Instantons: applications

Ref: will closely follow §§ 5(+3) and 6 of Coleman's 'The uses of instantons'

Two applications: I) the solution of the $U(1)$ problem in QCD
~~II) the decay of the false vacuum~~

I) 't Hooft's solution of the U(1) problem

$$\mathcal{L}_{QCD} = -\frac{1}{4g^2}(F_{\mu\nu}, F^{\mu\nu}) + \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f$$

The $U(1)$ problem is an apparent contradiction:

- In the limit $m_f \to 0$, chiral $SU(2)_L \times SU(2)_R$ is a perfect symmetry.
- In this limit $\mathcal{L}_{QCD}$ is also invariant under a chiral $U(1)$ symmetry,

$$\psi_f \to e^{-i\alpha\gamma_5}\psi_f \qquad f \in \{1,2\} \qquad w/\ \alpha \in \mathbb{R}$$

The associated conserved current is $j^5_\mu = \sum_f \bar{\psi}_f \gamma_\mu \gamma_5 \psi_f$

(Aside: this enhanced symmetry is special to QCD. For example the σ model does not have it in the chiral limit)

Now, there are two possibilities

$U(1)_A$ → manifest ⇒ all massive hadrons come in parity doublets. NOT the case.
$U(1)_A$ → spontaneously broken ⇒ pNGB

* $\boxed{\text{The } U(1) \text{ problem is: where is the } 4^{th} \text{ pNGB?}}$

Naively: is it the η meson? No.
∵ $U(1)_A$ is broken by the same effect (quark masses) as $SU(2)\times SU(2)$, then one would expect $m_\eta \sim m_\pi$ and Weinberg showed that $m_\eta \leq \sqrt{3}\, m_\pi$
[Weinberg, Phys. Rev. D11, 3583 (1975)]

Same argument can be applied to $SU(3) \times SU(3)$ in the limit $m_f \to 0$ w/ $f \in \{1,2,3\}$. In this case there should be 8+1 pNGBs but the 9^{th} is absent from the spectrum.

- This is the end of the story for η and η'. There is no connection to the $U(1)$ problem.

But, we know that j^5_μ is not conserved but has an ABJ anomaly

$$\partial^\mu j^5_\mu = \frac{N}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} (F_{\mu\nu}, F_{\rho\sigma})$$

But, as we will see, the RHS can itself be written as a divergence of the Chern-Simons current

$$G_\mu = 2 \varepsilon_{\mu\nu\rho\sigma} (A_\nu, \partial_\rho A_\sigma + \frac{2}{3} A_\rho A_\sigma)$$

$$\Rightarrow \partial^\mu G_\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} (F_{\mu\nu}, F_{\rho\sigma})$$

(By the way, $\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ s.t. $\tilde{\tilde{F}} = F$)

$$\partial^\mu G_\mu = (F_{\mu\nu}, \tilde{F}^{\mu\nu})$$

So can define a divergenceless current

$$J^5_\mu = j^5_\mu - 2 G_\mu$$

which is gauge variant but conserved.

Inserting one J^5_μ into Green's fn with quark multilinears leads to Chiral Ward identities.

So either $\langle \phi^{(1)} \dots \phi^{(m)} \rangle$ are alone $U(1)$ invariant or there are Goldstone poles in $\langle J^5_\mu \phi^{(1)} \dots \phi^{(m)} \rangle$ where $\phi^{(n)}$ is a quark multilinear.

Proper formulation of the problem: Kogut and Susskind [Phys. Rev. D11, 3594 (1976)] looked at the Schwinger model (QED with massless fermions in 1+1 dimensions). The Schwinger model is exactly soluble and has an anomalous gauge invariant axial current and a divergenceless but gauge variant current. Most importantly it has chiral symmetry breaking but without Goldstone poles.

Kogut and Susskind, in covariant gauge, that there are two free massless fields ϕ_+ and ϕ_-. One of them, ϕ_+, creates quanta with +ve norm and the other ϕ_-, creates quanta w/ −ve norm.
(Cf. A_0 and A_3 in QED)

- All gauge invariant operators couple to $\phi_+ + \phi_-$ which has zero propagator and thus no poles.

- The gauge variant but conserved current couples to $\partial_\mu(\phi_+ - \phi_-)$ and Goldstone poles appear in $\langle J^5_\mu \, \phi^{(1)} \dots \phi^{(m)} \rangle$ where they should.

(Coleman (? haven't traced the origin) calls this a Goldstone dipole but admits it is confusing since there is only one pole.)

