

Vortices:Literature:

- Books:
- (Manton, Sutcliffe) Topological Solitons
  - Quantum Field Theory (Ryder)
  - Cosmic Strings and other topological defects (Vilenkin, Shellard)
  - Quantum Theory of Fields Vol 2 (Weinberg)

Reviews:

- Lenz 0403286
- D. Tong 0509216 + TASI Lectures on Solitons 2005
- t' Hooft 0010225
- Hindmarsh 9411342
- Schaposnik 0611028

Talks:

- Francesco Benini: Strings 2018 June 25 a.m. minute (28:30)
- D. Tong: @ CERN 2016, Recent Progress in 3d gauge theories.

Index:

1) Introduction

2) The Nielsen-Olesen vortex.

- The Abelian Higgs model
- Finding the vortex solution (Ansatz & asymptotics).
- BPS solution
- Basic phenomenology

3) Other vortices

- Global vortex
- Chern-Simmons vortex
- Non-Abelian vortices

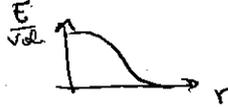
4) Applications

- Phase diagrams of superconductors
- Particle vortex dualities
- String dynamics
- Cosmic strings

# 1) Introduction:

• Vortex  $\sim$  classical field configuration  $\phi$  where  $\phi \neq 0$  everywhere except at one point around which  $\arg(\phi(\theta=0)) - \arg(\phi(\theta=2\pi)) = 2\pi n$

+ Localized core



$d=2+1$   
(particle-like)



$d=3+1$   
(string-like)

• why study vortices?

- One of the most common objects in physics.  
(as we will see).

• when were they discovered?

- 1957: Abrikosov finds them in the Ginzburg-Landau equation.

$$F_{GL} = |\vec{\nabla}\phi|^2 + m^2(\tau)|\phi|^2 + \lambda(\tau)|\phi|^4$$

- 1973: Nielsen-Olesen find them in their relativistic incarnation from QFT.

## 2) The Nielsen-Olesen vortex

### - The Abelian-Higgs model

• Theory with a complex scalar and a local  $U(1)$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - V(\phi)$$

with  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ,  $D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$  and

$$V(\phi) = m^2 |\phi|^2 + \lambda |\phi|^4$$

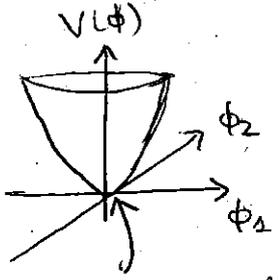
• Invariant under

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x) \quad \text{and} \quad \phi \rightarrow e^{i q \Lambda(x)} \phi$$

• Vacuum solution.

Given  $\phi = \phi_1 + i \phi_2$  we have that:

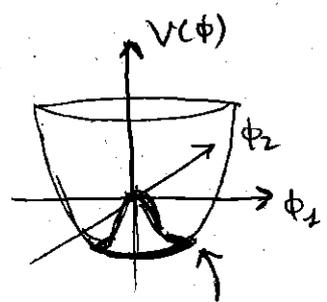
$$\underline{m^2 > 0}$$



vacuum manifold =  $S^0$

$$V_{min} \leftrightarrow \phi = 0$$

$$\underline{m^2 < 0}$$



vacuum manifold =  $S^1$

$$V_{min} \leftrightarrow \phi = v \cdot e^{i\theta}$$

• Working in  $d=2+1$ , the boundary of the space at infinity is a circle, i.e.  $S^1$ .

• For  $m^2 > 0$   $\phi(r=\infty): S^1_{\infty} \rightarrow S^0 \Rightarrow \pi_1(S^0) = 0 \Rightarrow$  no solitons (all maps in the same equiv. class).

• For  $m^2 < 0$   $\phi(r=\infty): S^1_{\infty} \rightarrow S^1 \Rightarrow \pi_1(S^1) = \mathbb{Z} \Rightarrow$  solitons possible (maps classified by  $n \in \mathbb{Z}$ )  
 $\Rightarrow$  vortices only if  $\text{SSB}$  &  $d=2+1$

- Finding the vortex solution:

• EOM:

①  $D^2 \phi = - \frac{\partial V}{\partial \phi}$

②  $ie (\phi \partial_\nu \phi^* - \phi^* \partial_\nu \phi) + 2e^2 A_\nu |\phi|^2 = \partial_\nu F_{\mu\nu}$

• Energy of the system:

