

# The interaction potential between point masses at the fifth Post-Newtonian order

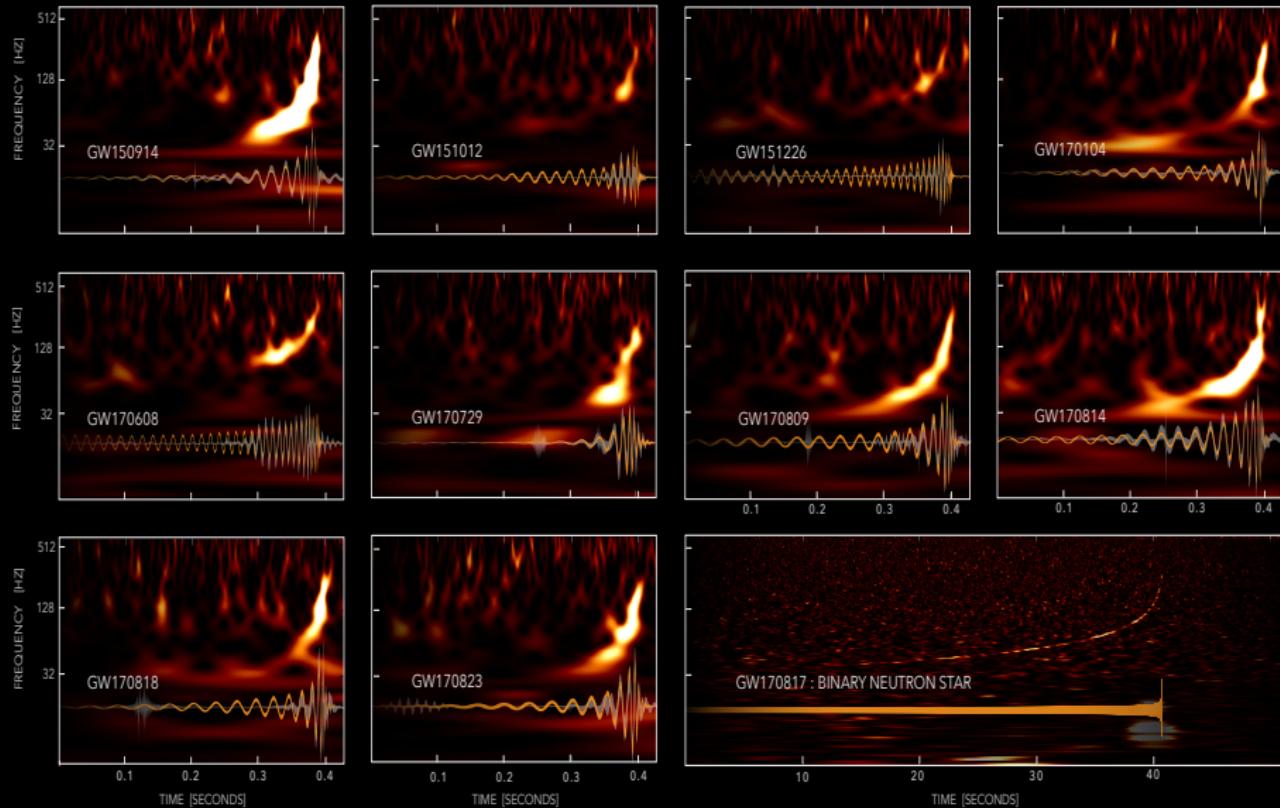
Andreas Maier



Berlin, 20 November 2019

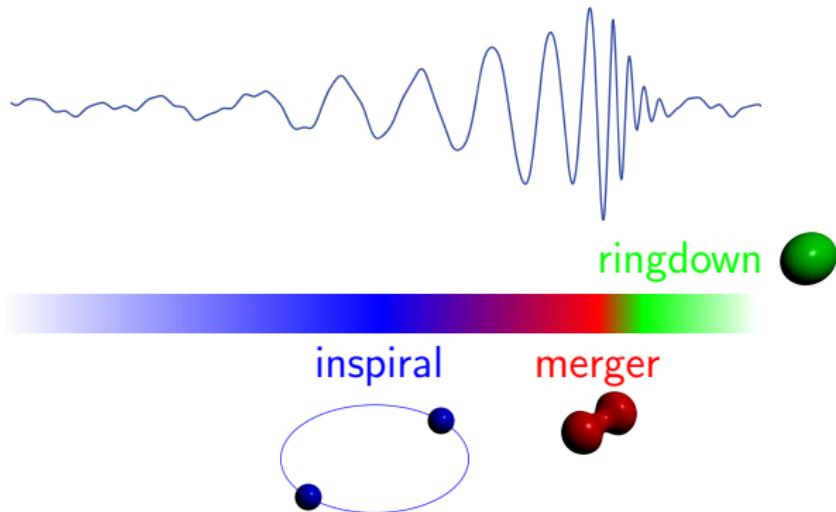
J. Blümlein, A. Maier, P. Marquard, arXiv:1902.11180