So, the proper formulation of question is:

$\boxed{\text{Is the } U(1) \text{ problem solved via SSB by a Goldstone "dipole"? 't Hooft: YES! [PRL 37, 8 (1976)]}}$

I.a) Conventions

*((From now on, work in Euclidean space))

Lie algebra: $[T^a, T^b] = c^{abc} T^c$ $\quad c \equiv$ structure constants

$$\mathrm{tr}\{T^a T^b\} \propto \delta^{ab}$$

$$(T^a, T^b) = \delta^{ab} \quad \text{(Cartan inner product)}$$

$\square$ of $SU(2)$: $T^a = -\dfrac{i\sigma^a}{2}$

$$(T^a, T^b) = -2\,\mathrm{tr}\{T^a T^b\} = \delta^{ab} \Rightarrow \mathrm{tr}\{T^a T^b\} = -\frac{1}{2}\delta^{ab}$$

$\square$ of $SU(3)$: $T^a = -i\dfrac{\lambda^a}{2}$

Gauge fields: $A_\mu \equiv g A^a_\mu T^a$ w/ $g \equiv$ gauge coupling const.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Pure YM: $S = \dfrac{1}{4g^2}\displaystyle\int d^4x \, (F_{\mu\nu}, F_{\mu\nu})$

Gauge transformations: GT $g(x): \mathbb{R}^4 \mapsto G$

$$g(x) = \exp\{\lambda^a(x) T^a\}$$

λ^a = arbitrary fns.

Under such a trans'n we have

$$\boxed{A_\mu \to g A_\mu g^{-1} + g\partial_\mu g^{-1} \quad \text{and} \quad F_{\mu\nu} \to g F_{\mu\nu} g^{-1}}$$

while S is invariant.

As we saw last time in Henrique's lecture

If $F_{\mu\nu} = 0 \to A_\mu$ is a gauge trans'n of 0

i.e., it is pure gauge. That is $\boxed{A_\mu = g\partial_\mu g^{-1}}$

Covariant derivative: $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]$

$\to$ Euclidean E.O.Ms $D_\mu F_{\mu\nu} = 0$

Given $\psi \to g(x)\psi$, then, $(D_\mu = \partial_\mu + A_\mu)$

$$D_\mu \psi \to g(x) D_\mu \psi \quad \text{i.e. transforms as } \psi \text{ does.}$$

I.b: The winding number

The convergence of the action integral is controlled by the behavior of A_μ at large r.

$\Rightarrow$ Assume can expand A_μ for $r \to \infty$ in powers of r

$\therefore$ For finite action, $F_{\mu\nu} \sim \mathcal{O}(\frac{1}{r^3})$ (faster than $\frac{1}{r^2}$)

which implies $A_\mu \sim \mathcal{O}(\frac{1}{r^2})$

But! vanishing $F_{\mu\nu} \neq$ vanishing A_μ

only that A_μ is pure gauge as we just saw.

$$\Rightarrow A_\mu = g\partial_\mu g^{-1} + \mathcal{O}(\tfrac{1}{r^2})$$

$\to$ Can still gauge transform with h,

$$A_\mu \to h A_\mu h^{-1} + h\partial_\mu h^{-1}$$

$$g \to hg + \mathcal{O}(\tfrac{1}{r^2})$$

So if can choose $h = g^{-1}$ and thus tranform $g \to \mathbb{1}$ at ∞

But cannot since h must be continuous on all nested hyperspheres from $r=0$ to $r=\infty$ $\to$ h = const. g.t.

might as well say $h = \mathbb{1}$.

$\Rightarrow$ A gauge invariant quantity associated to a finite action field configuration is a homotopy class of mappings $S^3 \mapsto G$.

From last time (modulo slight not'n changes)

$$\nu = -\frac{1}{24\pi^2} \varepsilon^{ijk} \int d\theta_1 d\theta_2 d\theta_3 \, \mathrm{tr}\{ g\partial_i g^{-1} \; g\partial_j g^{-1} \; g\partial_k g^{-1} \}$$

This quantity is a homotopy invariant. That is it is invariant under continuous deformations.

Recall the Chern-Simons current

$$G_\mu = \varepsilon_{\mu\nu\rho\sigma} (A_\nu, F_{\rho\sigma} - \tfrac{2}{3} A_\rho A_\sigma)$$

$$\int d^4x \, (F_{\mu\nu}, \tilde{F}_{\mu\nu}) = \int d^3S \, \hat{n}_\mu G_\mu \quad \text{by Gauss's thm.}$$

d^3S is the area element of the hypersphere and $\hat{n}_\mu$ is the normal to the surface.

The first term in $G_\mu \sim O(\frac{1}{r^4})$ and does not contrib while the second term is $\propto \nu$

$$\Rightarrow \boxed{\int d^4x \, (F, \tilde{F}) = 32\pi^2 \nu}$$

- For a gauge field theory ($SU(2)$), every field config. in 4d Euclidean space has an integer associated to it called the winding number or Pontryagin index.