$E = \int d^2x \left( -\frac{1}{4} F_{\mu\nu}^2 + |D_\nu \phi|^2 + V(\phi) \right)$

We want to find static solutions where the energy density  $\mathcal{E} \equiv \frac{E}{\text{volume}}$  is finite  $\forall \vec{x}$

and  $\phi$  acquires a winding number.  $\arg \phi(\theta=0) - \arg \phi(\theta=2\pi) = 2\pi n$

• Case  $n=0$

One can check that

$\phi_{\text{vortex}} = \sqrt{\frac{-m^2}{2\lambda}} \equiv v$  and  $A_\nu = 0$   
 $\frac{\partial V}{\partial \phi} |_{\phi=\phi_{\text{vortex}}}$

has finite  $\mathcal{E}$  density and satisfies the EOM.

• Case  $n \neq 0$

No exact solution is known, only numerical.

We can guess its form and general properties just requiring  $\lim_{r \rightarrow \infty} D_\nu \phi = 0$  in order to make the  $\mathcal{E}$  density finite.

• Case  $n \neq 0$ : conditions at  $r = \infty$

\*  $D_\nu \phi = 0$  EOM (1)  
 $\Rightarrow \frac{\partial V}{\partial \phi} = 0 \Rightarrow \boxed{|\phi| = v}$   
 $\Rightarrow (\phi_0: s_1^0 \rightarrow s_1)$

\*  $D_\nu \phi = 0 = \partial_\nu \phi + i e A_\nu \phi \Rightarrow$

$A_\nu = \frac{i}{e \phi} \partial_\nu \phi = \partial_\nu \left( \frac{i}{e} \ln \phi \right) \equiv \partial_\nu \chi \Leftrightarrow \boxed{A_\nu \text{ is pure gauge}}$

$\Rightarrow F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = \partial_\nu \partial_\mu \chi - \partial_\mu \partial_\nu \chi = 0 \Rightarrow \boxed{F_{\nu\mu} = 0}$

\* A static solution  $\Rightarrow \partial_0 \phi = 0 \Rightarrow \boxed{A_0 = 0}$

\* EOM (2) in polar coordinates and radial gauge ( $A_r = 0$ )  
( $r, \theta$ )

(2r)  $\phi \partial_r \phi^* = \phi^* \partial_r \phi$

(2\theta)  $\frac{i}{r} (\phi \partial_\theta \phi^* - \phi^* \partial_\theta \phi) + 2 e A_\theta |\phi|^2 = 0$

\* Ansatz for  $\phi$  satisfying all conditions:

$\phi = v \cdot e^{i n \theta} \Rightarrow \boxed{A_\theta = - \frac{n}{e r}}$

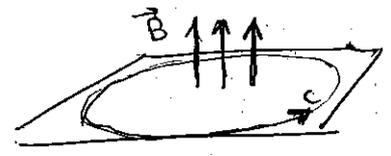
→ Notice that for  $n=0$  we recover the trivial solution

→ These conditions imply that there is a magnetic flux

stores

$$\oint_C \vec{A} \cdot d\vec{e} = \oint_C A_\theta \cdot r \cdot d\theta \stackrel{\downarrow}{=} \int_S \vec{B} \cdot d\vec{S} = - \frac{2\pi n}{e} = \Phi_B$$

$A_\theta = - \frac{n}{e r}$



the flux is quantized.

(in  $d=3+1$  theories with cylindrical symmetry, trivial in  $\hat{z}$ )

• Case  $n \neq 0$ : Vortex solution ansatz & asymptotics

\* It has been proven that solutions with the following ansatz:

$$\phi = \underbrace{f(r)}_{\in \mathbb{R}} e^{in\theta}, \quad A_\theta = A_\theta(r), \quad A_r = A_\phi = 0$$

exist, and are local minima of the  $E$ .

