Introduction to topology and homotopy theory

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1 Motivational example: Instantons in Yang-Mills theory

Consider SU(2) Yang-Mills theory on Euclidean $\mathbb{R}^4$ and action given by $S = -\frac{1}{2}\int_{\mathbb{R}^4} \operatorname{tr}(F_{\mu\nu}F^{\mu\nu})d^4x$. We want to find finite action solutions to the classical equations of motion. To do so, we first have to require that as our coordinate $|x| \to \infty$, the curvature $F_{\mu\nu} = \mathcal{O}(\frac{1}{|x|^2})$. This means that $F_{\mu\nu} \to 0$ in this limit and as such the gauge field A_μ becomes pure gauge, i.e. $A_\mu \to g^{-1}(x)\partial_\mu g(x)$. We must therefore asymptotically pick out a section $g(x) \in \mathrm{SU}(2)$ on the sphere S^3 at infinity.

To obtain physically equivalent states, we have to quotient out by gauge transformations that can be continuously deformed to the identity on the sphere at infinity, so that $\tilde{g}(x) \sim g(x)$ if we can find an $h(x)$ such that $\tilde{g}(x) = h(x)g(x)$ and $h(x)$ can be deformed to the identity on the asymptotic S^3.

If we quotient by this action, the physically different choices for $g(x)$ are classified by the homotopy group $\pi_3(\mathrm{SU}(2)) \simeq \mathbb{Z}$. This set of integers is also what defines the instanton number of a classical solution to the equations of motion.

GOAL: The goal for this talk will be to understand what, for example, these groups π_n are and understand some calculational tools. Further applications will be discussed in next talks.

2 What is a topology?

To discuss these homotopy groups, we first have to discuss some topology. The field of topology is very extensive and I can only hope to give you an introduction and some examples to understand the basic concepts in this talk.

Motivation to introduce topology

The field of topology is used to make the notion of 'nearness' mathematically precise. If one has a space with a metric, one can define rigourously what it means for two points to be near, but it turns out that many results do not require a metric. They only depend on topology. In some sense, a topological space is therefore a generalisation of a metric space. A topology should be understood as all the opens of a space. More precisely, a topology is the following

Definition *A topological space (X, τ) is a pair of a set X and a family τ of subsets of X such that*

- *The empty set $\emptyset$ and X are both contained in τ*
- *If $U, V \in \tau$, then $U \cap V \in \tau$*
- *If $\{U_i\}_{i \in I} \subset \tau$ for some possibly infinite index set I, then $\bigcup_{i \in I} U_i \in \tau$*

For concreteness, let us give some examples

Examples

1. Any set X with $\tau = \{\emptyset, X\}$ defines a topological space. This is a strange topology if one is used to metric spaces, because if our set X contains at least two points, these two points can never be separated by non-intersecting open neighbourhoods. This can always be done in a metric space, for example. (Such a topology is known as non-Haussdorff.)
2. The Euclidean $\mathbb{R}^n$ with opens given by (infinite unions) of open balls in $\mathbb{R}^n$
3. Topological manifolds: Any space X such that around each point $x \in X$, we can find a neighbourhood U_x that has the same topology as $\mathbb{R}^n$ (homeomorphic, see next section). For example, $X = S^n \subset \mathbb{R}^{n+1}$ inherits this topology from $\mathbb{R}^{n+1}$.

Notion of having the same topology

To define what it means to have the same topology, we have to define what it means for a map $f : (X, \tau_X) \to (Y, \tau_Y)$ between topological spaces to be continuous. The definition of a continuous function between metric spaces is defined such that the pre-image of an open set is again an open set. We therefore mimic this definition of continuity:

Definition *A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called continuous iff for each open $U \in \tau_Y$, the pre-image $f^{-1}(U) \in \tau_X$.*

If the function f is a bijection and the inverse f^{-1} is also continuous, we call f a homeomorphism and we say that the topological spaces (X, τ_X) and (Y, τ_Y) are homeomorphic. Let us give some examples

Examples

1. The real line $\mathbb{R}$ with Euclidean topology and the semi-infinite interval $(0, \infty)$ are homeomorphic. One can see this by applying the map $\exp : \mathbb{R} \to (0, \infty)$ with inverse $\log : (0, \infty) \to \mathbb{R}$. Both of these maps are continuous.

