SKYRMIONS

Outline:

1. Chiral Lagrangians.
2. Topology + Derrick's theorem
3. The Skyrmion sol'n
4. Quantization: Baryons as topological solitons
   (This will mostly be a sketch of the ideas)

References:

- Balachandran: Skyrmions, Lectures given at Yale ASI 1985 (avail. via KEK scanned docs)
- Shifman: Advanced Topics in QFT ch. 16
- Weinberg: Quantum Theory of Fields II §23.1
- Manton & Sutcliffe: Topological Solitons ch. 9

---

1 Chiral Lagrangians

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{i=1}^{N_F} \bar{\psi} i \not{D} \psi$$

$$= \quad \cdots \quad + \sum_{i=1}^{N_F} \left(\bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \right)$$

Where $P_L \psi = \psi_L$ and $P_R \psi = \psi_R$ w/ $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$

$\mathcal{L}_{QCD}$ is invariant under

$$U(N_F)_L \otimes U(N_F)_R = SU(N_F)_L \otimes SU(N_F)_R \otimes U(1)_V \otimes U(1)_A$$

And we know that $SU(N_F)_L \otimes SU(N_F)_R \xrightarrow{SSB} SU(N_F)_V$

$$\psi_{L/R} \xrightarrow{SU(N_F)_V} V \psi_{L/R}$$

The $U(1)$ trans'ns are; $U(1)_V$: $\psi_{L/R} \to e^{i\theta} \psi_{L/R}$

$U(1)_A$: $\psi_{L/R} \to e^{\mp i\theta_5} \psi_{L/R}$

OR $\psi \to e^{i\theta_5 \gamma_5} \psi$

The Noether current associated to the $U(1)_A$ trans is

$j^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$ and it is not conserved

$$\partial_\mu j^\mu_5 = -\frac{N_F}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}G^{a\mu\nu}G^{a\rho\sigma}$$

<u>SSB:</u> $G \equiv SU(N_F)_L \otimes SU(N_F)_R \longrightarrow H \equiv SU(N_F)_V$

By Goldstone's thm. $\rightarrow$ have $N_F^2 - 1$ NGBs

- Want to construct a low energy Lagrangian that describes the interactions of these NGBs.

It must satisfy:

a. Be invariant under G

b. Have $(N_F^2 - 1)$ d.o.f.

c. The subgroup of G which leaves any value of the fields invariant must be $\cong H$

<u>For b.</u> $\mathcal{M}_{G/H} = \mathcal{M}_{SU(N_F)}$ which has $N_F^2 - 1$ d.o.f ✓

use an $(N_F^2 - 1)$ component field $U(\vec{x},t) \in SU(N_F)$

Under G, $U(\vec{x},t) \longrightarrow L\,U(\vec{x},t)\,R^\dagger$

<u>For c.</u> Say, e.g., $U = \mathbb{1}$ so $U \rightarrow L\,\mathbb{1}\,R^\dagger = \mathbb{1}\,(LR^\dagger)$

For U to be invariant, $L = R = V \in H$

To construct the Lag., $\mathcal{L}_\chi$, it is useful to use the Maurer-Cartan form (Left invariant here)

$$\boxed{\mathcal{L}_\mu = U^\dagger\partial_\mu U} \xrightarrow{G} R\,\mathcal{L}_\mu R^\dagger$$

$$\Rightarrow \mathcal{L}_\chi = \frac{F_\pi^2}{4}\mathrm{tr}\{\mathcal{L}_\mu\mathcal{L}^\mu\} + \frac{1}{32e^2}\mathrm{tr}\{[\mathcal{L}_\mu,\mathcal{L}_\nu]^2\}$$

$F_\pi = \frac{f_\pi}{\sqrt{2}}$ where f_π can be determined from $\pi \rightarrow \mu\nu$ decay.

Power counting: $\partial \sim \mathcal{O}(p)$ so $\mathrm{tr}\{\mathcal{L}_\mu\mathcal{L}^\mu\} \sim \mathcal{O}(p^2)$

$\mathrm{tr}\{[\mathcal{L}_\mu,\mathcal{L}_\nu]^2\} \sim \mathcal{O}(p^4)$

- What happens to the Baryonic current in $\mathcal{L}_{QCD}$ when we go to $\mathcal{L}_\chi$?