I.C: Euclidean fermions

- We need to define the integration variables in the path integral.

For bosons, we integrated over commuting c-number fields.* This suggests that for fermions, we $\int$ over anti-commuting classical fields.

* Note: the logic here is that we treated the c-number fields commuted at arbitrary separations which suggested thinking of them as classical fields.

$$\{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = \boxed{\{\psi(x), \bar{\psi}(y)\}} = 0$$

Last relation is crucial and implies that ψ and $\bar{\psi}$ must be treated as independent variables

i.e. $\bar{\psi} \neq \psi^\dagger A$ for some matrix A.

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$$

$S = -\int d^4x\, \bar\psi\,(i\partial\!\!\!/ - i m)\,\psi$

For $m \to 0$ S is invariant under chiral trans'ns
$\psi \to e^{-i\alpha\gamma_5}\psi$ and $\bar\psi \to \bar\psi e^{-i\alpha\gamma_5}$

• Integration: over Grassman variables

$\int da\, f(a)$: ① Linearity: $\int da\, f(a) + g(a) = \int f + \int g$

(For one variable $a^2 = 0$) ② Transl'n invariance

$$\int da\, f(a+b) = \int da\, f(a)$$

↳ only have $\int da\, a = 1$ and $\int da\, 1 = 0$

For two variables $\int da\, d\bar a \begin{pmatrix} \bar a a \\ \bar a \\ a \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Define integration over fermi fields same as we did for bose fields

• Introduce arbitrary complete O-N set of c-num. functions ψ_r and $\bar\psi_r$ such that:

$$\int d^4x\, \psi_r^\dagger \psi_s = \int d^4x\, \bar\psi_r \bar\psi_s^\dagger = \delta_{rs}$$

and expand the integration variables in this basis

$\psi = \sum_r a_r \psi_r$ and $\bar\psi_r = \sum_r \bar a_r \bar\psi_r$

then define $[d\psi][d\bar\psi] = \prod_r da_r\, d\bar a_r$

Example: evaluate $\int [d\psi][d\bar\psi]\, e^{-S}$ w/ $S = -\int d^4x\, \bar\psi A \psi$

for a linear operator A. If A commutes with $A^\dagger$ then can choose ψ_r to be eigenfns. of A

$A\psi_r = \lambda_r \psi_r$

so $S = -\sum_r \lambda_r \bar a_r a_r$

and $\int [d\psi][d\bar\psi]\, e^{-S} = \prod_r \lambda_r = \det A$

N.B.: this is the inverse of the answer were we to do the $\int$ for bosons.

I.d: the solution! But for "baby" QCD as Coleman dubs it. That is ($N_c = 2$ and $N_f = 1$)

$$S = \int d^4x \left[\frac{1}{4g^2}(F,F) - i\bar{\psi}\not{D}\psi\right] \quad (\text{for now } m_\psi = 0)$$

From the analysis of last time for pure YM (mod not'n)

$$\frac{E(\theta)}{V} = -2K\cos\theta\, e^{-S_0},$$

and $\langle\theta|(F,\tilde{F})|\theta\rangle = -64\pi^2 i K e^{-S_0}\sin\theta$,

remain un-altered by inclusion of the fermion.

However, K gains a factor $\det\left(\frac{i\not{D}}{i\not{\partial}}\right) = \det\left[\frac{i(\not{\partial}+\not{A})}{i\not{\partial}}\right]$

Looks innocuous but wait!

$i\not{D}$ has a vanishing eigenvalue and so $\frac{E(\theta)}{V}$ and $\langle\theta|(F,\tilde{F})|\theta\rangle$ vanish!

To see this, assume for simplicity that $i\not{D}$ has a purely discrete spectrum (which is of course not true) (Cf. note 32 in Coleman)

$$i\not{D}\psi_r = \lambda_r\psi_r$$

and so $i\not{D}\gamma_5\psi_r = -\lambda_r\gamma_5\psi_r$ ($\because \gamma_5$ anti-commutes with γ_μ)

Because $i\not{D}$ is Hermitian $\forall \lambda_r \in \mathbb{R}$

So, if $\lambda_r \neq 0 \Rightarrow$ get pairs of opposite sign eigenvalues

But for $\lambda_r = 0$, can take ψ_r to be eigenfn. of γ_5

$$\gamma_5\psi_r = \chi_r\psi_r$$

and since $\gamma_5^2 = 1$, $\chi_r = \pm 1$ denote them by $n_\pm$

One can then prove that $\boxed{n_- - n_+ = \nu}$

$\Rightarrow$ Not only is there a zero eigenvalue in the field of the instanton, but also in any gauge field of non-zero winding number regardless of whether it solves the Euclidean EOMs.