# GRAVITATIONAL-WAVE TRANSIENT CATALOG-1



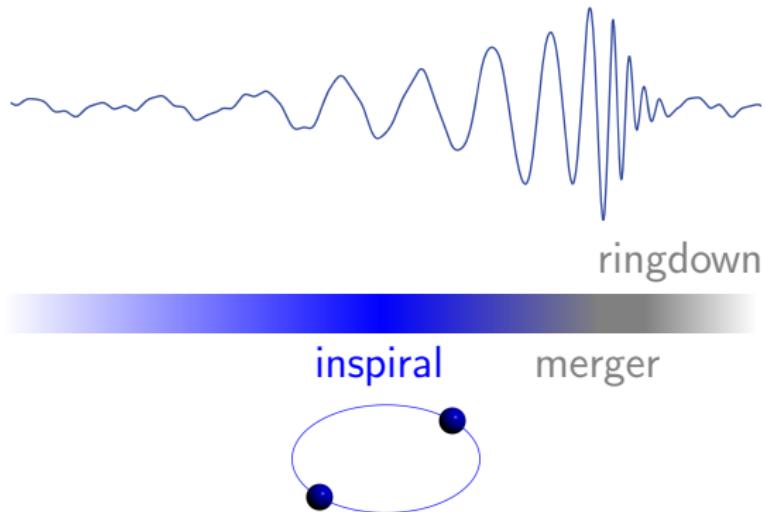
# Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]



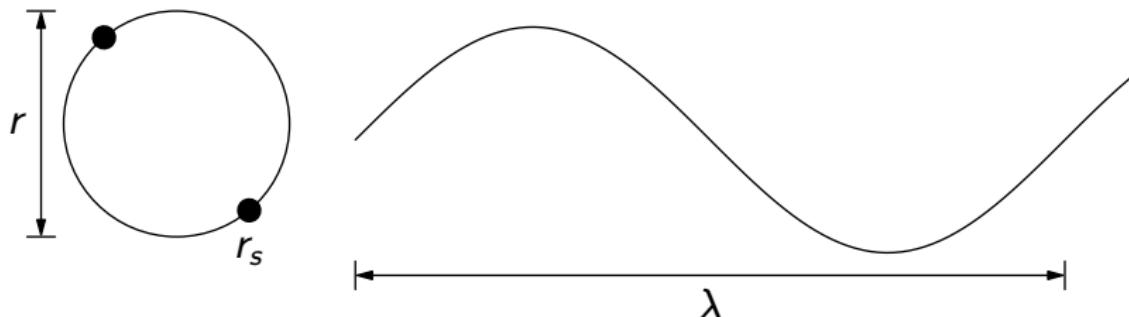
# Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]



# Compact binary systems

## Power counting

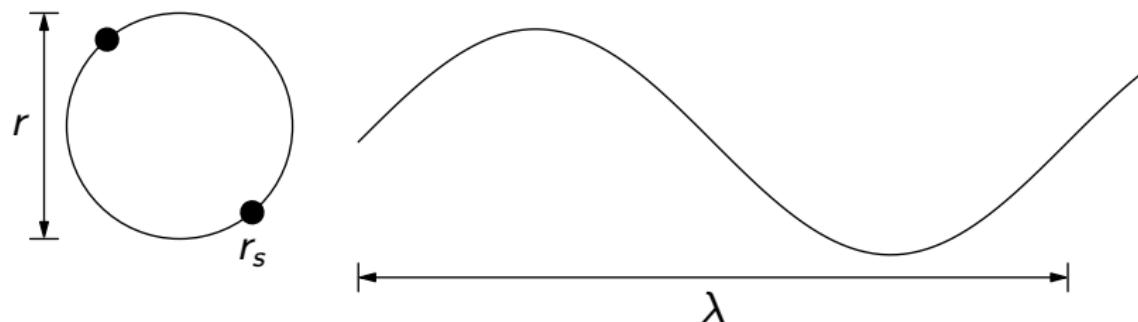


- Masses comparable:  $m \equiv m_1 \sim m_2$   
Generalisation to different masses straightforward
- Nonrelativistic system:  $v \ll 1$
- Virial theorem:  $mv^2 \sim \frac{Gm^2}{r}$

*Post-Newtonian (PN) expansion:*  
Combined expansion in  $v \sim \sqrt{Gm/r} \ll 1$

# Post-Newtonian expansion

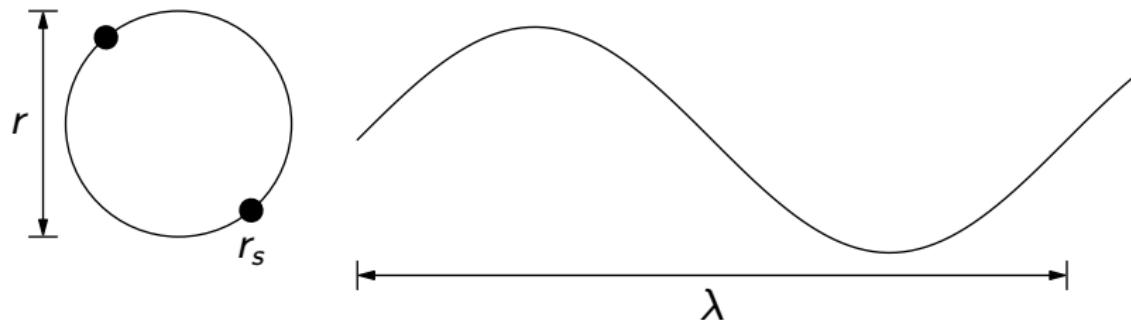
## Scales



- $\omega \approx \frac{2v}{r} \Rightarrow \boxed{\lambda \sim \frac{r}{v}}$
- $r_s = 2GM \Rightarrow \boxed{r_s \sim rv^2}$

