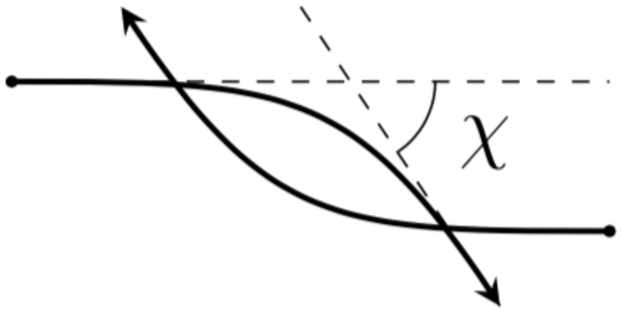
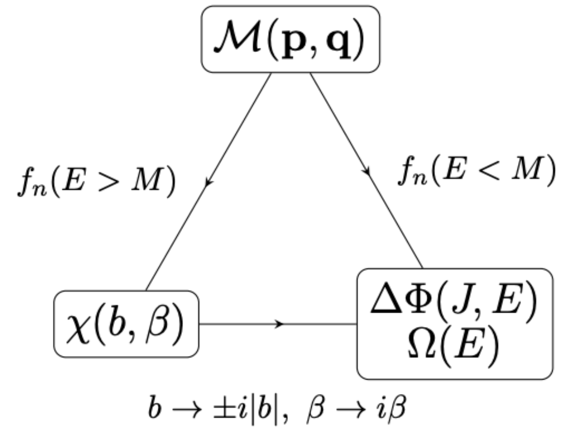
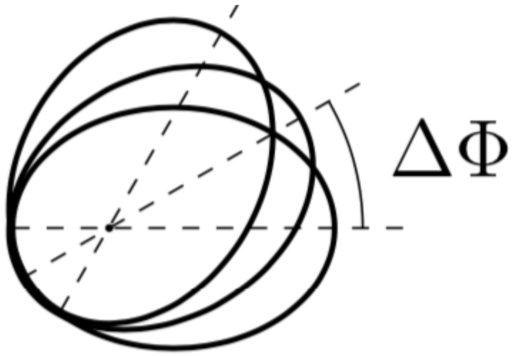
From Boundary Data to Bound States



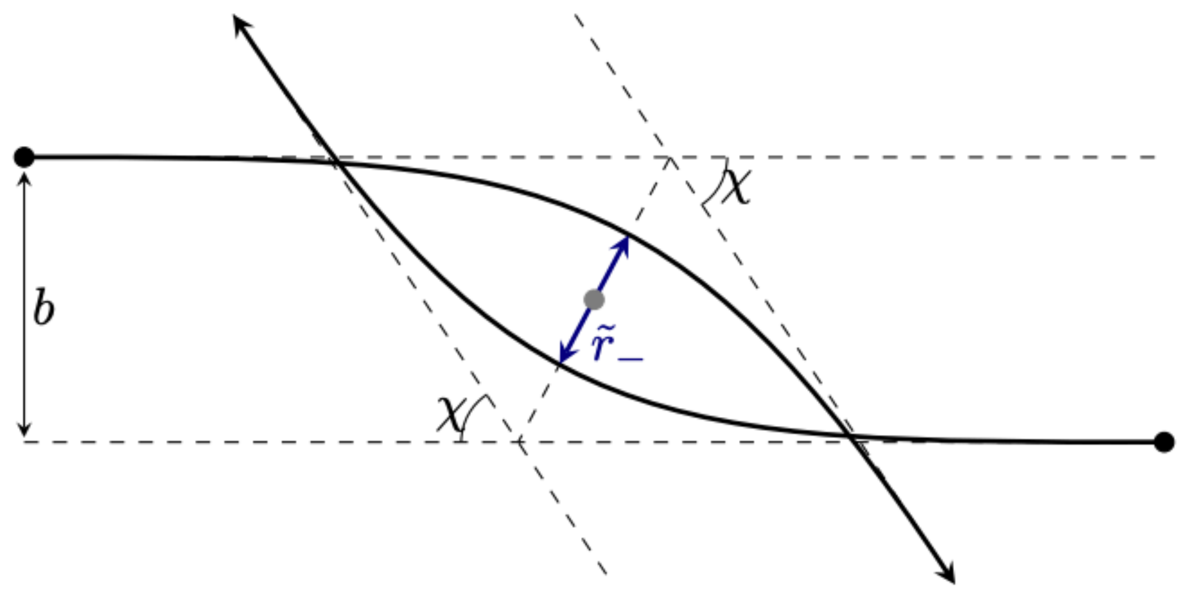




Rafael A. Porto

Based on 1910.03008 and 1911.XXXXX
(with Gregor Kalin)

