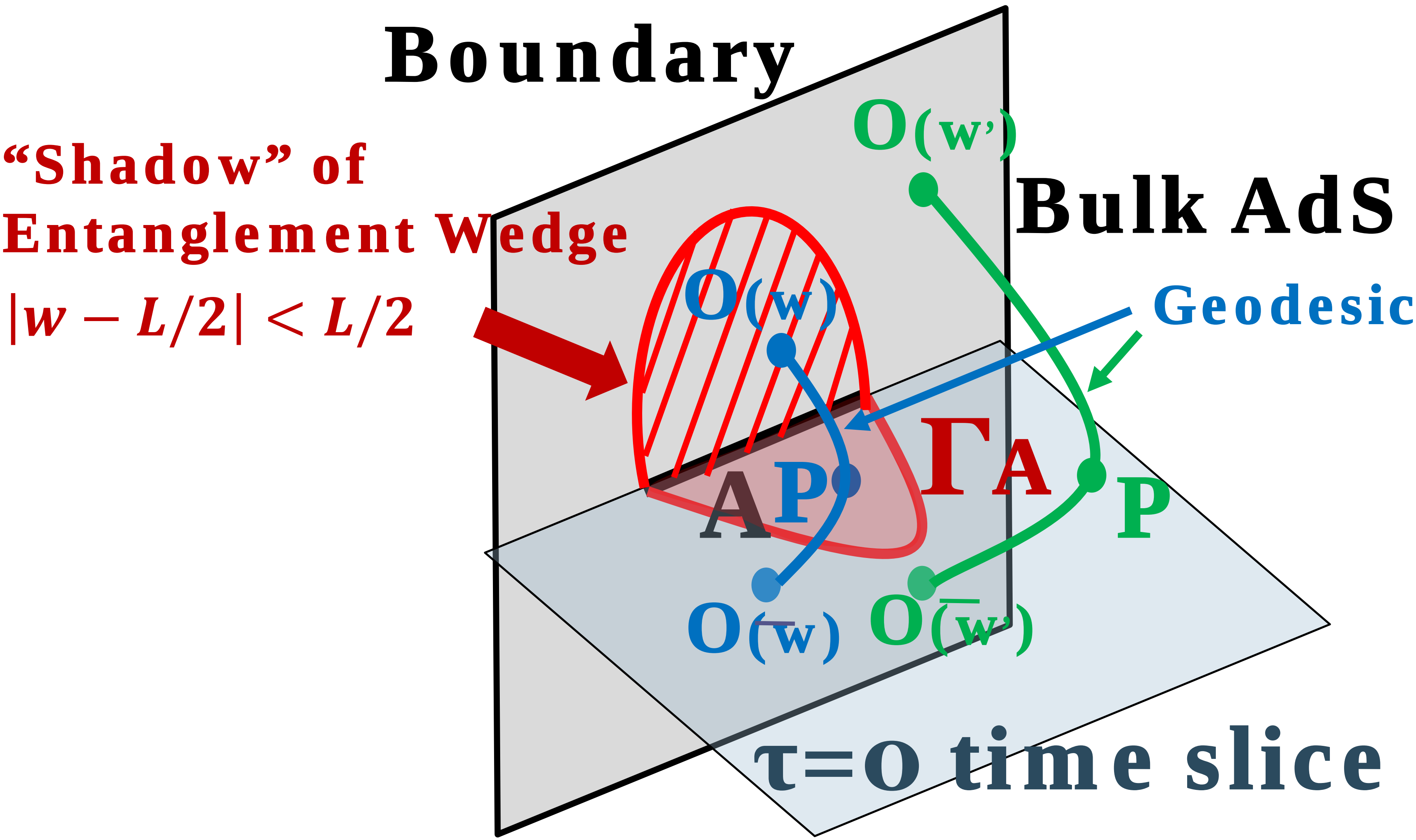
$I(\rho,\rho') = 0$ except w=w'

We can distinguish !