# Post-Newtonian expansion

## Scales at LIGO/VIRGO



- $10 \text{ km} \lesssim \lambda \lesssim 10000 \text{ km}$
- $10 \text{ km} \lesssim r_s \lesssim 100 \text{ km}$
- Inspiral:  $0.1 \lesssim v \lesssim 0.5$ ,  $100 \text{ km} \lesssim r \lesssim 1000 \text{ km}$

	black holes	neutron stars
masses	$\sim 10\text{--}50 m_\odot$	$\sim 1 m_\odot$
radiated energy	$\sim 1\text{--}5 m_\odot$	$\geq 0.04 m_\odot$
redshift	$\sim 0.1\text{--}0.5$	$\sim 0.01$

# Post-Newtonian expansion

## Theory status

Conservative dynamics, no spin: complete results up to 4PN ( $v^8$ )

- ADM Hamiltonian formalism [Damour, Jaranowski, Schäfer 2016]
- Fokker Lagrangian in harmonic coordinates  
[Bernard, Blanchet, Bohé, Faye, Marchant, Marsat 2017]
- Non-relativistic effective field theory  
[Foffa, Mastrolia, Porto, Rothstein, Sturani, Sturm 2017–2019]

## Partial results at 5PN

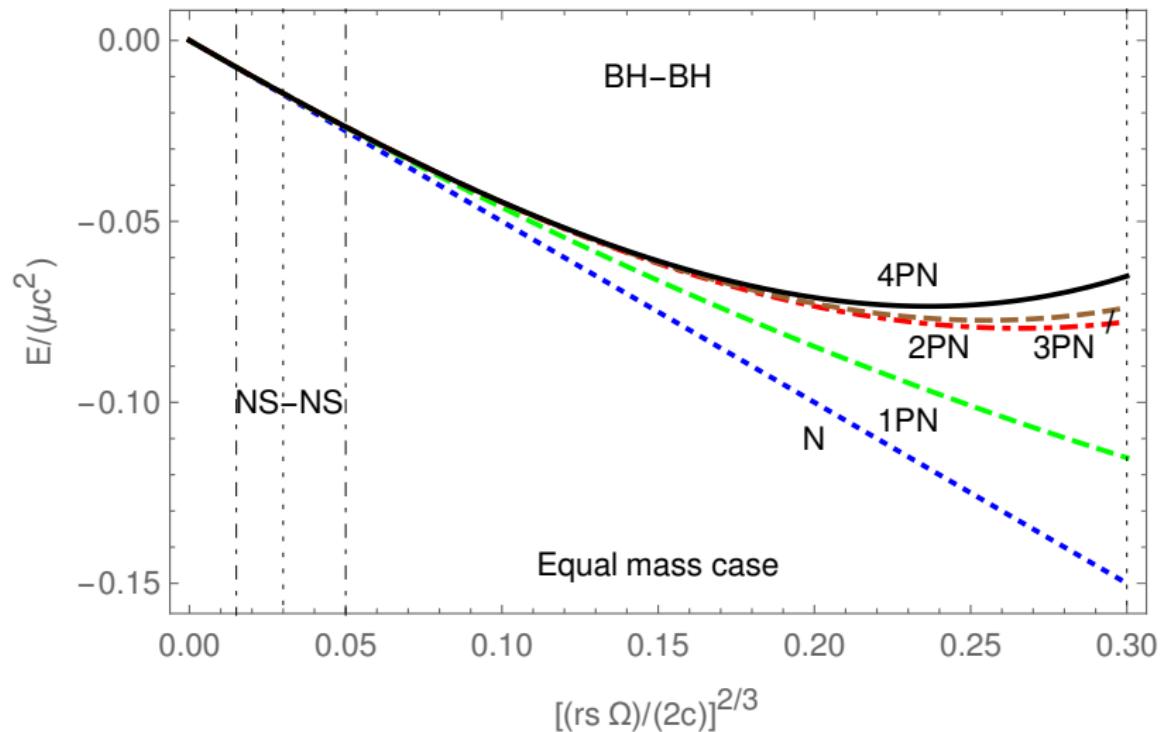
[Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 2019; Blümlein, Maier, Marquard 2019]

[Bern, Cheung, Roiban, Shen, Solon, Zeng 2019; Bini, Damour, Geralico 2019]

Method in this talk:  
non-relativistic effective field theory [Goldberger, Rothstein 2004]

# Post-Newtonian expansion

## Energy



# General Relativity

- Here: point-like objects
  - No spin
  - No finite-size effects (neutron stars: 5PN, black holes: 6PN)
- Harmonic gauge fixing:  $\partial_\mu(\sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}) = 0$   
 $g = \det(g^{\mu\nu})$
- Dimensional regularisation:  $d = 3 - 2\epsilon$

$$S_{\text{GR}} = S_{\text{EH}} + S_{\text{GF}} + S_{\text{pp}}$$

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} R$$

$$S_{\text{GF}} = -\frac{1}{32\pi G} \int d^{d+1}x \sqrt{-g} \Gamma_\mu \Gamma^\mu$$

$$S_{\text{pp}} = -\sum_i m_i \int d\tau_i = -\sum_i m_i \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_i^\mu}{\partial t} \frac{\partial x_i^\nu}{\partial t}}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\Gamma^\mu = g^{\alpha\beta} \Gamma^\mu{}_{\alpha\beta}$$

# Non-relativistic effective theory

[Goldberger, Rothstein 2004]

Similar to non-relativistic QCD

[Caswell, Lepage 1985; Pineda, Soto 1997; Luke, Manohar, Rothstein 2000; ...]