From Hamiltonians to Angles



$$\chi(b,E) = -\pi + 2b \int_{r_{\min}}^{\infty} \frac{\mathrm{d}r}{r\sqrt{r^2\bar{\mathbf{p}}^2(r,E) - b^2}}\,, \qquad \chi(J,E) = -\pi + 2J \int_{r_{\min}}^{\infty} \frac{\mathrm{d}r}{r^2\sqrt{\mathbf{p}^2(r,E) - J^2/r^2}}\,.$$

$$\bar{\mathbf{p}} = \mathbf{p}/p_\infty.$$

PM expansion:

$$\mathbf{p}^2(r,E) = p_\infty^2(E) + \sum_i P_i(E)\left(\frac{G}{r}\right)^i = p_\infty^2(E)\left(1 + \sum_i f_i(E)\left(\frac{GM}{r}\right)^i\right)$$

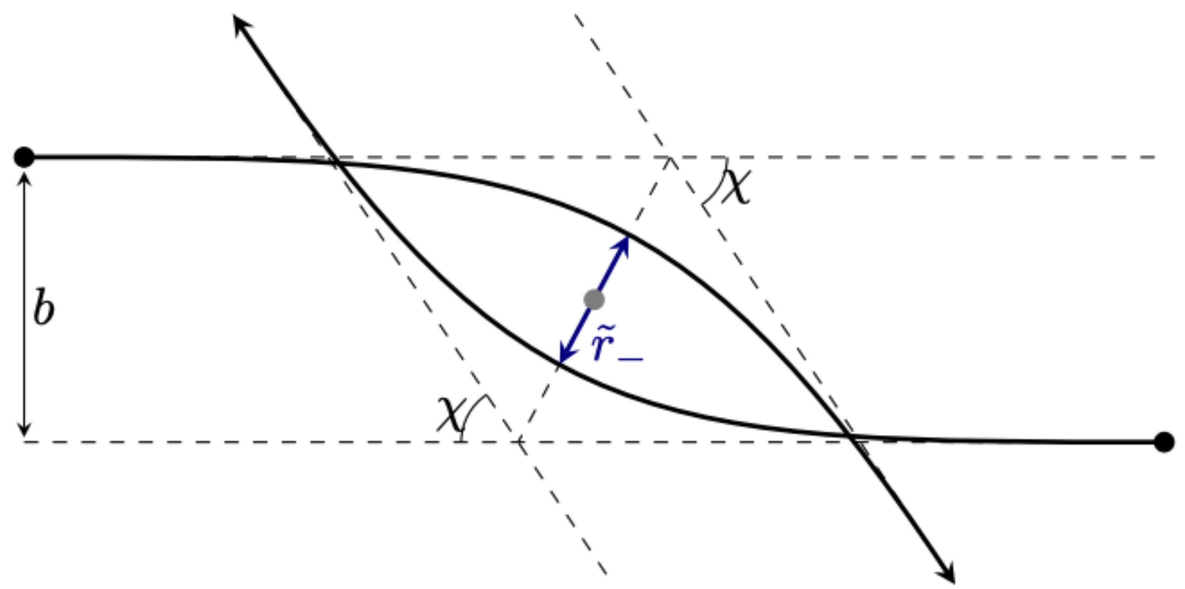
$$\frac{1}{2}\chi(b,E) = \sum_n \chi_b^{(n)}(E)\left(\frac{GM}{b}\right)^n = \sum_n \chi_j^{(n)}(E)\frac{1}{j^n}\,,$$

$$\gamma \equiv \frac{1}{2}\frac{E^2 - m_1^2 - m_2^2}{m_1 m_2} = 1 + \mathcal{E} + \frac{1}{2}\nu\mathcal{E}^2\,,$$

$$\Gamma \equiv E/M = \sqrt{1 + 2\nu(\gamma - 1)} = 1 + \nu\mathcal{E}\,.$$

$$\chi_j^{(n)} = \hat{p}_\infty^n \chi_b^{(n)}\,, \qquad \hat{p}_\infty^2 = \frac{\gamma^2 - 1}{\Gamma^2}\,. \qquad j = \frac{J}{GM\mu}$$

From Angles to Dynamics



Firsov Formula

$$\bar{\mathbf{p}}^2(r,E) = \exp\left[\frac{2}{\pi}\int_{r|\bar{\mathbf{p}}(r,E)|}^{\infty}\frac{\chi_b(\tilde{b},E)\mathrm{d}\tilde{b}}{\sqrt{\tilde{b}^2 - r^2\bar{\mathbf{p}}^2(r,E)}}\right],$$

Iteration Formula II:

$$f_n = \sum_{\sigma\in\mathcal{P}(n)} g_\sigma^{(n)} \prod_\ell \left(\widehat{\chi}_b^{(\sigma_\ell)}\right)^{\sigma^\ell}$$