Well, in mesonic sector $B=0$. But, for topo. non-trivial configs. there $\exists$ a "projection" of this current given by

$$j_B^\mu = -\frac{\varepsilon^{\mu\nu\rho\sigma}}{24\pi^2}\, tr\{L_\nu L_\rho L_\sigma\}$$

with conserved charge $B = \frac{\varepsilon^{ijk}}{24\pi^2}\int d^3x\, tr\{L_i L_j L_k\} \in \mathbb{Z}$

This current is conserved <u>topologically.</u>

---

② <u>Topology</u>

Consider the potential energy in d-space dims.

$$S[\pi] = \int d^d x \left\{ \frac{1}{2} \sum_{ab} g_{ab}\, \partial_i \pi_a \partial_i \pi_b + \cdots \right\}$$

For finite S, $\partial_i \pi_a$ must vanish at $|\vec{x}| \to \infty$ faster than $|\vec{x}|^{-d/2}$ s.t. $\pi_a \to \text{const.} + \mathcal{O}(|\vec{x}|^{(2-d)/2}) = \pi_a^\infty$

Since Goldstones parametrize the coset space G/H, we can cut any point make a global tran'n so that $\pi_a^\infty = 0$

$\pi_a(\vec{x}) : S^d \mapsto G/H$

↑ d-dim. space with $|\vec{x}| \to \infty$ identified as a single point = surf. of a d-dim. sphere

The field configs. can be classified according to the topologically distinct mappings $S^d \mapsto G/H$

The set of classes of such topo. distinct mappings, $S^d \mapsto \mathcal{M}$ with one point of S^d mapped into a fixed point of $\mathcal{M}$ is known as the dth <u>homotopy group</u> $\pi_d(\mathcal{M})$

Homotopy groups of particular interest for us are

$$\pi_3(SU(2)) = \mathbb{Z}$$

and $\pi_4(SU(2)) = \mathbb{Z}_2$

<u>Skyrmions</u> $\equiv$ topologically non-trivial field configs. of the potential energy with $d=3$.

<u>Derrick's theorem:</u>

$$\pi_a^R(\vec{x}) \equiv \pi_a\left(\frac{\vec{x}}{R}\right) \qquad R \in \mathbb{R}_+$$

@ $\mathcal{O}(p^2)$: $S[\pi^R] = R^{d-2} S[\pi]$

For $d > 2$ $\quad R^{d-2} S[\pi]$ decreases continuously as $R \to 0$

@ $\mathcal{O}(p^4)$: $\quad S[\pi] = T[\pi] + D[\pi]$

$$S[\pi^R] = R^{d-2} T[\pi] + R^{d-4} \cdot D[\pi]$$

So S can reach a minimum at finite R if $\boxed{2 < d < 4}$ $\supset$ $\underline{\underline{3}}$.

---

③ <u>The Skyrmion sol'n with $N_F = 2$</u>

Recall: Solitons are consequences of the topology of the field configurations. This topology is sensitive to the <u>boundary conditions</u> at spatial ∞ which in turn are consequences of requiring the energy integral to be finite.

(Recall: $\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L}$)

$$E = \int d^3x \, \mathcal{H}$$

$$= \int d^3x \left\{ \frac{F_\pi^2}{4} \mathrm{tr}\{L_0^2\} + \frac{F_\pi^2}{4} \mathrm{tr}\{L_i^2\} + \frac{1}{16e^2} \mathrm{tr}\{[L_0, L_i]^2\} + \frac{1}{32\pi^2} \mathrm{tr}\{[L_i, L_j]^2\} \right.$$

Sufficient condition for existence of finite energy is $U(\vec{x},t) \to U_0 = \text{const.}$ as $|\vec{x}| \to \infty$ such that $U_0^{-1} U = \mathbb{1} + \mathcal{O}(\frac{1}{r^2})$

(?) Is this energy bounded from below for a static field configuration? $\quad (L_i = i L_i^a \tau^a)$

$$\dots \doteq E = \int d^3x \left\{ \frac{F_\pi^2}{2} L_i^a L_i^a + \frac{1}{4e^2} \varepsilon^{abc} \varepsilon^{dek} L_i^a L_j^b L_i^d L_j^e \right\}$$

$\to$ complete the square

$$= \frac{1}{2} \int d^3x \left\{ \left(F_\pi L_i^a + \frac{1}{2e} \varepsilon^{abc} \varepsilon_{ijk} L_j^b L_k^c \right)^2 - \frac{F_\pi}{e} \varepsilon^{abc} \varepsilon_{ijk} L_i^a L_j^b L_k^c \right\}$$

$$= 6\pi^2 \frac{F_\pi}{e} B + \underbrace{\frac{1}{2} \int d^3x \left(\quad \right)^2}_{\geq 0}$$

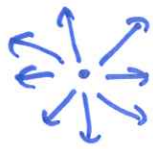
So, lower bound $\equiv \boxed{M = 6\pi \frac{F_\pi}{e}}$ $\quad (B=1)$

But in fact Skyrmion sol'n does not saturate the bound an $M_* = 1.23 M$.