2. The cylinder $S^1 \times \mathbb{R}$ is homeomorphic to the punctured plane $\mathbb{R}^2 - \{0\}$. If we parametrise $\mathbb{R}^2 - \{0\}$ as $\mathbb{C} - \{0\}$, we can apply the map $(t, x) \mapsto e^{x+it}$. This is continuous and has continuous inverse.

3. The Euclidean space $\mathbb{R}^n$ and the sphere S^n are not homeomorphic! One can see this for example by the fact that S^n is compact, while $\mathbb{R}^n$ is not

4. The Euclidean space $\mathbb{R}^n$ and a single point $\{*\}$ are also not homeomorphic, since $\{*\}$ is compact. However, these two spaces are homotopy equivalent to each other! We will discuss this later.

Topological invariants

Topological invariants are properties or objects of topological spaces that only depend on the topology up to homeomorphism. If such invariants for different spaces do not coincide, we can therefore immediately tell that these spaces are not homeomorphic to each other. Examples include compactness, connectedness, the Euler characteristic, orientability of spaces, etc.

Topological invariants can be intrinsic to topology (e.g. compactness and connectedness) or obtained by associating some other, often algebraic, structure to a topology (e.g. the Euler characteristic and orientability). The homotopy groups fall in the second class.

3 What are homotopies?

If we can associate an algebraic structure (e.g. a group) to topological spaces X and Y in some way, often a continuous map $f : X \to Y$ induces a map $f_* : A(X) \to A(Y)$ on the associated algebraic structures. Often these structures can be discrete, for example when $A(X)$ is a discrete group. In this case, one would expect f to induce the same map f_* even if we continuously deform f. This is the idea behind what we call a homotopy.

Definition *Given two topological spaces X and Y with continuous maps $f_0, f_1 : X \to Y$, we call $F : X \times [0, 1] \to Y$ a homotopy from f_0 to f_1 if it is continuous and $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We write $f_0 \sim f_1$ if such a map F exists.*

Intuitively, we now want to call two spaces X and Y homotopy equivalent if they have isomorphic algebraic structures. That is, if we have maps $f : X \to Y$ and $g : Y \to X$, they induce inverses $g_* f_* = \mathrm{id}_{A(X)}$ and $f_* g_* = \mathrm{id}_{A(Y)}$. To make this precise, we say

Definition *Two topological spaces X and Y are called homotopy equivalent if there exist two maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.*

One should think about homotopy equivalent spaces as being obtained from each other by 'squishing' or 'stretching'.

Examples

1. If two spaces are homeomorphic, they are also homotopy equivalent. The converse is not necessarily true.
2. $\mathbb{R}^n$ is homotopy equivalent to a single point $\{*\}$. One can check that $p(\mathbf{x}) = *$ and $i(*) = \mathbf{0}$ indeed satisfy $p \circ i = \mathrm{id}_*$ and $i \circ p \sim \mathrm{id}_{\mathbb{R}^n}$ through $F(\mathbf{x}, t) = t\mathbf{x}$.

4 What are homotopy groups?

Consider a topological space X and all continuous functions $f : [0,1]^n \to X$ such that the boundary $\partial([0,1]^n)$ is mapped to a single point $x_0 \in X$ (known as the base point). In other words, we are looking at all functions mapping S^n into X with a given base point.

If we consider this space of functions up to homotopies, we obtain a space we denote by $\pi_n(X, x_0)$. It turns out that this space defines a group by concatenating maps together. We therefore call this *the n-th homotopy group of X based at x_0*. The homotopy groups are an algebraic structure associated to a topological space X and indeed, they only depend on X up to homotopy equivalence.

Additionally, if the space X is path-connected (which is equivalent to being connected in the standard sense if we are looking at topological manifolds), the groups only depend on x_0 up to conjugation. We therefore often leave out the base point x_0.

Example: Calculating $\pi_1(S^1)$, i.e. the fundamental group of S^1

Consider S^1 as the set $\{e^{2\pi i t} | t \in \mathbb{R}\}$. To calculate $\pi_1(S^1)$, we will look at all maps $f : [0,1] \to S^1$ such that $f(0) = f(1) = 1$. Such a map can always be written as $f(t) = e^{2\pi i g(t)}$ for some $g : [0,1] \to \mathbb{R}$ if we require $g(1) - g(0) = n \in \mathbb{Z}$. Without loss of generality we may assume $g(0) = 0$ so that $g(1) = n$.