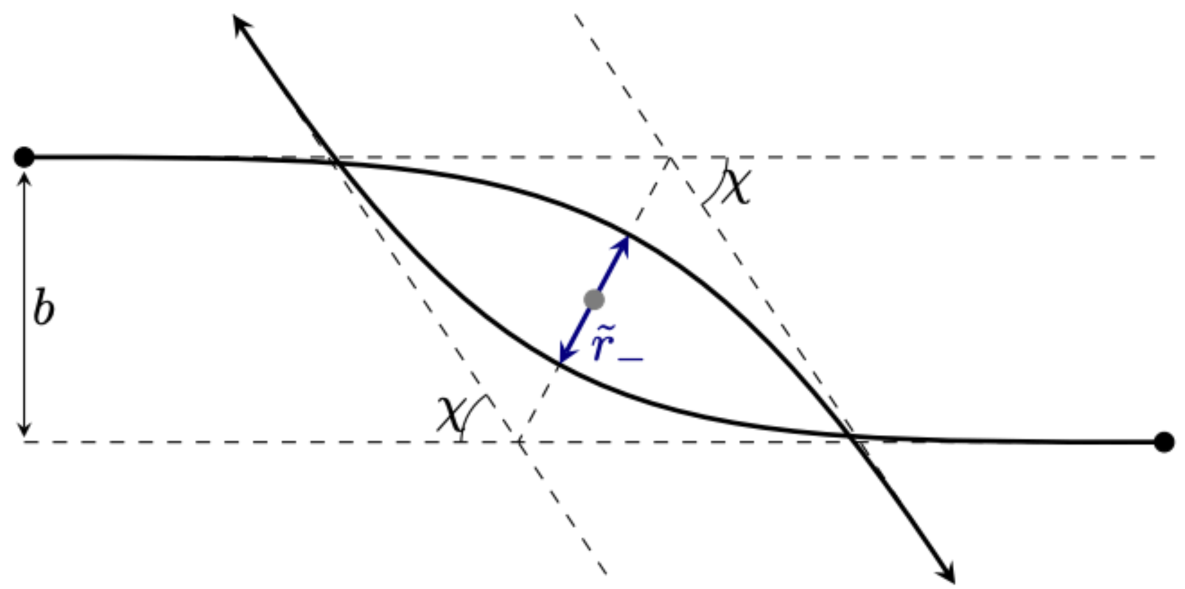
$$\widehat{\chi}_b^{(n)} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\chi_b^{(n)}.$$

$$n = \sigma_\ell \sigma^\ell$$

$$g_\sigma^{(n)} = \frac{2(2-n)^{\Sigma^\ell - 1}}{\prod_\ell (2\sigma^\ell)!!}.$$

$$\Sigma^\ell \equiv \sum_\ell \sigma^\ell.$$

From Angles to Dynamics



Firsov Formula

$$\bar{\mathbf{p}}^2(r,E) = \exp\left[\frac{2}{\pi}\int_{r|\bar{\mathbf{p}}(r,E)|}^{\infty} \frac{\chi_b(\tilde{b},E)\mathrm{d}\tilde{b}}{\sqrt{\tilde{b}^2 - r^2\bar{\mathbf{p}}^2(r,E)}}\right],$$

Iteration Formula II:

$$\chi_b^{(n)} = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right)\sum_{\sigma\in\mathcal{P}(n)} \frac{1}{\Gamma\left(1+\frac{n}{2}-\Sigma^\ell\right)}\prod_\ell \frac{f_{\sigma_\ell}^{\sigma^\ell}}{\sigma^\ell!},$$

$$\chi_b^{(2)} = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{3}{2}\right)\left(\frac{1}{\Gamma(0)}\frac{f_1^2}{2!} + \frac{1}{\Gamma(1)}\frac{f_2^1}{1!}\right) = \frac{\pi}{4}f_2\,.$$

$$\chi_b^{(3)} = \frac{\sqrt{\pi}}{2}\Gamma(2)\left(\frac{1}{\Gamma(3/2)}\frac{f_3^1}{1!} + \frac{1}{\Gamma(1/2)}\frac{f_2^1 f_1^1}{1!1!} + \frac{1}{\Gamma(-1/2)}\frac{f_1^3}{3!}\right) = f_3 + \frac{1}{2}f_2 f_1 - \frac{1}{24}f_1^3\,.$$

Reconstructing the Hamiltonian

$$\sqrt{\boldsymbol{p}^2 - \sum_{i=1}^{\infty} P_i(E)\left(\frac{G}{r}\right)^i + m_1^2} + \sqrt{\boldsymbol{p}^2 - \sum_{i=1}^{\infty} P_i(E)\left(\frac{G}{r}\right)^i + m_2^2} = \sum_{i=0}^{\infty} \frac{c_i(\boldsymbol{p}^2)}{i!}\left(\frac{G}{r}\right)^i .$$