\* Notice that for  $\phi$  to be single valued at  $r=0 \Rightarrow \boxed{f(r) \xrightarrow{r \rightarrow 0} 0}$

\* Expanding around  $r = \infty$

$$|\phi| = v - \delta\phi(r)$$

$$A_\theta = -\frac{n}{e \cdot r} - \delta A_\theta(r)$$

and solving the EOM perturbatively (linearized EOM), we find that

$$|\phi| \sim v(1 - c_s K_0(m_h \cdot r)) \sim v(1 - \tilde{c}_s \frac{e^{-m_h \cdot r}}{\sqrt{r}})$$

$$A_\theta \sim -\frac{n}{e \cdot r} - c_v K_2(m_r \cdot r) \sim -\frac{n}{e \cdot r} - \tilde{c}_v \frac{e^{-m_r \cdot r}}{\sqrt{r}}$$

$$\Rightarrow \vec{B} = B_z \hat{z} \sim \frac{e^{-m_r \cdot r}}{\sqrt{r}} \hat{z} \Rightarrow \left\{ \begin{array}{l} \xi_s = \frac{1}{m_h} \\ \xi_r = \frac{1}{m_r} \end{array} \right. \quad \text{characteristic lengths.}$$

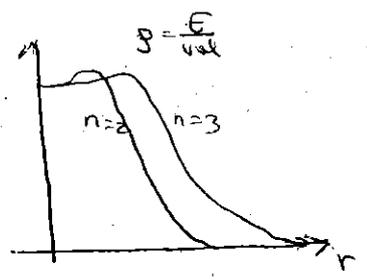
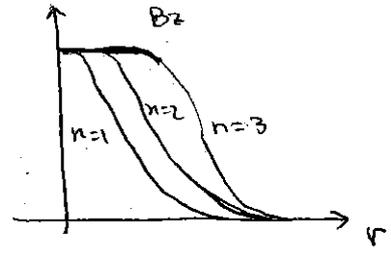
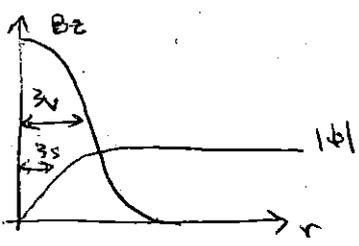
\* Expanding near  $r=0$  in polynomials

$$|\phi| \sim r^n$$

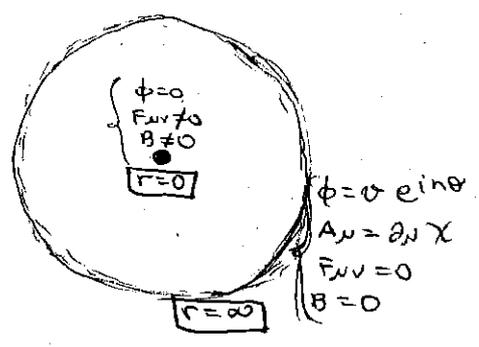
$$A_x \sim -y \cdot \tilde{f}(r^2)$$

$$A_y \sim x \cdot \hat{f}(r^2)$$

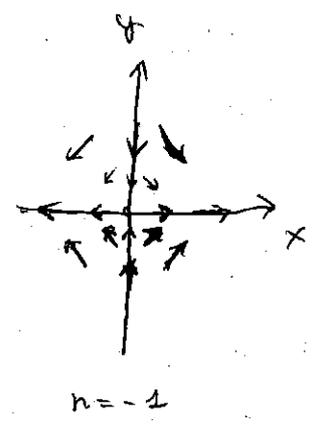
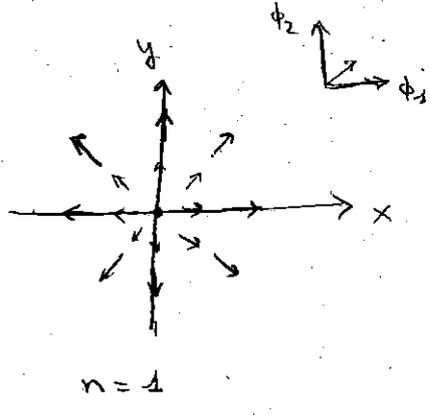
• Case  $n \neq 0$ : Numerical solutions



schematic solution



$$\phi = \phi_1 + i\phi_2 = f(r) \cdot (e^{in\theta} + i f(r) s(n))$$



BPS solution

- \* The Abelian-Higgs vortices found have special physical and mathematical properties when  $m_A = m_S$  i.e. when  $\lambda = e^2$
- \* For convenience we rescale the fields and couplings such that the Energy can be written as

$$E = \frac{1}{2} \int (B^2 + |D_i \phi|^2 + \frac{\tilde{\lambda}}{4} (1 - |\phi|^2)^2) d^3x$$

where we assumed again a static solution.

- \*  $m_h = m_A$  now corresponds to  $\tilde{\lambda} = 1$ .