Full theory:  
General relativity

$$S_{\text{GR}} = S_{\text{EH}} + S_{\text{GF}} + S_{pp} \quad \longrightarrow$$

potential gravitons:

$$k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{1}{r}$$

radiation gravitons:

$$k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{v}{r}$$

Effective theory:  
NRGR

$$S_{\text{NRGR}} = \int dt \frac{1}{2} m_i v_i^2 + \frac{G m_1 m_2}{r} + \dots$$

classical potentials

radiation gravitons

# Potential matching

## Expansion of action

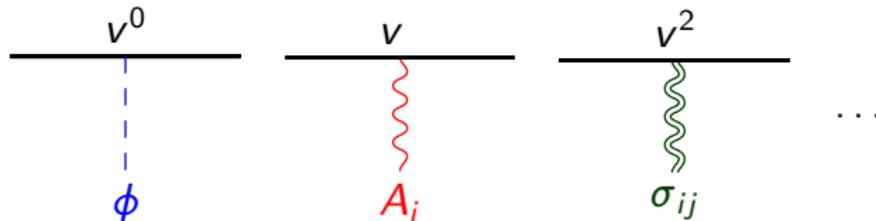
Expand  $S_{\text{GR}}$  in  $v \sim \sqrt{Gm/r} \ll 1$ , e.g.

$$S_{\text{pp}} = - \sum_i m_i \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_i^\mu}{\partial t} \frac{\partial x_i^\nu}{\partial t}} = - \sum_i m_i \int dt \sqrt{-g_{00}} + \mathcal{O}(v_i)$$

Coupling to spatial components of metric suppressed

Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]

$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-2\frac{d-1}{d-2}\phi} (\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix}$$



# Potential matching

## Diagrammatic expansion

Equate amplitude in effective and full theory:

$$\begin{aligned} & \text{Diagram with } q\downarrow \text{ and } -iV \text{ (orange)} + \frac{1}{2!} \text{ Diagram with } 2 \text{ orange vertical lines} + \frac{1}{3!} \text{ Diagram with } 3 \text{ orange vertical lines} + \dots \\ = & \text{Diagram with } 1 \text{ blue vertical line} + \text{Diagram with } 1 \text{ red wavy line} + \text{Diagram with } 1 \text{ blue triangle} + \text{Diagram with } 2 \text{ blue vertical lines} + \text{Diagram with } 2 \text{ blue X's} + \dots \end{aligned}$$

All momenta potential,  $p_0 \sim \frac{v}{r} \ll p_i \sim \frac{1}{r}$   
↪ expand propagators:

$$\frac{1}{\vec{p}^2 - p_0^2} = \frac{1}{\vec{p}^2} + \frac{p_0^2}{\vec{p}^4} + \mathcal{O}(v^4)$$

# Potential matching

## Diagrammatic expansion

$$V = i \log \left( 1 + \frac{\text{Diagram 1}}{\text{Diagram 0}} + \frac{\text{Diagram 2}}{\text{Diagram 0}} + \frac{\text{Diagram 3}}{\text{Diagram 0}} + \dots \right)$$
$$= i \left( \frac{\text{Diagram 1}}{\text{Diagram 0}} + \underbrace{\frac{\text{Diagram 2}}{\text{Diagram 0}} + \frac{\text{Diagram 3}}{\text{Diagram 0}} + \dots}_{\text{1PN}} \right)$$

The diagrams are represented by horizontal lines with vertical dashed lines indicating internal degrees of freedom. The first diagram shows two vertical dashed lines. The second diagram shows a red wavy line. The third diagram shows three vertical dashed lines. The 1PN part shows the sum of the second and third diagrams.

# Potential matching

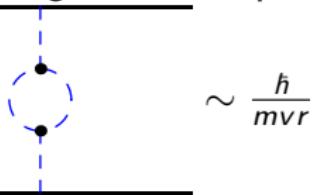
## Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

# Potential matching

## Diagram selection

- No pure graviton loops (quantum corrections)

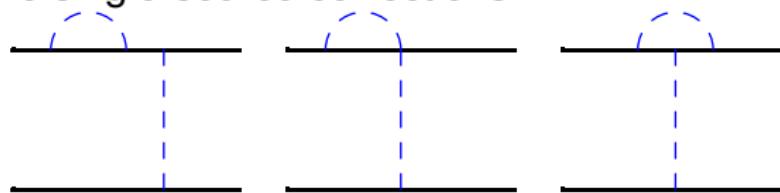


- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

# Potential matching

## Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections



Absorbed into renormalisation of sources

- No source-reducible diagrams [Fischler 1977]

# Potential matching

## Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

$$\frac{\text{---} \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \text{---}}{\Theta(t_2 - t_1)} = \frac{1}{2} \left( \frac{\text{---} \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \text{---}}{\Theta(t_2 - t_1)} + \frac{\text{---} \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \text{---}}{\Theta(t_1 - t_2)} \right) = \frac{1}{2} \text{---} \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \text{---}$$

# Potential matching

## Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

$$\begin{array}{c} \text{---} \\ | \quad | \\ | \quad | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \quad | \\ \times \quad | \\ | \quad | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ | \quad | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \quad | \\ | \quad | \\ \text{---} \end{array}$$