ρ=
w-plane
w1
w2
w1
w2
0
L
A
ρ'=
w-plane
w3
w4
w3
w4
0
L
A
|w-L/2|=L/2
z=z(w)
$z^2 = \frac{w}{w-L}$
Tr[ρρ’]=
z3
z2
z3
z2
z4
z1
z4
z1
z-plane

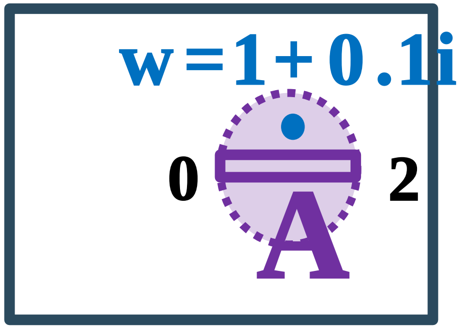
Wedge in CFT = Shadow of Entanglement Wedge

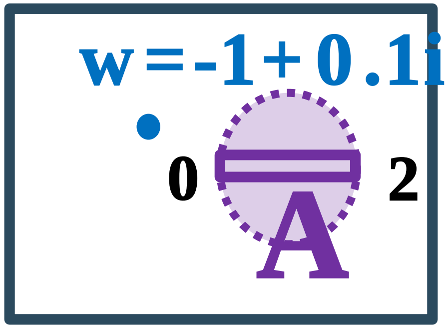


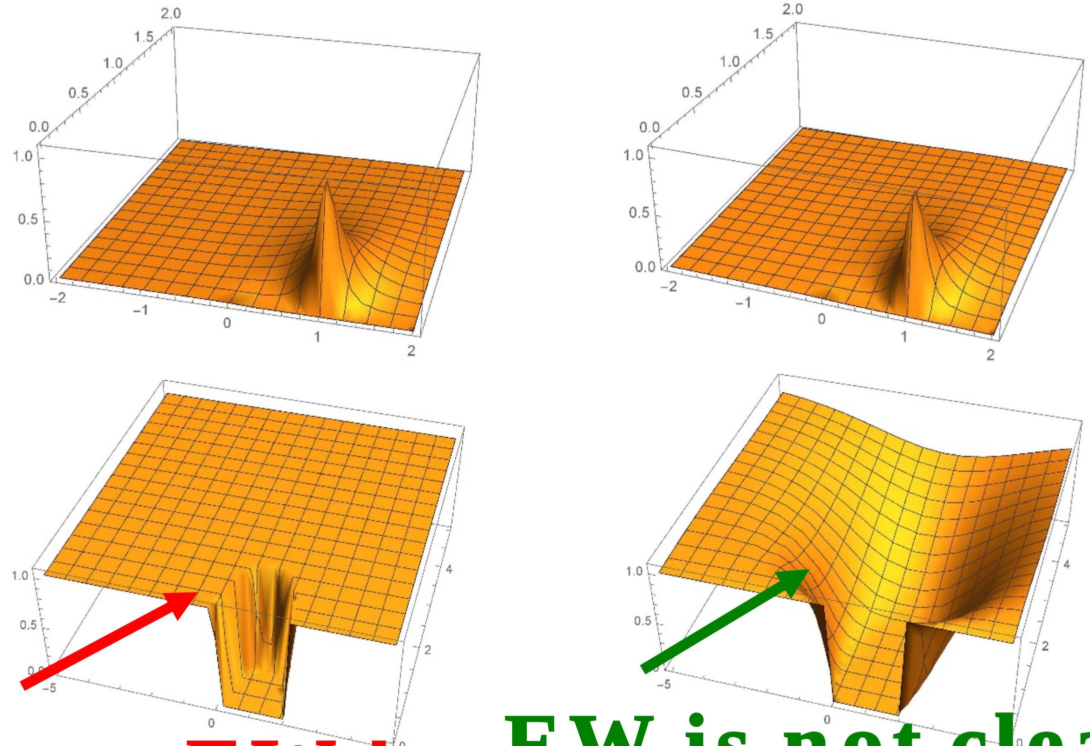
Plots of I(ρ(w),ρ(w’)) as a function of w’

Holographic CFT
(Strongly interacting)

Free Scalar CFT







Sharp EW !

EW is not clear !

We get sharp EWs only for holographic CFTs !

(4-4) Bures Information Metric

We can also evaluate the **Fidelity** by taking n=m=1/2 limit of $A_{n,m} = \mathrm{Tr}[(\rho^m \rho' \rho^m)^n]$.