Recursion relation:

$$c_i(\boldsymbol{p}^2) = \sum_{k=1}^{k=i} \frac{\sqrt{\pi}}{2\Gamma\left(\frac{3}{2}-k\right)} \frac{E_1(\boldsymbol{p}^2)^{2k-1} + E_2(\boldsymbol{p}^2)^{2k-1}}{(E_1(\boldsymbol{p}^2)E_2(\boldsymbol{p}^2))^{2k-1}} B_{i,k}\left(\mathcal{G}_1(\boldsymbol{p}^2), \ldots, \mathcal{G}_{i-k+1}(\boldsymbol{p}^2)\right) .$$

The $B_{i,k}$'s are partial Bell polynomials, and $\mathcal{G}_m(\mathbf{p}^2)$ is given by[6]

$$\mathcal{G}_m(\mathbf{p}^2) = -\sum_{s=0}^{m}\sum_{\ell=0}^{s} \frac{m!}{s!} P_{m-s}^{(\ell)}(c_0(\mathbf{p}^2)) B_{s,\ell}\left(c_1(\mathbf{p}^2), \ldots, c_{s-\ell+1}(\mathbf{p}^2)\right) .$$

$$\frac{c_k}{k!} = -\frac{1}{2}\left(\frac{1}{E_1} + \frac{1}{E_2}\right) P_k(E) + \cdots ,$$

From Amplitudes to Impetus...

$$\mathcal{M}(\mathbf{q},\mathbf{p}) \equiv \Re\,\mathcal{M}^{\rm cl}_{\text{IR-fin}}(\mathbf{q},\mathbf{p}), \qquad \mathcal{M}_n = 4E_1E_2\mathcal{M}_n^{\rm EFT}$$

Fourier Transform (relativistic normalization)

$$\widetilde{\mathcal{M}}(r,E) \equiv \frac{1}{2E}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\,\mathcal{M}(\mathbf{q},\mathbf{p}^2 = p_\infty^2(E))e^{-i\mathbf{q}\cdot\mathbf{r}}\,,$$

$$p_\infty^2\left(1+\sum_{i=1} f_i\frac{(GM)^i}{r^i}\right) = p_\infty^2 + \sum_{n=1}\widetilde{\mathcal{M}}_n\frac{G^n}{r^n}\,,$$

Bern et al. found (indirectly) to 5PM order:

$$\widetilde{\mathcal{M}}_n(E) = P_n(E) = p_\infty^2 M^n f_n(E)\,,$$

Is this true in general?

Impetus Formula:

$$\mathbf{p}^2(r,E) = p_\infty^2(E) + \widetilde{\mathcal{M}}(r,E)\,,$$

+Radiation-Reaction

Original problem:

$$H_{\rm PM}|\psi\rangle = \left(c_0(\mathbf{p}^2) + \sum_{i=1} \frac{c_i(\mathbf{p}^2)}{i!}\frac{G^i}{r^i}\right)|\psi\rangle = E|\psi\rangle$$

Effective QM problem:

$$\left(\mathbf{p}^2 + U_{\rm eff}(\mathbf{p}^2, r)\right)|\psi\rangle = p_\infty^2(E)|\psi\rangle\,.$$

Recall the scattering Amplitude:

$$4\pi f(\boldsymbol{p},\boldsymbol{p}') = -\langle \boldsymbol{p}'|U_{\rm eff}|\psi_{\boldsymbol{p}}(p_\infty)\rangle\,,$$

Instantaneous scattered momentum:

$$\boldsymbol{p}_{\rm sc}^2(r, p_\infty^2) = \psi_{\boldsymbol{p}}^\dagger(\boldsymbol{r}, p_\infty)\,(-\nabla^2 - p_\infty^2)\psi_{\boldsymbol{p}}(\boldsymbol{r}, p_\infty)\,.$$

two contributions (linear and quadratic in the amplitude)

$$\boldsymbol{p}_{\rm sc}^2(r, E) = I_{(1)}(r, E) + I_{(2)}(r, E)\,,$$

$$\Re\left[\left(\psi_{\boldsymbol{p}}^\dagger(\boldsymbol{r}, p_\infty) - \phi_{\boldsymbol{p}}^\dagger(\boldsymbol{r}, p_\infty)\right)(-\nabla^2 - p_\infty^2)\psi_{\boldsymbol{p}}(\boldsymbol{r}, p_\infty)\right]$$

$$\Re\left[\phi_{\boldsymbol{p}}^\dagger(\boldsymbol{r}, p_\infty)\,(-\nabla^2 - p_\infty^2)\psi_{\boldsymbol{p}}(\boldsymbol{r}, p_\infty)\right]$$