The energy eq'n is highly non-linear.

$\Rightarrow$ make an ansatz (static sol'n)

$$U_0(\vec{x}) = \exp\left\{ i F(r) \frac{\vec{x} \cdot \vec{\tau}}{r} \right\}$$

This is called the Polyakov hedgehog 

B.C.s $\quad F(r \to 0) = n\pi$, $\quad F(r \to \infty) = m\pi \quad$ w/ $n, m \in \mathbb{Z}$

$$\Rightarrow B = -\frac{1}{\pi} \Big[F(r) - \underbrace{\frac{1}{2} \sin 2F(r)}_{=0 \text{ by B.C.s}} \Big]_0^\infty$$

and so $F(0) = \pi$ for $B = 1$

For this sol'n,

$$M_* = \dots = \frac{2\pi F_\pi}{e} \int_0^\infty d\rho \left\{ \rho^2 F'^2 + 2\sin^2 F + \sin^2 F \left(2F'^2 + \frac{\sin^2 F}{\rho^2} \right) \right\}$$

where $\rho = e F_\pi r$

$$\rightsquigarrow M_* = 6\pi^2 \frac{F_\pi}{e} \times 1.23 \quad \text{(numerically)}$$

- The Skyrmion profile:

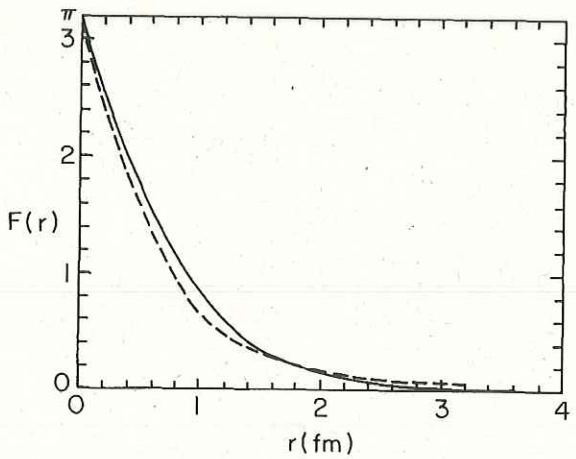


Fig. 4. Profile of the chiral angle $F(r)$ for a hedgehog configuration solution to (56) and subject to the boundary conditions (57). The continuous curve represents $F(r)$ for $m_\pi \neq 0$, as discussed in ref. [22a]. The dashed curve refers to the chiral case $m_\pi = 0$ of ref. [21].

Figure from Zahed & Brown: The Skyrme Model

④ Quantization

For skyrmions to describe a single Baryon, they need to obey Fermi-Dirac statistics.

For $N_F > 2$: WZW (WZNW) ← Novikov

$$S_{WZ} = \frac{-i N_c}{240 \pi^2} \int_{\partial = M_4} d^5x \, \mathrm{tr}\{L_\mu L_\nu L_\alpha L_\beta L_\gamma\} \, \varepsilon^{\mu\nu\alpha\beta\gamma}$$

This action includes a term linear in time derivatives which breaks T and P symmetries down to:

$$t \to -t$$
$$\vec{x} \to -\vec{x}$$
$$U \to U^+$$

Now consider a skyrmion at rest for a long time T and one that is slowly rotated by 2π in the same time T.

- Without the WZW term $\longrightarrow$ no difference
- With the WZW term $\rightarrow$ the linear term in time derivatives gives a factor $(-1)^{N_c}$!

*) We know that for odd N_c, baryons are fermions and for even N_c, they are bosons. This behavior is reproduced with the inclusion of WZW term.

However, for $N_F = 2$, $S_{WZ} = 0$ so we need different reasoning to see that skyrmions in $N_F = 2$ theories can describe baryons.

$\Rightarrow$ Homotopy groups to the rescue!
Recall that $\pi_4[SU(2)] = \mathbb{Z}_2$

Finkelstein & Rubenstein [Connection between spin, statistics and kinks: J Math. Phys. 9, 1762 (1968)] showed that the exchange of two $B = 1$ skyrmions is a loop homotopic to a 2π rotation of a skyrmion which agrees with the expected spin statistics.