The Fidelity $A_{1/2,1/2} = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]$ behaves as the previous quantity $I(\rho, \rho')$ does in the single interval case.

Thus, fidelity (or Bures distance) reproduces correct EWs.

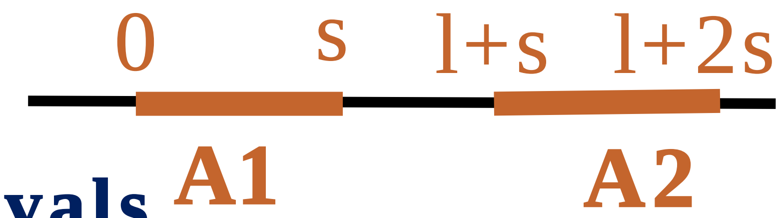
The Bure metric is computed as

$$\text{Inside wedge} \quad : \; ds_B^2 = \frac{h}{\tau^2}\left(\mathrm{d}x^2 + d\tau^2\right)$$
$$\text{Outside wedge} \; : \; ds_B^2 = 0$$

$\Rightarrow$ **Agree with EW geometry**

(4-5) Double Interval Case

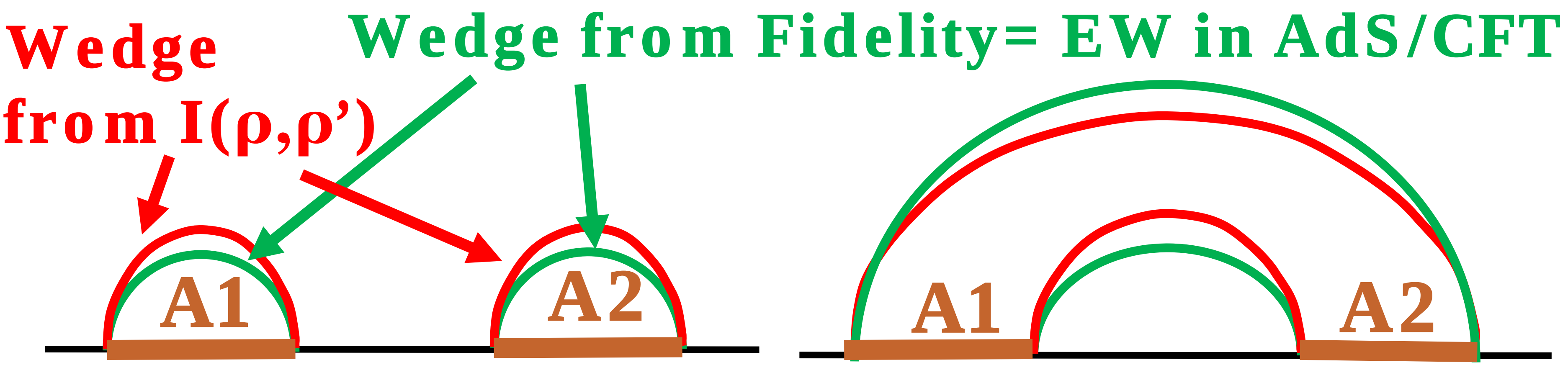
Consider a subsystem A
which consists of two intervals.



In this case, two distinguishability measures

(i) $I(\rho,\rho') = \dfrac{\mathrm{Tr}[\rho\rho']}{\sqrt{\mathrm{Tr}[\rho^2]\mathrm{Tr}[\rho'^2]}}$ (ii) $\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]$ Fidelity (Bures)