IR safe:

$$I_{(1)}^{\rm IR}(r, E) + I_{(2)}^{\rm IR}(r, E) = 0\,.$$

Conservative sector w/out Radiation-Reaction

$$\boldsymbol{p}^2_{\rm sc}(r, p^2_\infty) = 4\pi \Re \int \frac{{\rm d}^3\boldsymbol{q}}{(2\pi)^3} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\, f_{\text{IR-fin}}(p^2_\infty, \boldsymbol{q})$$

Matching NR and Relativistic amplitudes:

$$\frac{d\sigma}{d\Omega} = |f(p^2_\infty, \boldsymbol{q})|^2 = \frac{1}{(4\pi)^2(2E)^2}|\mathcal{M}(p^2_\infty, \boldsymbol{q})|^2$$

$$4\pi\, \Re\, f^{\rm cl}_{\text{IR-fin}}(p^2_\infty, \boldsymbol{q}) = \frac{1}{2E}\Re\, \mathcal{M}^{\rm cl}_{\text{IR-fin}}(p^2_\infty, \boldsymbol{q})\,,$$

Impetus Formula:

$$\mathbf{p}^2(r, E) = p^2_\infty(E) + \widetilde{\mathcal{M}}(r, E)\,,$$

+Radiation-Reaction

Kalin Porto

… to Deflection Angle

$$\widetilde{\mathcal{M}}_n(E) = P_n(E) = p_\infty^2 M^n f_n(E)\,,$$

$$\chi_b^{(n)} = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right) \sum_{\sigma\in\mathcal{P}(n)} \frac{1}{\Gamma\left(1+\frac{n}{2}-\Sigma^\ell\right)} \prod_\ell \frac{f_{\sigma_\ell}^{\sigma^\ell}}{\sigma^\ell!}\,,$$

Directly from the amplitude to all orders!

$$4\,\chi_j^{(4)} = 2\hat{p}_\infty^4 \sqrt{\pi}\,\Gamma\left(\frac{5}{2}\right)\left(\frac{1}{\Gamma(2)}\frac{f_4^1}{1!} + \frac{1}{\Gamma(1)}\frac{f_2^2}{2!} + \frac{1}{\Gamma(1)}\frac{f_1^1 f_3^1}{1!1!} + \frac{1}{\Gamma(-1)}\frac{f_1^4}{4!}\right)$$

$$4\chi_j^{(4)} = \frac{3\pi\hat{p}_\infty^4}{4}\left(f_2^2 + 2f_1 f_3 + 2f_4\right) = \frac{3\pi}{4M^4\mu^4}\left(\widetilde{\mathcal{M}}_2^2 + 2\widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_3 + 2p_\infty^2\widetilde{\mathcal{M}}_4\right)$$

(keep an eye on these expressions!)

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… to Deflection Angle

$$\widetilde{\mathcal{M}}_n(E) = P_n(E) = p_\infty^2 M^n f_n(E)\,,$$

$$\chi_b^{(n)} = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right) \sum_{\sigma\in\mathcal{P}(n)} \frac{1}{\Gamma\left(1+\frac{n}{2}-\Sigma^\ell\right)} \prod_\ell \frac{f_{\sigma_\ell}^{\sigma^\ell}}{\sigma^\ell!}\,,$$

In general f_i's enter to all PM orders:

$$\chi_b^{(2n)}[f_{1,2}] = \frac{\sqrt{\pi} f_2^n \Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(n+1)}, \quad n = 1, 2, \ldots$$

$$\chi_b^{(2n+1)}[f_{1,2}] = \frac{1}{2} f_1 f_2^n \, {}_2F_1\left(\frac{1}{2}, -n; \frac{3}{2}; \frac{f_1^2}{4f_2^2}\right), \quad n = 0, 1, \ldots\,,$$