# Potential matching

## Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

$$\begin{array}{c} \text{---} \rightarrow \\ | \quad | \\ \text{---} \rightarrow + \quad \text{---} \times \quad = \quad \text{---} \rightarrow \\ | \quad | \\ \text{---} \rightarrow + \quad \text{---} \rightarrow \\ \hline \end{array}$$
$$= \frac{1}{2} \left( \begin{array}{c} \text{---} \rightarrow \\ | \quad | \\ \text{---} \rightarrow + \quad \text{---} \rightarrow \\ | \quad | \\ \text{---} \rightarrow - \quad \text{---} \rightarrow \\ | \quad | \\ \text{---} \rightarrow + \quad \text{---} \rightarrow \\ | \quad | \\ \text{---} \rightarrow - \end{array} \right)$$

# Potential matching

## Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

$$\begin{aligned} & \text{---} \rightarrow \quad \text{---} \rightarrow = \text{---} \rightarrow + \text{---} \rightarrow \\ & \text{---} \rightarrow \quad \text{---} \rightarrow \quad \text{---} \leftarrow + \text{---} \leftarrow \\ & = \frac{1}{2} \left( \text{---} \rightarrow + \text{---} \leftarrow + \text{---} \rightarrow + \text{---} \leftarrow \right) \\ & = \frac{1}{2} \text{---} = \frac{1}{2} \left( \text{---} \right)^2 \end{aligned}$$

# Potential matching

## Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

$$\begin{aligned} & \text{Diagram 1} + \text{Diagram 2} = \text{Diagram 3} + \text{Diagram 4} \\ &= \frac{1}{2} \left( \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \right) \\ &= \frac{1}{2} \text{Diagram 9} = \frac{1}{2} \left( \text{Diagram 10} \right)^2 \end{aligned}$$

$$-iV = \log \left( 1 + \text{Diagram 11} + \frac{1}{2} \left( \text{Diagram 10} \right)^2 + \dots \right) = \text{Diagram 12} + \dots$$

# Potential matching

## Known results

Confirmation of previous results:

- 1PN: [Goldberger, Rothstein 2004]
- 2PN: [Gilmore, Ross 2008]
- 3PN: [Foffa, Sturani 2011] [Blümlein, Maier, Marquard 20XX]
- 4PN:
  - “static” contribution  $\nu = 0$ :  
[Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaradowski 2017]
  - $\nu \neq 0$ : [Foffa, Sturani 2019; Foffa, Porto, Rothstein, Sturani 2019]

New:

- **5PN static contribution:**

[Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 27 Feb 2019; Blümlein, Maier, Marquard 28 Feb 2019 ]

# Potential matching

## Static 5PN calculation

$$-iV_{\text{5PN}}^S =$$

The diagrammatic expansion of the static 5PN potential  $V_{\text{5PN}}^S$  is shown as a sum of terms. Each term consists of two horizontal lines representing energy levels and a blue wavy line representing an interaction between them. The first term shows a single wavy line connecting the two levels. Subsequent terms show more complex interactions involving multiple wavy lines and vertices, with additional terms indicated by ellipses at the bottom right.

# Potential matching

## Number of diagrams

	QGRAF	source irred	no source loops	no tadpoles	sym
N	1	1	1	1	1
1PN	2	2	2	2	1
2PN	19	19	19	15	5
3PN	360	276	258	122	8
4PN	10081	5407	4685	1815	50
5PN	332020	128080	101570	27582	154

# Potential matching

## Feynman rules

$$\text{---} \stackrel{p}{\text{---}} = -\frac{i}{2c_d \vec{p}^2}$$

$$\begin{array}{c} i_1 i_2 \\ \text{~~~} \text{~~~} \\ p \\ \text{~~~} \text{~~~} \\ j_1 j_2 \end{array} = -\frac{i}{2\vec{p}^2} (\delta_{i_1 j_1} \delta_{i_2 j_2} + \delta_{i_1 j_2} \delta_{i_2 j_1} + (2 - c_d) \delta_{i_1 i_2} \delta_{j_1 j_2})$$

$$\overline{\frac{m_i}{\text{---}}} \stackrel{n}{\text{---}} = -i \frac{m_i}{m_{\text{Pl}}^n}$$

$$\begin{array}{c} p_1 \\ \text{---} \text{~~~} \text{~~~} \\ \text{~~~} \text{~~~} \\ p_2 \\ \text{---} \text{~~~} \text{~~~} \\ i_1 i_2 \end{array} = i \frac{c_d}{2m_{\text{Pl}}} (V_{\phi\phi\sigma}^{i_1 i_2} + V_{\phi\phi\sigma}^{i_2 i_1})$$

$$V_{\phi\phi\sigma}^{i_1 i_2} = \vec{p}_1 \cdot \vec{p}_2 \delta^{i_1 i_2} - 2p_1^{i_1} p_2^{i_2}$$

$$\begin{array}{c} p_1 \\ \text{---} \text{~~~} \text{~~~} \\ \text{~~~} \text{~~~} \\ p_2 \\ \text{---} \text{~~~} \text{~~~} \\ j_1 j_2 \\ \text{---} \text{~~~} \text{~~~} \\ i_1 i_2 \end{array} = i \frac{c_d}{16m_{\text{Pl}}^2} (V_{\phi\phi\sigma\sigma}^{i_1 i_2 j_1 j_2} + V_{\phi\phi\sigma\sigma}^{i_2 i_1 j_1 j_2} + V_{\phi\phi\sigma\sigma}^{i_1 i_2 j_2 j_1} + V_{\phi\phi\sigma\sigma}^{i_2 i_1 j_2 j_1})$$

$$V_{\phi\phi\sigma\sigma}^{i_1 i_2 j_1 j_2} = \vec{p}_1 \cdot \vec{p}_2 (\delta^{i_1 i_2} \delta^{j_1 j_2} - 2\delta^{i_1 j_1} \delta^{i_2 j_2}) - 2(p_1^{i_1} p_2^{i_2} \delta^{j_1 j_2} + p_1^{j_1} p_2^{j_2} \delta^{i_1 i_2}) + 8\delta^{i_1 j_1} p_1^{i_2} p_2^{j_2}$$