\* For  $\tilde{\lambda} = 1$  we can write the E as:

$$E = \frac{1}{2} \int \left[ (B - \sqrt{V})^2 + (D_x \phi^* - i D_y \phi^*) \cdot (D_x \phi + i D_y \phi) + B - i (\partial_x (\phi^* D_y \phi) - \partial_y (\phi^* D_x \phi)) \right] d^2x =$$

$\Phi = \int \vec{B} \cdot d\vec{s} = 2\pi n$ 
 $\sim \int d\vec{s} \cdot \vec{\nabla} \times (\phi^* \vec{D} \phi) = \oint_{\infty} (\phi^* \vec{B} \phi) \cdot d\vec{\ell} = 0$   
 $\uparrow$   
 $D_x \phi = 0$  at  $r = \infty$

$$E = \frac{1}{2} \int \left[ \underbrace{(\dots)^2 + 1}_{\text{positive}} \right] + \pi n \Rightarrow \boxed{E \geq \pi |n|}$$

Bogomol'nyi bound.

Bound saturated if:

$$\left. \begin{cases} D_x \phi + i D_y \phi = 0 \\ B = \frac{1}{2} (1 - |\phi|^2)^2 \end{cases} \right\} \text{BPS equations.}$$

If  $\phi, A$  satisfy BPS  $\Rightarrow$  satisfy EOM (stable) (exist).

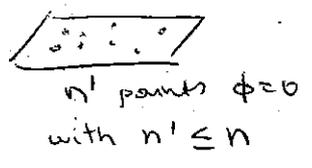
Vortices with  $\tilde{\lambda} = 1$  (critical coupling)

$\rightarrow$  The only static solutions that satisfy the EOM are those also satisfying the BPS bound.  $\Rightarrow \boxed{E_n = \pi |n|}$

$\rightarrow$  There are no solutions for  $n < 0$ .

$\rightarrow n = 1$  has a unique solution centered at  $r = 0$  with  $E_{n=1} = \pi$ .

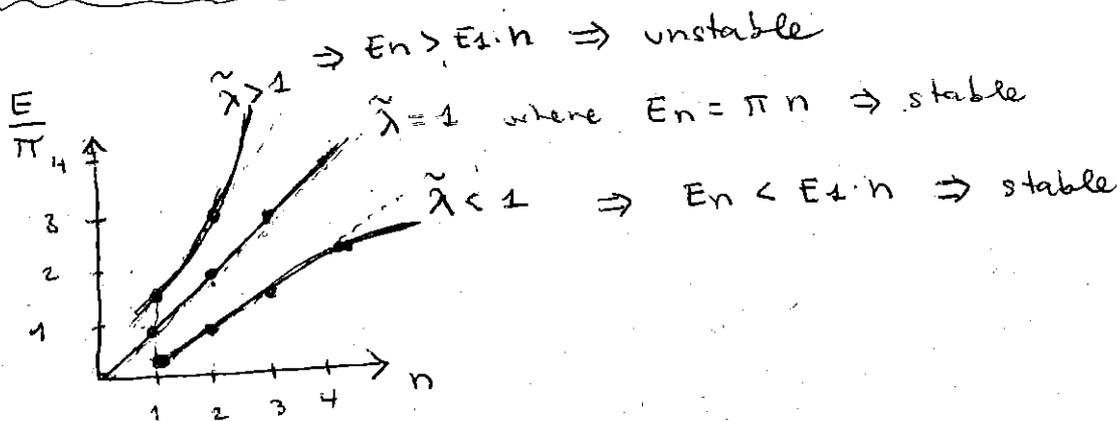
$\rightarrow n > 1$  solution has  $n$  vortices at positions  $\{\vec{x}_1, \dots, \vec{x}_n\}$  which can coincide completely determined by them.



• Notice  $\boxed{E_n = n E_1}$

- Basic phenomenology

• stability of n-vortex solution



• Forces between vortices

\* For  $v \ll c$  interactions not dissipative  $\Rightarrow$

$$\vec{F} = -\vec{\nabla} E_{int}$$

where  $E_{int}$  is the energy due to the vortices' interactions.

\* For two vortices with  $n=1$

$$E_{int} = E_{total} - 2E_{n=1}$$

If distance  $d \gg \frac{1}{m\hbar}, \frac{1}{m\sigma}$ , we can compute it perturbatively.