The instanton obeys the seem rule by having one eigenfn. of vanishing eigenvalue w/ $\chi = -1$ and none w/ $\chi = +1$.

For completeness the e.fn with vanishing e. value is

$$\psi_0(x - X, \rho) = \frac{\rho}{[\rho^2 + (x-X)^2]^{3/2}}\, u \,,$$

for an instanton centered at X with size ρ and u is a constant spinor.

Implications: all θ vacua have the same energy and vanishing $\langle\theta|(F\tilde{F})|\theta\rangle$.

Consider the denominator free* Green's fn.

*(note: for massless quarks $i\not{D}$ in den. of det vanishes

$$\langle\langle \phi^{(1)}(x) \dots \rangle\rangle^A = \int [d\psi][d\bar{\psi}]\, e^{-S}\, \phi^{(1)}(x) \dots$$

they obey the chiral Ward identities

$$\left[\frac{\partial}{\partial\alpha} + 2iv\right]\langle\langle \phi^{(1)}(x) \dots \rangle\rangle^A = 0$$

The Green's fns of baby QCD are then

$$\langle\theta|\phi^{(1)}(x)\dots|\theta\rangle = \frac{\int [dA]\, e^{-S_g}\, e^{i\nu\theta}\langle\langle \phi^{(1)}(x)\dots\rangle\rangle^A}{\int [dA]\, e^{-S_g}\, e^{i\nu\theta}\langle\langle 1\rangle\rangle^A} \Rightarrow \left[\frac{\partial}{\partial\alpha} + 2\frac{\partial}{\partial\theta}\right]\langle\theta|\dots|\theta\rangle = 0$$

where S_g is the pure YM action.

$\Rightarrow$ The effect of a chiral $U(1)$ tran'n can be "undone" by a change in θ.

For "chiral eigenfields" $\sigma_\pm = \frac{1}{2}\bar{\psi}(1 \pm \gamma_5)\psi$

$$\text{if } \langle\theta|\sigma_\pm(x)|\theta\rangle = \frac{\int [dA][d\psi][d\bar{\psi}]\, e^{-S}\, e^{i\nu\theta}\, \sigma_\pm(x)}{\int [dA][d\psi][d\bar{\psi}]\, e^{-S}\, e^{i\nu\theta}} \neq 0$$

then signals SSB of $U(1)_A$.

This indeed happens and can show that (in dilute gas approx)

$$\langle\theta|\sigma_\pm(0)|\theta\rangle = e^{-8\pi^2/g^2}\, e^{\mp i\theta}\, \frac{1}{g^8}\int_0^\infty \frac{d\rho}{\rho^5}\, f(\rho M)\, \frac{\det'(i\not{D})}{\det(i\not{\partial})}$$

So SSB occurs, are there Goldstone poles?

If there are, we would find them in

$$\langle\theta|\sigma_+(x)\,\sigma_-(0)|\theta\rangle \quad \rightarrow \text{ there are none among the physical states.}$$

For the gauge-variant Green's function, the situation is very different.

$$\langle\theta|J_\mu^5(x)\,\sigma_-(0)|\theta\rangle = \underbrace{\langle\theta|j_\mu^5(x)\,\sigma_-(0)|\theta\rangle}_{\text{this term does not have any poles}} + \frac{1}{16\pi^2}\underbrace{\langle\theta|G_\mu(x)\,\sigma_-(0)|\theta\rangle}_{\text{this one does}}$$

In a covariant gauge, there is a Goldstone pole iff

$$\int d^4x\, \partial_\mu \langle \theta | G_\mu(x)\, \sigma_-(0) | \theta \rangle \neq 0 .$$

Recall that $\int d^4x\, \partial_\mu G_\mu(x) = 32\pi^2 \nu$, we need only include $\nu = 1$ so we find,

$$\int d^4x\, \partial_\mu \langle \theta | G_\mu(x)\, \sigma_-(0) | \theta \rangle = 32\pi^2 \langle \theta | \sigma_-(0) | \theta \rangle \neq 0$$

- Summary: in the dilute gas approximation, we found,
  - SSB of $U(1)_A$
  - No Goldstone poles in gauge invariant Green's functions.
  - No Goldstone poles in the propagator of a gauge-variant conserved current J^5_μ
  - A Goldstone pole in the Green's fn. for one gauge variant current and one gauge invariant operator $\langle \theta | J^5_\mu \sigma_- | \theta \rangle$

$\boxed{\text{This is the "Goldstone dipole" mechanism of Kogut and Susskind.}}$

For real QCD, similar analysis but some diff. due to N_f mainly $n_- + n_+ = 2\nu$. Also, since $N_f = 2$, will have two nonvanishing e.values thus will need to look for non-vanishing expectation values of 4-quark operators.