# Potential matching

## Feynman rules

Diagram:

$$\begin{aligned}
 & \text{Diagram: } \text{Two external lines } p_1 \text{ and } p_2 \text{ enter from the left. They interact at a central vertex labeled } i_1 i_2. \text{ Two internal lines emerge from the vertex, labeled } k_1 k_2. \\
 & \text{Equation: } \frac{i}{32m_{\text{Pl}}} (\tilde{V}_{\sigma\sigma\sigma}^{i_1 i_2 j_1 j_2, k_1 k_2} + \tilde{V}_{\sigma\sigma\sigma}^{i_2 i_1 j_1 j_2, k_1 k_2}) \\
 & \tilde{V}_{\sigma\sigma\sigma}^{i_1 i_2 j_1 j_2, k_1 k_2} = V_{\sigma\sigma\sigma}^{i_1 i_2 j_1 j_2, k_1 k_2} + V_{\sigma\sigma\sigma}^{i_1 i_2 j_2 j_1, k_1 k_2} + V_{\sigma\sigma\sigma}^{i_1 i_2 j_1 j_2, k_2 k_1} + V_{\sigma\sigma\sigma}^{i_1 i_2 j_2 j_1, k_2 k_1} \\
 & V_{\sigma\sigma\sigma}^{i_1 i_2 j_1 j_2, k_1 k_2} = (\vec{p}_1^2 + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_2^2) \left( -\delta^{i_1 j_2} (2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) \right. \\
 & \quad \left. + 2[\delta^{i_1 j_1} (4\delta^{i_2 k_1} \delta^{j_2 k_2} - \delta^{i_2 j_2} \delta^{k_1 k_2}) - \delta^{i_1 i_2} \delta^{j_1 k_1} \delta^{j_2 k_2}] \right) \\
 & \quad + 2 \left\{ 4(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2}) \delta^{i_1 j_1} \delta^{j_2 k_1} \right. \\
 & \quad + 2[(p_1^{i_1} + p_2^{i_1}) p_2^{i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} - p_1^{k_1} p_2^{k_2} \delta^{i_1 j_1} \delta^{i_2 j_2}] \\
 & \quad + \delta^{j_1 j_2} [p_1^{k_1} p_2^{k_2} \delta^{i_1 i_2} + 2(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2}) \delta^{i_1 k_1} - (p_1^{i_1} + p_2^{i_1}) p_2^{i_2} \delta^{k_1 k_2}] \\
 & \quad + p_2^{i_2} (4p_1^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} + p_1^{j_1} (2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) \\
 & \quad \quad \left. + 2[\delta^{i_1 j_1} (p_1^{i_2} \delta^{k_1 k_2} - 2p_1^{k_2} \delta^{i_2 k_1}) - p_1^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1}] \right) \\
 & \quad + p_1^{i_2} (p_1^{j_1} (2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) - 4p_2^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} \\
 & \quad \quad \left. + 2[p_2^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} + \delta^{i_1 j_1} (2p_2^{k_2} \delta^{i_2 k_1} - p_2^{i_2} \delta^{k_1 k_2})] \right) \} \\
 & c_d = 2 \frac{d-1}{d-2}, \quad m_{\text{Pl}} = \sqrt{32\pi G}
 \end{aligned}$$

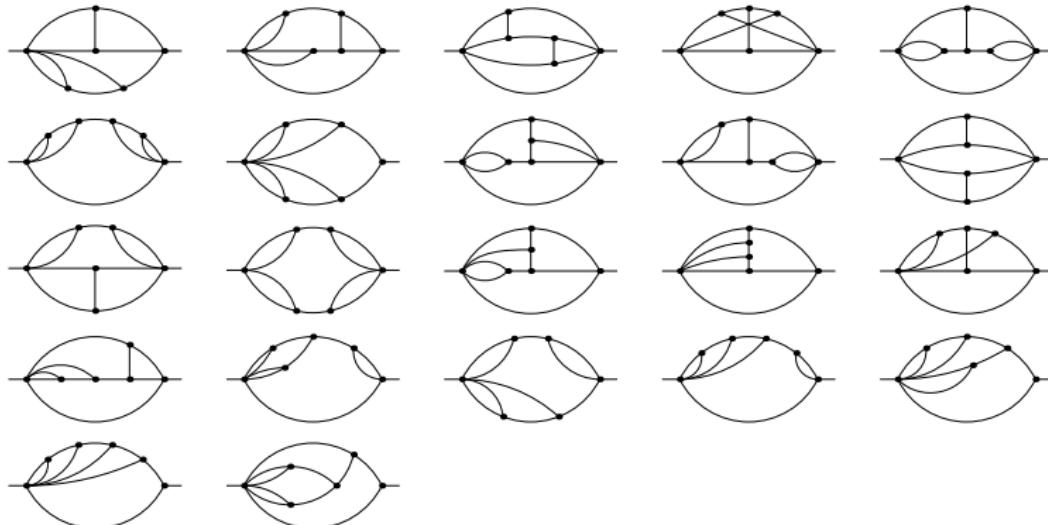
# Potential matching

## Diagram families

Algebraic manipulations (FORM [Vermaseren et al.] )