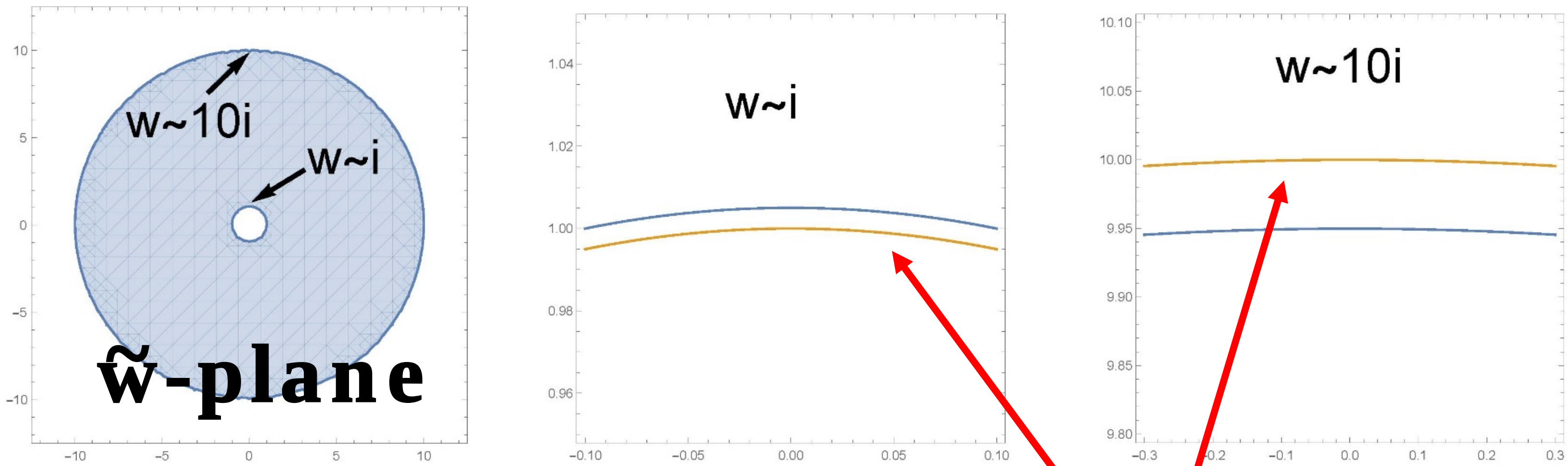
predict two different wedges:



Disconnected EW Case Connected EW Case

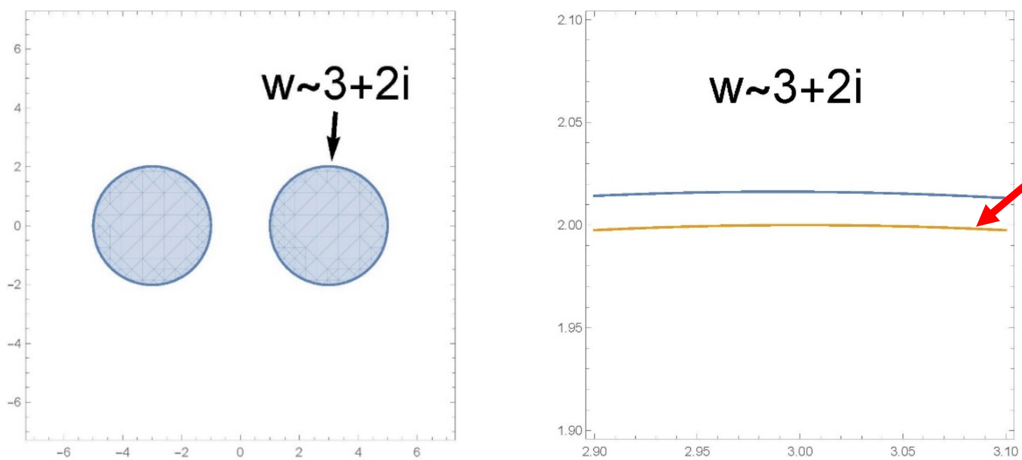
Numerical Plots of Wedges from CFT

(i) Connected Phase (κ=0.1)



(ii) Disconnected Phase (κ=0.2)

Actual EW



⑤ Conclusions

- The holographic counterpart of entanglement entropy is given by minimal surface areas in AdS.
- The entanglement wedges provide a bulk counterpart of a sub-region in a holographic CFT.
- We presented a new framework to derive the geometry of entanglement wedges purely from CFTs.
- The Bures distance (Fidelity) correctly reproduces the expected entanglement wedges, while the Renyi-like distance $\boldsymbol{I(\rho,\rho')}$ leads to a slightly deformed wedges.
- The Bures information metric is proportional to the metric of time slice of AdS. The same is true for BTZ.

Future Directions

- Why do wedges depend on the distance measures ?