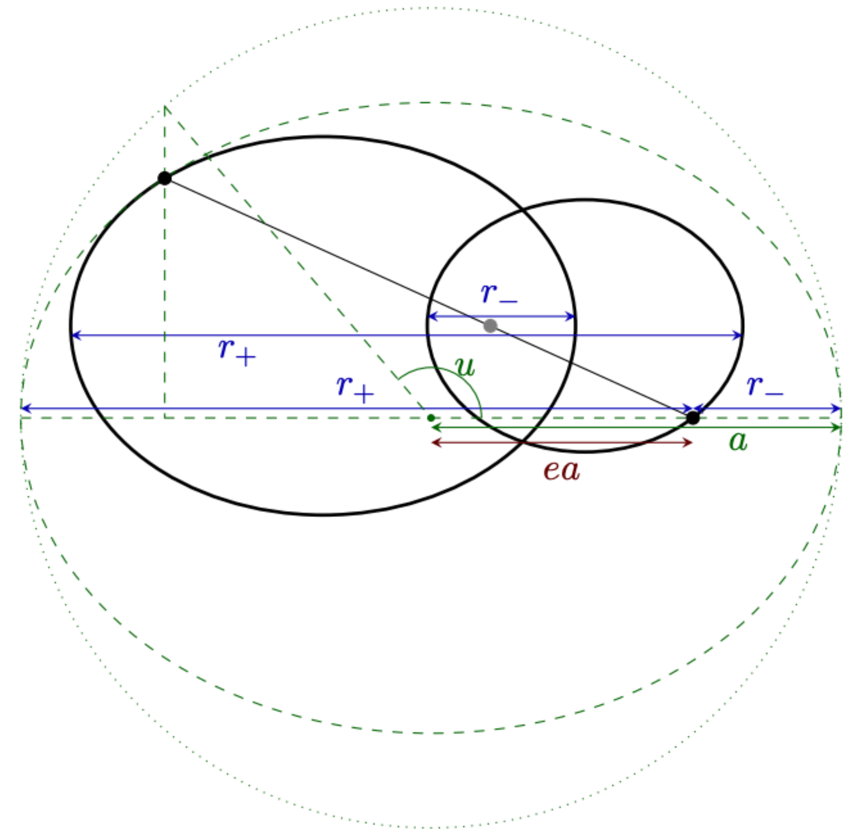
$$\frac{\chi[f_{1,2}]+\pi}{2} = \frac{1}{\sqrt{1-\mathcal{F}_2 y^2}}\left(\frac{\pi}{2} + \arctan\left(\frac{y}{2\sqrt{1-\mathcal{F}_2 y^2}}\right)\right) \qquad \mathcal{F}_2 \equiv f_2/f_1^2$$

… to Dynamical Invariants

Damour Schäfer

$$\mathcal{S}_r(J,\mathcal{E}) \equiv \frac{1}{\pi}\int_{r_-}^{r_+} p_r \mathrm{d}r = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{\boldsymbol{p}^2(r,\mathcal{E}) - J^2/r^2}\, \mathrm{d}r\,,$$

$$\frac{\Phi}{2\pi} = 1 + \frac{\Delta\Phi}{2\pi} = -\frac{\partial \mathcal{S}_r(J,\mathcal{E})}{\partial J}\,.$$



We can write in terms of the amplitude via continuation to negative binding energy

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$$\mathcal{S}_r(J,\mathcal{E}) = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{p_\infty^2(\mathcal{E}) + \widetilde{\mathcal{M}}(r,\mathcal{E}) - J^2/r^2}\, \mathrm{d}r\,,$$

… to Dynamical Invariants

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$$\mathcal{S}_r(J,\mathcal{E}) = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{p_\infty^2(\mathcal{E}) + \widetilde{\mathcal{M}}(r,\mathcal{E}) - J^2/r^2}\, \mathrm{d}r\,,$$

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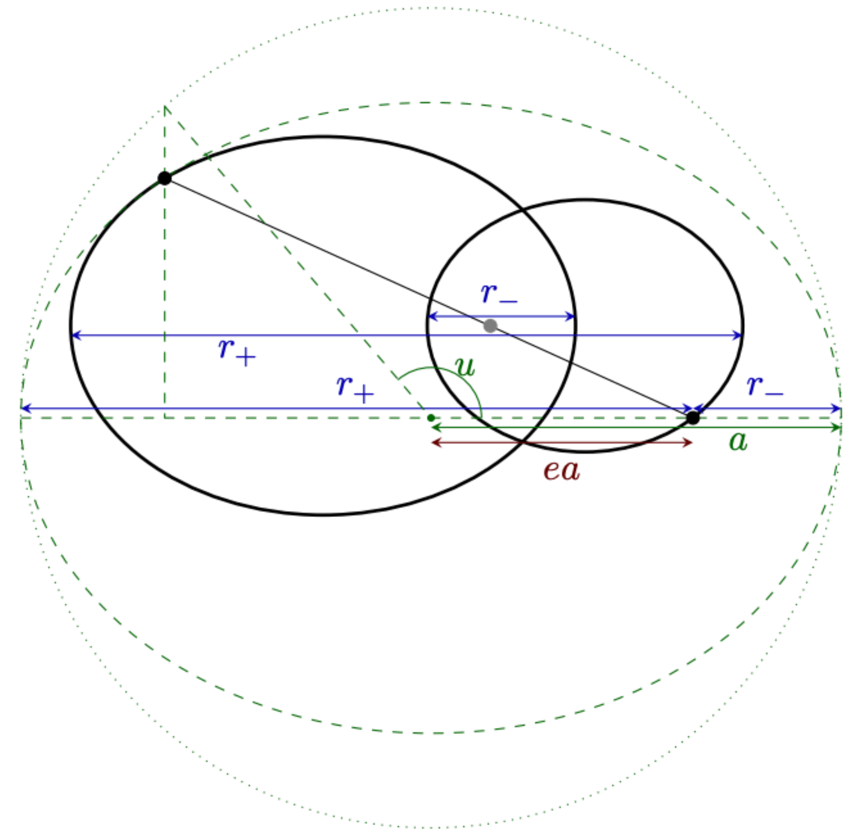