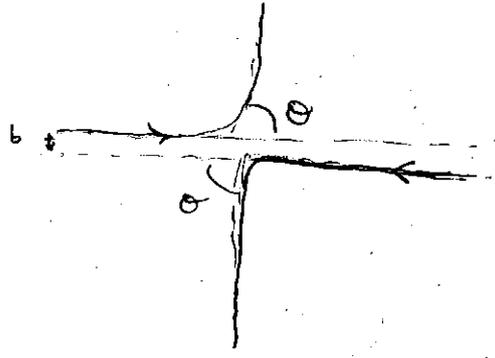
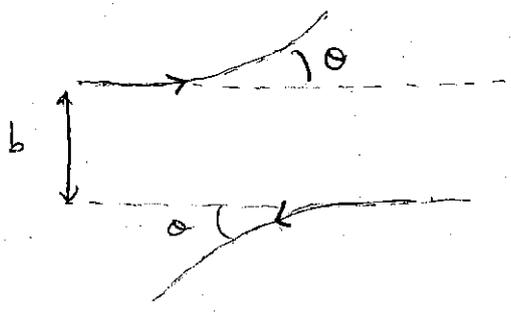
$$E_{int} = -c_s^2 K_0\left(\frac{r}{m\hbar}\right) + c_m^2 K_0\left(\frac{r}{m\sigma}\right)$$

$$= -c_s^2 K_0(\sqrt{\lambda} \cdot r) + c_m^2 K_0(r)$$

- For  $\tilde{\lambda} = 1$   $c_s = c_m \Rightarrow E_{int} = 0 \Rightarrow \vec{F} = 0$
- For  $\tilde{\lambda} > 1$   $c_s \gtrsim c_m$  but  $E_{int} > 0 \Rightarrow \vec{F}$  repulsive
- For  $\tilde{\lambda} < 1$   $c_s \lesssim c_m$  but  $E_{int} < 0 \Rightarrow \vec{F}$  attractive

• scattering

- \* For  $\tilde{\lambda} \approx 1$  it is possible to calculate scattering processes.
- \* At critical coupling the dynamics of vortices motions can be mapped to geodesics followed in the manifold space given by all  $\{\vec{x}_1, \dots, \vec{x}_n\}$  characterizing an n-vortex solution; e.g. For  $n=2$ .



\* As  $b \rightarrow 0$   $\theta \rightarrow \frac{\pi}{2}$  ! (true in general, can be derived ~~from~~  $\tilde{\lambda} = 1$ )

3) other vortices:

• The global vortex

- \* It is possible to have static solutions for the Abelian-Higgs model with winding number  $n \neq 0$  even when we decouple the gauge fields.

$$\mathcal{L} = |\partial\phi|^2 - V(\phi)$$

- \* In  $d=2+1$   $E \xrightarrow{r \rightarrow \infty} \log r$
- \* In finite systems, these solutions can  $\exists$  and be minima.  
e.g. Bose-Einstein condensates, superfluid vortices.

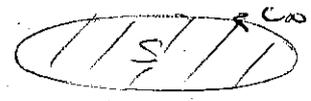
Chern-Simmons vortices:

- \* Even with only a complex scalar, there are many variations in the models.
- \* Always presence of

$$L_{CS} = \frac{1}{4} \epsilon^{\mu\nu\sigma} F_{\mu\nu} A_\sigma$$

- \* Generic prediction: modification of EOM corresponding to Gauss law.

$$\vec{\nabla} \cdot \vec{E} + B = \rho_e \quad ; \quad \leftarrow \text{electric charge density}$$



$$\Rightarrow \int_S \vec{\nabla} \cdot \vec{E} \, dS = \int_S (\rho_e - B) \cdot dS = Q - \Phi = \oint_{\partial S} \vec{E} \cdot \hat{n} \cdot d\ell = 0$$

$$\Rightarrow \boxed{Q = \Phi}$$

In the presence of vortices  $\Phi$  is quantized

$\Rightarrow$  The total charge of the system is quantized!

- \* Applications in condensed matter & fluids. (ideal).