  Bures distance $\Rightarrow$probe only Low energy (code subspace)

  $\boldsymbol{I}(\boldsymbol{\rho}, \boldsymbol{\rho}') \sim \mathbf{Tr}[\boldsymbol{\rho}\boldsymbol{\rho}']$ $\Rightarrow$probe both low and high energy

- Other Information Distances ?

  [Trace distance, Quantum Fisher Metric,...]

- Quantum Corrections to EW ?
- Connection to Tensor Network interpretation of AdS/CFT ?
- Any hints for holography for de-Sitter spaces ?
- Any hints for a proof of AdS/CFT ?

:

Special thanks to Organizers for this great workshop !

Klaus Blaum, MPIK
Allen Caldwell, MPP
Ignacio Cirac, MPQ
Manfred Lindner, MPIK
Dieter Luest, MPP

Backup Slides

Earlier Justification of Entanglement Wedges

Usually, EW is explained by combining several ideas:

(1) HKLL Reconstruction (CFT dual of bulk fields)

[Hamilton-Kabat-Lifschytz-Lowe 06]

(2) Quantum Corrected HEE (or Hol. Relative entropy)

[Faulkner-Lewkowycz-Maldacena 13, Jafferis-Lewkowycz-Maldacena-Suh 15]

(3) Relation to Quantum Error Correcting Codes

[Almheri-Dong-Harlow 2014, Dong-Harlow-Wall 2016]

These arguments assume the AdS spacetime and its dynamics from the beginning, based on AdS/CFT.

Our new argument in this talk is purely based on on CFT computations. ⇒EWs emerge from CFT !

I(ρ,ρ’) in 2d Holographic CFTs

The 4-pt function $F(z_1, z_2, z_3, z_4) = \langle O^\dagger O O^\dagger O \rangle$ is

given by applying the large N factorization:

$$F \approx |z_1 - z_2|^{-4h} |z_3 - z_4|^{-4h} + |z_2 - z_3|^{-4h} |z_1 - z_4|^{-4h}.$$

$\Downarrow$ $\Downarrow$

Trivial Wick Contraction

$\Rightarrow$ Always $I(\rho, \rho') = 1$

$\Rightarrow$ $\rho = \rho'$.

(i.e. Indistinguishable)

Non-Trivial Wick Contraction

$\Rightarrow$**In general,** $I(\rho, \rho') < 1$

(i.e. distinguishable)

Having in mind the information metric,
we take w ≈ w'.

[i] Case 1: the trivial Wick contraction is dominant: $|z_2 - z_3| < |z_1 - z_4| \Rightarrow |w - L/2| > L/2$

w ∉ Entanglement Wedge

$\rho_A(w)$ and $\rho_A(w')$ are indistinguishable !
Always we find I(ρ,ρ')=1.

[ii] Case 2: the non-trivial Wick contraction is domiant: $|z_2 - z_3| > |z_1 - z_4| \Rightarrow |w - L/2| < L/2$

w∈ Entanglement Wedge

$\rho_A(w)$ and $\rho_A(w')$ are distinguishable !
I(ρ,ρ')=1 only when w=w'.

Bures Metric in Holographic CFTs

We introduce $\boldsymbol{A_{n,m} = \mathbf{Tr}[(\rho^m \rho' \rho^m)^n]}.$

We compute Bures metric via analytical continuation:

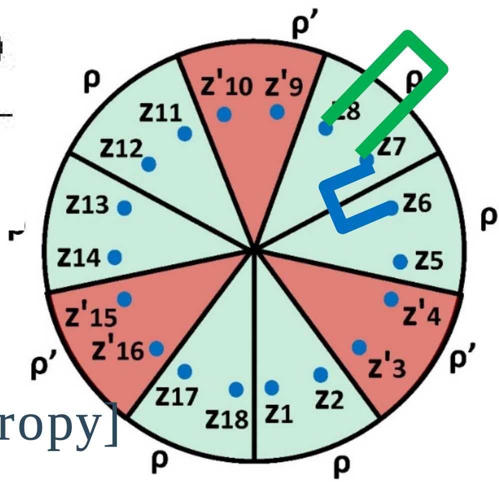
$$\boldsymbol{A_{n=\frac{1}{2},m=\frac{1}{2}} = \mathbf{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right].}$$