Post-Minkowskian Expansion

$$-\sum_{n=0}^{\infty}\sum_{\sigma\in\mathcal{P}(n)} \frac{(-1)^{\Sigma^\ell}\Gamma\left(\Sigma^\ell - \frac{1}{2}\right)}{2\sqrt{\pi}} \mathcal{S}_{\{n+2\Sigma^\ell,\Sigma^\ell\}}(J,\mathcal{E}) \prod_\ell \frac{D_{\sigma_\ell}^{\sigma^\ell}(\mathcal{E})}{\sigma^\ell!}$$

$$A(\mathcal{E}) \equiv p_\infty^2(\mathcal{E})\,,$$
$$2B(\mathcal{E}) \equiv \widetilde{M}_1(\mathcal{E})G$$
$$C(J,\mathcal{E}) \equiv \widetilde{M}_2(\mathcal{E})G^2 - J^2\,,$$
$$D_n(\mathcal{E}) \equiv \widetilde{M}_{n+2}(\mathcal{E})G^{n+2}\,,$$

$$\mathcal{S}_{\{m,q\}} = -\,i\,\delta_{m,0}(2q-1)B(\mathcal{E})A(\mathcal{E})^{-q-\frac{1}{2}} + \sum_{k\ \mathrm{even}} \frac{(-1)^q i^{k+m+1} 2^k \Gamma\left(\frac{1}{2}(m+k-1)\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}(2+m-k-2q)\right)\Gamma\left(q-\frac{1}{2}\right)} \frac{A(\mathcal{E})^{\frac{1}{2}(m-k-2q)}B(\mathcal{E})^k}{C(J,\mathcal{E})^{\frac{1}{2}(m+k-1)}}$$

… to Dynamical Invariants

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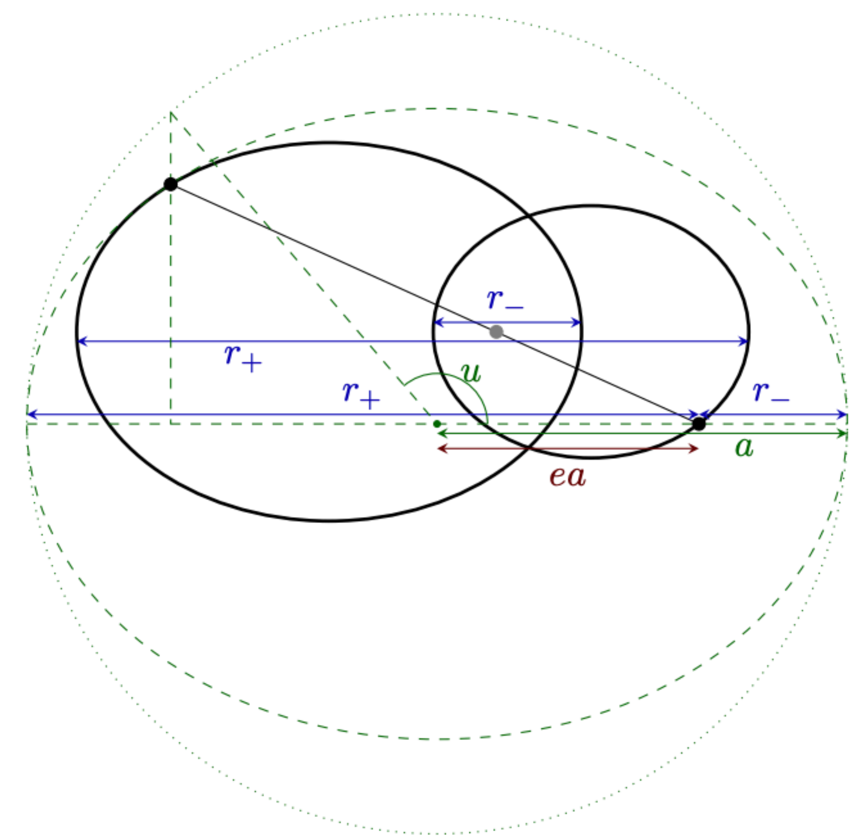
$$\mathcal{S}_r(J,\mathcal{E}) = \frac{1}{\pi}\int_{r_-}^{r_+}\sqrt{p_\infty^2(\mathcal{E}) + \widetilde{\mathcal{M}}(r,\mathcal{E}) - J^2/r^2}\,\mathrm{d}r\,,$$