→ massless propagators:

$$P_f(q) = \int \frac{d^d l_1}{\pi^{d/2}} \cdots \frac{d^d l_5}{\pi^{d/2}} \frac{\mathcal{N}(q, l_1, \dots, l_5)}{\vec{p}_1^{2a_1} \cdots \vec{p}_{10}^{2a_{10}}}$$



# Potential matching

Integration-by-parts relations [Chetyrkin, Tkachov 1981]

$$0 = \int \frac{d^d I_1}{\pi^{d/2}} \cdots \frac{d^d I_L}{\pi^{d/2}} \frac{\partial}{\partial |I_i|} \cdot \textcolor{brown}{P} I(q, I_1, \dots, I_L)$$

# Potential matching

Integration-by-parts relations [Chetyrkin, Tkachov 1981]

$$0 = \int \frac{d^d I_1}{\pi^{d/2}} \cdots \frac{d^d I_L}{\pi^{d/2}} \frac{\partial}{\partial |I|} \cdot P I(q, I_1, \dots, I_L)$$

Example:

$$P(a, b) = \text{Diagram} = \int \frac{d^d I}{\pi^{\frac{d}{2}}} \frac{1}{I^{2a}} \frac{1}{(\vec{I} + \vec{q})^{2b}} = P(b, a)$$

Two identities:

$$\begin{aligned} 0 &= \int \frac{d^d I}{\pi^{\frac{d}{2}}} \frac{\partial}{\partial |I|} \cdot P \frac{1}{I^{2a}} \frac{1}{(\vec{I} + \vec{q})^{2b}} \\ &= (d - 2a - b)P(a, b) - bP(a - 1, b + 1) + q^2 b P(a, b + 1) \end{aligned}$$

$$0 = \int \frac{d^d I}{\pi^{\frac{d}{2}}} \frac{\partial}{\partial |I|} \cdot P \frac{1}{I^{2a}} \frac{1}{(\vec{I} + \vec{q})^{2b}} = \dots$$

Reduce to *master integral*  $P(1, 1)$

# Potential matching

Laporta's algorithm [Laporta 2000]

- ① Insert discrete integer values for  $a, b$ :

$$a = 1, b = 1 : \quad 0 = (d - 3)P(1, 1) + q^2 P(1, 2)$$

$$a = 1, b = 2 : \quad 0 = (d - 4)P(1, 2) + 2q^2 P(1, 3)$$

$$a = 2, b = 1 : \quad 0 = (d - 6)P(1, 2) + q^2 P(2, 2)$$

$$a = 2, b = 2 : \quad 0 = (d - 6)P(2, 2) + 2q^2 P(2, 3) - 2P(1, 3)$$

⋮

- ② Solve linear system of equations

# Potential matching

## Reduction to master integrals

Reduction to master integrals (crusher [Marquard, Seidel]):

$$V_{\text{5PN}}^S = \tilde{c}_0 \text{ (diagram)} + \tilde{c}_1 \text{ (diagram)} + \tilde{c}_2 \text{ (diagram)} + \dots$$

$$\stackrel{\text{IBP}}{=} c_0 \text{ (diagram)} + c_1 \text{ (diagram)} + c_2 \text{ (diagram)}$$

$$+ c_3 \text{ (diagram)} + \mathcal{O}(\epsilon)$$

$\tilde{c}_i, c_j$ : Laurent series in  $\epsilon = \frac{3-d}{2}$ , polynomials in  $m_1, m_2, r^{-1}, G$

# Potential matching

## Calculation of master integrals

Master integrals factorise, e.g.

$$\text{Diagram} = \left[ \frac{q}{\text{Diagram}} \right]_{q^2=1}^2 \times \text{Diagram}$$

Diagram: A horizontal line with two vertices. Four curved lines radiate from the left vertex. The right vertex has a momentum arrow labeled  $q$ .

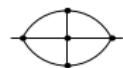
Diagram: An oval loop with two vertices. The right vertex has a momentum arrow labeled  $q$ . The label  $d-3$  is written below the oval.

$$\text{Diagram} = \left[ \frac{q}{\text{Diagram}} \right]_{q^2=1} \times \text{Diagram}$$

Diagram: A horizontal line with two vertices. Two curved lines radiate from each vertex. The right vertex has a momentum arrow labeled  $q$ .