To evaluate Anm, consider the map:

$$\boxed{\boldsymbol{z^k = \frac{w}{w-L}},\ k=(2m+1)n}$$

$$A_{n,m} = \frac{\langle O^\dagger(w_1)O(w_2)\cdots O^\dagger(w_{2k-1})O(w_{2k})\rangle \cdot Z^{(k)}}{\prod_{i=1}^{k}\langle O^\dagger(w_{2i-1})O(w_{2i})\rangle \cdot (Z^{(1)})^k}$$

$$\boldsymbol{A_{n=3,m=1} =}$$



[See also Lashkari 2014, 2015, Ugajin 2016 for relative entropy]

Trivial Wick Contraction (Outside EW: |w-L/2|>L/2)

$$\langle O^\dagger(z_1)O(z_2)\cdots O^\dagger(z_{2k-1})O(z_{2k})\rangle$$
$$\simeq \prod_{j=1}^{k}\langle O^\dagger(z_{2j-1})O(z_{2j})\rangle \simeq \prod_{j=1}^{k}|z_{2j-1}-z_{2j}|^{-4h}$$

➡ **$A_{1/2,1/2}=1$,**
$D_B(\rho,\rho')=0$
Trivial metric

Non-Trivial Contraction (Inside EW: |w-L/2|<L/2)

$$\langle O^\dagger(z_1)O(z_2)\cdots O^\dagger(z_{2k-1})O(z_{2k})\rangle$$
$$\simeq \prod_{j=1}^{k}\langle O^\dagger(z_{2j-2})O(z_{2j-1})\rangle \simeq \prod_{j=1}^{k}|z_{2j-2}-z_{2j-1}|^{-4h}.$$

In the limit $n \to 1/2$ and $m \to 1/2$, this leads to

$$A_{1/2,1/2} \simeq |w-\bar{w}|^{2h}|w'-\bar{w}'|^{2h}|w'-\bar{w}|^{-4h},$$

$$dD_B^2 \simeq \frac{h}{\tau^2}(dx^2+d\tau^2).$$

➡ **Reproduce the time slice of AdS**

Other examples of Bure Metrics in Hol. CFTs

[1] Hol. CFT on Cylinder

$$ds_B^2 = \frac{h}{(\sinh\tau)^2}\left(d\tau^2 + dx^2\right)$$

Agrees with Global AdS

[2] Hol. CFT at Finite temp.

$$ds_B^2 = \frac{h(2\pi/\beta)^2}{\left(\sin\frac{2\pi}{\beta}\tau\right)^2}\left(d\tau^2 + dx^2\right)$$

Agrees with Global BTZ

Double Intervals Case

Consider the case where the subsystem A consists of double intervals in 2d CFTs. We choose A as A=A1∪A2, A1=[0,s] and A2=[l+s,l+2s].

Comformal Map:
from a cpx plane with two slits to a cylinder
[e.g. Rajabpour 2015]

$$\boxed{z = f(w) = \frac{J(\kappa^2)}{2} - \frac{J(\kappa^2)}{2K(\kappa^2)}\,\mathrm{sn}^{-1}(\widetilde{w}, \kappa^2),}$$