It turns out that $\tilde{g} \sim g$ iff $\tilde{g}(0) = 0$ and $\tilde{g}(1) = n$. In particular, this defines all maps f up to homotopy by the integer n, which is also known as the degree of the map f. In conclusion, we find $\pi_1(S^1) \simeq \mathbb{Z}$.

Generically, it is very difficult to calculate homotopy groups. One method to do so is through exact sequences.

5 What are exact sequences?

An exact sequence is a sequence of homomorphisms between (for our purposes) groups A_i of the form $... \to^{f_n} A_n \to^{f_{n+1}} A_{n+1} \to ...$ such that $\mathrm{im}(f_n) = \ker(f_{n+1})$.

In some sense, any exact sequence is built up from short exact sequences, i.e. exact sequences consisting of just three groups $0 \to A_0 \to^{f_1} A_1 \to^{f_2} A_2 \to 0$.

In the case of a short exact sequence, we find that

- The map f_1 is injective, because $\ker(f_1) = 0$
- The map f_2 is surjective, because $\mathrm{im}(f_2) = A_2$

In particular, this means that there exists a map $\tilde{f}_2 : A_1/\mathrm{im}(f_1) \to A_2$ induced from f_2 which satisfies $\ker(\tilde{f}_2) = 0$, since $\mathrm{im}(f_1) = \ker(f_2)$. Therefore $\tilde{f}_2$ is both injective and surjective.

If $\tilde{f}_2$ has an inverse which is also a group homomorphism, it defines an isomorphism $A_1/\mathrm{im}(f_1) \simeq A_2$. If we look at the special case where $A_0 = 0$ is trivial, we indeed find $A_1 \simeq A_2$.

Example Consider the sequence of homomorphisms $0 \to \mathbb{Z} \to^{\times n} \mathbb{Z} \to^{\mathrm{mod}\, n} \to \mathbb{Z}/n\mathbb{Z} \to 0$. Then one checks that indeed $\ker(\times n) = 0$ and $\mathrm{im}(\mathrm{mod}\, n) = \mathbb{Z}/n\mathbb{Z}$. Additionally, if we pick $m \in \mathbb{Z}$, then $mn \mod n = 0$ and conversely, if some $k \in \mathbb{Z}$ satisfies $k \mod n = 0$, then $k = mn$ for some $m \in \mathbb{Z}$. Therefore, this sequence is exact.

It turns out that if topological spaces fit together in a fibration, denoted by $F \to E \to B$, we can associate the following exact sequence of homotopy groups to this fibration:

$$... \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to ... \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0$$

In some sense, a fibration should be understood as the generalisation of a fibre bundle. The map $E \to B$ defines a fibre bundle with fibre F over base space B. The fibres are only required to be homotopy equivalent to each other.

Let us consider two examples of such calculations:

Example 1: The Hopf fibration

It is known that one can define a fibration $\mathrm{U}(1) \to \mathrm{SU}(2) \to \mathrm{SU}(2)/\mathrm{U}(1)$, which is a fibration of $S^3 \simeq \mathrm{SU}(2)$ over $S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$ with fibre $S^1 \simeq \mathrm{U}(1)$. We already showed that $\pi_1(S^1) \simeq \mathbb{Z}$, but one can similarly show that $\pi_n(S^n) \simeq \mathbb{Z}$ through the degree of the map (or through the fibration $\mathrm{SO}(n) \to \mathrm{SO}(n+1) \to S^n$). Additionally, it is also known that $\pi_{n\neq 1}(S^1) \simeq 0$ and $\pi_m(S^n) \simeq 0$ for $m < n$. Our long exact sequence therefore splits as follows

$$... \to 0 \to \pi_{n>2}(S^3) \to \pi_{n>2}(S^2) \to 0 \to ... \to 0 \to \pi_2(S^2) \to \pi_1(S^1) \to 0 \to ... \to 0$$

and we find $\pi_{n>2}(S^3) \simeq \pi_{n>2}(S^2)$ and $\pi_2(S^2) \simeq \pi_1(S^1)$.

Example 2: Coset spaces

If we have a group G with closed subgroup $H < G$, the map $G \to G/H$ defines a principal fibre bundle with fibre H. This therefore also defines a fibration $H \to G \to G/H$ and we can apply the same machinery to relate the homotopy groups of H, G and G/H. Note that the Hopf fibration is a special case of this example. This exact sequence then takes the form

$$... \to \pi_n(H) \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to ... \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 0$$

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