Diagram: An oval loop with two vertices. Both vertices have a dot above them. The right vertex has a momentum arrow labeled  $q$ . The label  $2d-8$  is written below the oval.

$$\begin{aligned}\text{Diagram} &= \int \frac{d^d l}{\pi^{d/2}} \frac{1}{((q-l)^2)^a} \frac{1}{(l^2)^b} \\ &= \frac{1}{(q^2)^{a+b-d/2}} \frac{\Gamma(\frac{d}{2}-a) \Gamma(\frac{d}{2}-b) \Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}\end{aligned}$$



known [Lee, Mingulov 2015; Damour, Jaradowski 2017]

# Potential matching

## Results for master integrals

$$\text{Diagram} = e^{5\epsilon\gamma_E} \frac{\Gamma\left(6 - \frac{5d}{2}\right) \Gamma^6\left(-1 + \frac{d}{2}\right)}{\Gamma(-6 + 3d)}$$

$$\text{Diagram} = e^{5\epsilon\gamma_E} \frac{\Gamma\left(7 - \frac{5d}{2}\right) \Gamma(3 - d) \Gamma\left(2 - \frac{d}{2}\right) \Gamma^7\left(-1 + \frac{d}{2}\right) \Gamma(5 - 2d)}{\Gamma\left(5 - \frac{3}{2}d\right) \Gamma(-2 + d) \Gamma\left(-3 + \frac{3}{2}d\right) \Gamma(-7 + 3d)}$$

$$\text{Diagram} = e^{5\epsilon\gamma_E} \frac{\Gamma\left(7 - \frac{5d}{2}\right) \Gamma^2(3 - d) \Gamma^7\left(-1 + \frac{d}{2}\right) \Gamma\left(-6 + \frac{5d}{2}\right)}{\Gamma(6 - 2d) \Gamma^2\left(-3 + \frac{3d}{2}\right) \Gamma(-7 + 3d)}$$

$$\text{Diagram} = 6\pi^{7/2} \left[ \frac{2}{\epsilon} - 4 - 4\ln(2) - (48 + 8\ln(2) - 4\ln^2(2) - 105\zeta_2)\epsilon + \mathcal{O}(\epsilon^2) \right]$$

$$\begin{aligned} V_{5\text{PN}}^S &\stackrel{\epsilon=0}{=} \frac{G^6}{r^6} (m_1 m_2) \pi^{-7/2} \left\{ \frac{15}{32} (m_1^5 + m_2^5) \left[ \text{Diagram} \right]_{\epsilon^0} + \frac{91}{4} m_1 m_2 (m_1^3 + m_2^3) \left[ \text{Diagram} \right]_{\epsilon^0} \right. \\ &+ m_1^2 m_2^2 (m_1 + m_2) \left( \left[ \frac{293}{4} \text{Diagram} - \frac{45}{16} \text{Diagram} + \frac{45}{32} \text{Diagram} \right]_{\epsilon^0} \right. \\ &\left. \left. + \left[ \frac{519}{16} \text{Diagram} - \frac{627}{32} \text{Diagram} + 2 \text{Diagram} \right]_{\epsilon^{-1}} \right) \right\} \end{aligned}$$

# Potential matching

## Result

$$V_N^S = -\frac{G}{r} m_1 m_2$$

$$V_{1\text{PN}}^S = \frac{G^2}{2r^2} m_1 m_2 (m_1 + m_2)$$

$$V_{2\text{PN}}^S = -\frac{G^3}{r^3} m_1 m_2 \left[ \frac{1}{2} (m_1^2 + m_2^2) + 3m_1 m_2 \right]$$

$$V_{3\text{PN}}^S = \frac{G^4}{r^4} m_1 m_2 \left[ \frac{3}{8} (m_1^3 + m_2^3) + 6m_1 m_2 (m_1 + m_2) \right]$$

$$V_{4\text{PN}}^S = -\frac{G^5}{r^5} m_1 m_2 \left[ \frac{3}{8} (m_1^4 + m_2^4) + \frac{31}{3} m_1 m_2 (m_1^2 + m_2^2) + \frac{141}{4} m_1^2 m_2^2 \right]$$

$$V_{5\text{PN}}^S = \frac{G^6}{r^6} m_1 m_2 \left[ \frac{5}{16} (m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2) \right]$$

# Potential matching

## Velocity corrections

Full corrections include velocities and *higher time derivatives*:

$$\begin{aligned}\mathcal{L}_{2\text{PN}} = & + \frac{G^3}{r^3} m_1 m_2 \left[ \frac{1}{2} (m_1^2 + m_2^2) + 3m_1 m_2 \right] \\ & + G m_1 m_2 r \left[ \frac{15}{8} \vec{a}_1 \vec{a}_2 - \frac{1}{8} (\vec{a}_1 \vec{r}) (\vec{a}_2 \vec{r}) \right] \\ & + (\text{terms depending on } \vec{v}_1, \vec{v}_2)\end{aligned}$$

Can be eliminated using

- *Total time derivatives*  $\delta \mathcal{L} \propto \frac{d}{dt} F(\vec{r}, \vec{v}_1, \vec{v}_2)$
- *Equations of motion*  $\delta \mathcal{L} \propto \left( \vec{a}_1 + \frac{G m_2}{r^3} \vec{r} \right) \left( \vec{a}_2 - \frac{G m_1}{r^3} \vec{r} \right)$

$$\begin{aligned}\mathcal{L}_{2\text{PN}} = & - \frac{G^3}{r^3} m_1 m_2 \left[ \frac{1}{4} (m_1^2 + m_2^2) + \frac{5}{4} m_1 m_2 \right] \\ & + (\text{terms depending on } \vec{v}_1, \vec{v}_2)\end{aligned}$$

# Conclusions

- Inspiral phase of compact binary systems described well by *Post-Newtonian (PN) expansion*  $v \sim \sqrt{Gm/r} \ll 1$
- Effective field theories and calculational methods from particle physics very effective for high PN orders
- Static gravitational potential known at five loops (5PN)