where we introduced

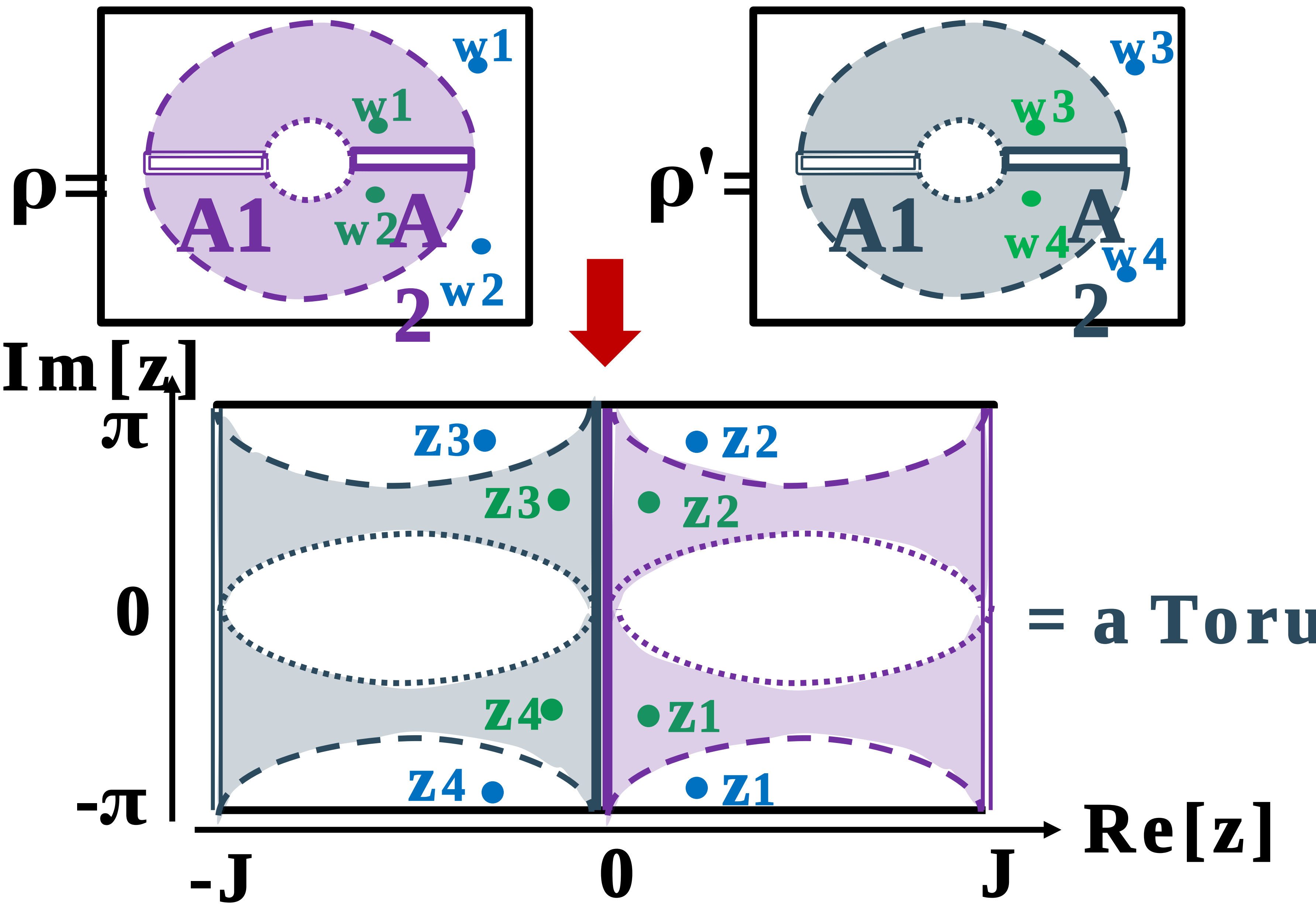
$$\tilde{w} = \frac{2}{l}\left(w - s - \frac{l}{2}\right), \quad \mathbf{J}(\kappa^2) = 2\pi \frac{K(\kappa^2)}{K(1-\kappa^2)},$$

$$K(\kappa^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}, \quad \kappa = \frac{l}{l+2s}.$$

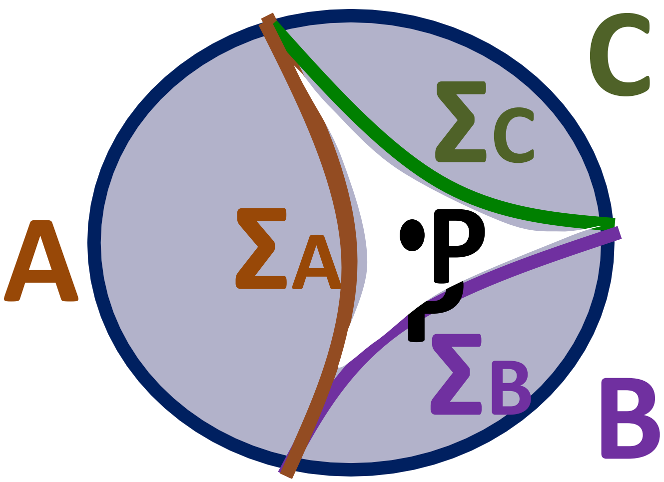
The function $\mathrm{sn}^{-1}(\tilde{w}, \kappa^2)$ is the Jacobi elliptic function:

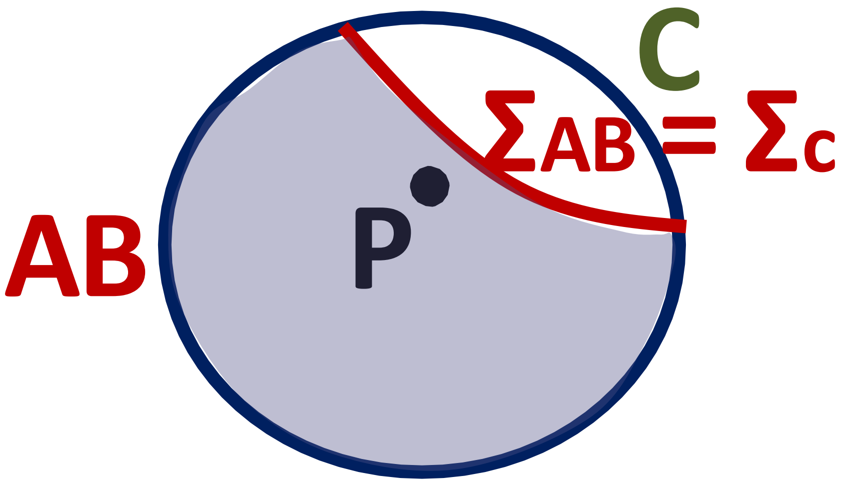
$$\mathrm{sn}^{-1}(\tilde{w}, \kappa^2) = \int_0^{\tilde{w}} \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}.$$

Computation of Tr[ρρ’]



Relation to Quantum Error Correcting Codes





We can reconstruct the bulk information at P from ρ_{AB}.

But we cannot do so from ρ_A, ρ_B, ρ_C .

➡ Property of **Quantum Error Correcting Codes**

Physical Space = all CFT states = quantum gravity

$\cup$

$\boxed{\textbf{Code Subspace}}$ **= Low energy states in CFT = GR**

Protected by QEC

[Almheiri-Dong-Harlow 2014]

Einstein Equation from QE

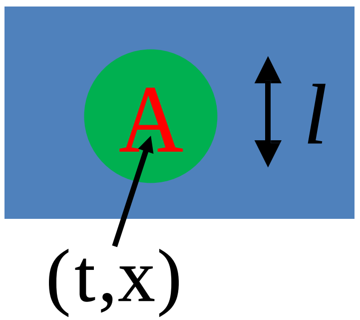
First Law of EE

[Casini-Huerta-Myers 2013,
Bhattachrrya-Nozaki-Ugajin-Takayanagi 2013]

$$\Delta S_A \cong \Delta H_A$$

[HA=-logρA: Modular Hamiltonian]

$$\left(\partial_l^2 - \partial_l - \partial_x^2 - \frac{3}{l^2}\right)\Delta S_A(t,\vec{x},l) = \langle O\rangle\langle O\rangle$$



$$\phi \leftrightarrow O$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$$

Kinetic term **C.C.** **Matter contributions**

The 1st law of EE explains the perturbative Einstein eq.

[Raamsdonk et.al. 2013, Faulkner et.al 2013, 2017, Sarosi-Ugajin. 2017]

Definition of Entanglement of Purification

Consider all purifications $\left|\Psi\right\rangle_{A\tilde{A}B\tilde{B}}$ of ρ_{AB} in the extended Hilbert space: $H_A \otimes H_B \to H_A \otimes H_B \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$.

Entanglement of Purification (EoP) is defined by

$$\boxed{\mathrm{E_P}(\rho_{AB}) = \underset{\text{All purifications } |\Psi\rangle \text{ of } \rho_{AB}}{\mathrm{Min}} S_{A\tilde{A}}\left(\left|\Psi\right\rangle_{A\tilde{A}B\tilde{B}}\right)}$$

$$\rho_{AB} = \mathrm{Tr}_{\tilde{A}\tilde{B}}\left[\left|\Psi\right\rangle\left\langle\Psi\right|\right].$$

Entanglement Entropy

Note: $\mathrm{E_p}(\rho_{AB}) \geq 0$ and $\mathrm{E_p}(\rho_{AB}) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B$.