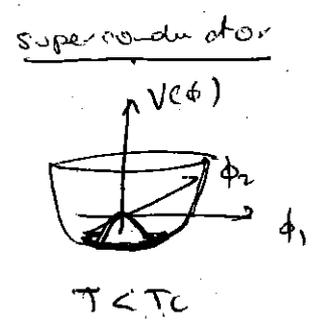
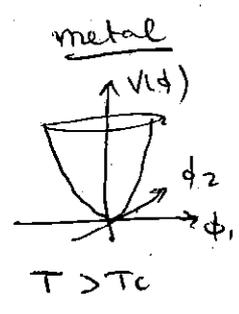
Non-Abelian vortices:

- \* It is possible to have vortex solutions for non-Abelian symmetry groups.
- \* Their existence will depend on the homotopy of  $\vec{P}_0 = S_1 \rightarrow G/H$

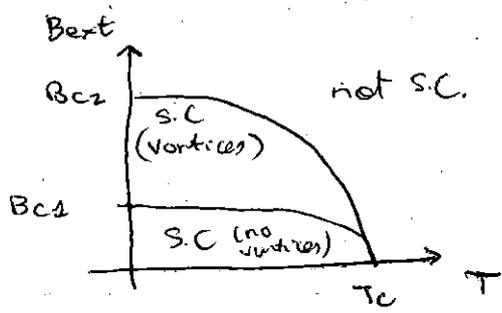
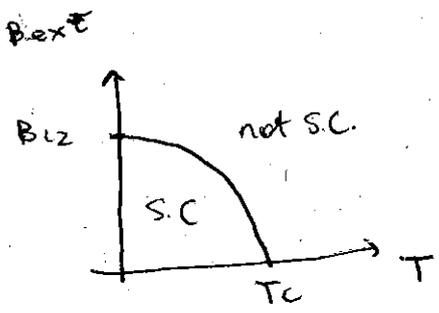
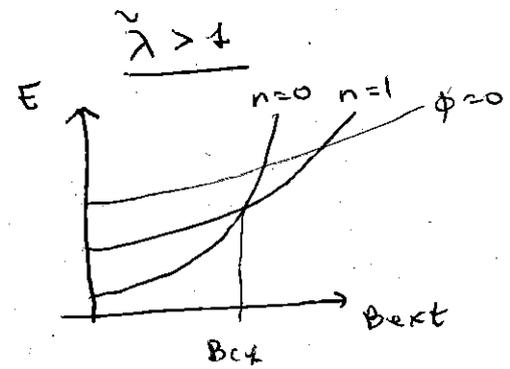
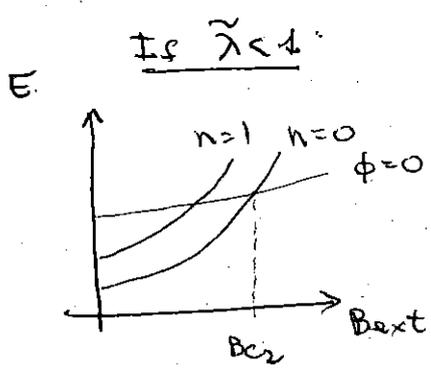
4) Applications:

Phase diagrams of superconductors:

Qualitative understanding of phase transition and phase diagrams, by computing the Energy of  $n=0$  and  $n=1$  configurations in the presence of an external magnetic field.



For  $T < T_c$ :



Particle / vortex dualities:

→ In  $d=1+1$ : duality GL / Calabi-Yau

written: "Phases of  $\mathcal{N}=2$  theories in 2 dimensions" (>1000 cites)

In  $1+1$  vortices play the role of instantons, used to compute instanton contributions on the worldsheet via GL vortices.

→ In  $d=2+1$  dualities between vortices & boson/fermion.

Applications in study of confinement and phase transitions.

e.g. boson / fermion.

$O(2)$  vector model

$$\mathcal{L} = |\partial\phi|^2 + |\phi|^4 + m^2|\phi|^2$$

dual  
↔

$O(2)$  local vector model

$$\mathcal{L} = |F_{\mu\nu}|^2 + |\tilde{\phi}|^2 + |\tilde{\phi}|^4 - m^2|\tilde{\phi}|^2$$

broken / gapped phase

↔

free photon / Higgsed phase

For  $m^2 > 0$ , particles

↔

finite  $E$  vortices

$\phi$

↔

monopole operator

2nd order phase transitions (IR)

⇒

2nd order phase transition

↑ not trivial!

Applications in condensed matter,

e.g. phase transition in XY model (Perkin 78, Dasgupta Halperin 82)

boson / fermion

$V(\phi)$  with c.s.

$$\mathcal{L} = \hbar c s + \delta A \psi$$

↔

Free fermion

$$\mathcal{L} = \psi \not{\partial} \psi$$

Applications in:

- Plateaus in Hall effect
- Surface states on topological insulators
- Arrays of  $d=1+1$  wires

- String dynamics & cosmic strings

- \* Vortices can create cosmic strings appearing during phase transitions.
- \* Their properties and cosmological effects will be explained in the next two workshop seminars.