$$\frac{\Delta\Phi}{2\pi} = \frac{\widetilde{\mathcal{M}}_2 G^2}{2J^2} + \frac{3(\widetilde{\mathcal{M}}_2^2 + 2\widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_3 + 2p_\infty^2\widetilde{\mathcal{M}}_4)G^4}{8J^4} + \mathcal{O}(G^6)\,,$$



Post-Minkowskian Expansion

$$\widetilde{\mathcal{M}}_1 = 2M\mu^2\left(\frac{2\gamma^2-1}{\Gamma}\right)$$

Cheung et al.

$$\widetilde{\mathcal{M}}_2 = \frac{3M^2\mu^2}{2}\left(\frac{5\gamma^2-1}{\Gamma}\right)$$

Bern et al.

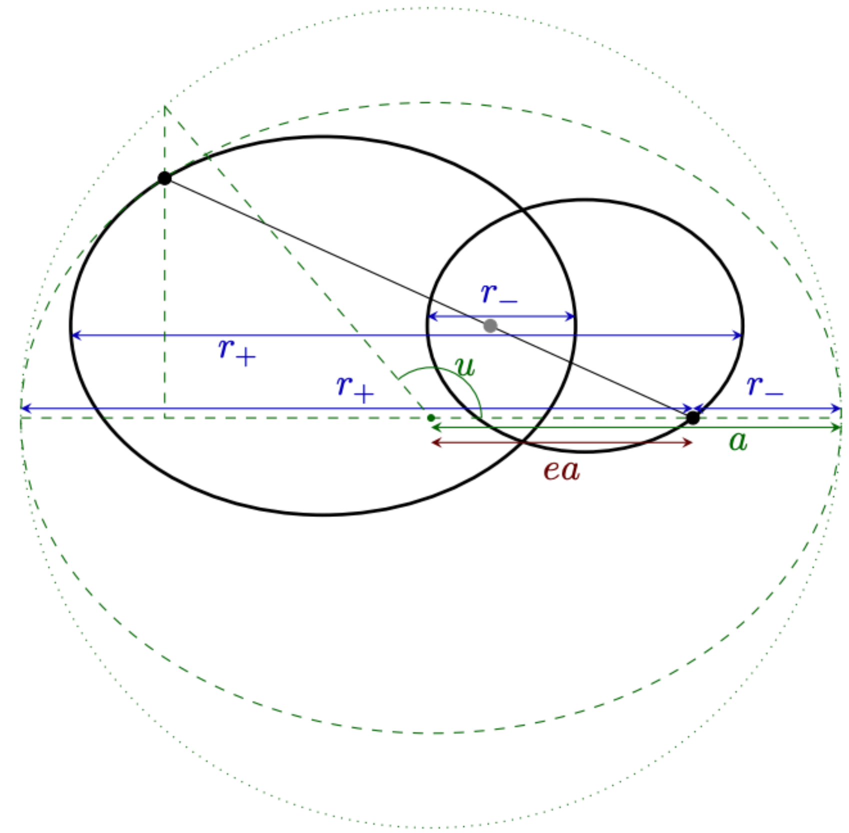
$$\widetilde{\mathcal{M}}_3 = -\frac{M^3\mu^2}{6\Gamma}\left(3 - 54\gamma^2 - 48\nu(3+12\gamma^2-4\gamma^4)\frac{\arcsin\sqrt{\frac{1-\gamma}{2}}}{\sqrt{1-\gamma^2}} + \nu\left(-6+206\gamma+108\gamma^2+4\gamma^3 - \frac{18\Gamma(1-2\gamma^2)(1-5\gamma^2)}{(1+\Gamma)(1+\gamma)}\right)\right)$$

Notice the power counting in 1/J

… to Dynamical Invariants

$$\mathcal{S}_r(J,\mathcal{E}) = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{p_\infty^2(\mathcal{E}) + \widetilde{\mathcal{M}}(r,\mathcal{E}) - J^2/r^2}\,\mathrm{d}r\,,$$

$$\frac{\Delta\Phi}{2\pi} = \frac{\widetilde{\mathcal{M}}_2 G^2}{2J^2} + \frac{3(\widetilde{\mathcal{M}}_2^2 + 2\widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_3 + 2p_\infty^2\widetilde{\mathcal{M}}_4)G^4}{8J^4} + \mathcal{O}(G^6)\,,$$



Post-Minkowskian Expansion

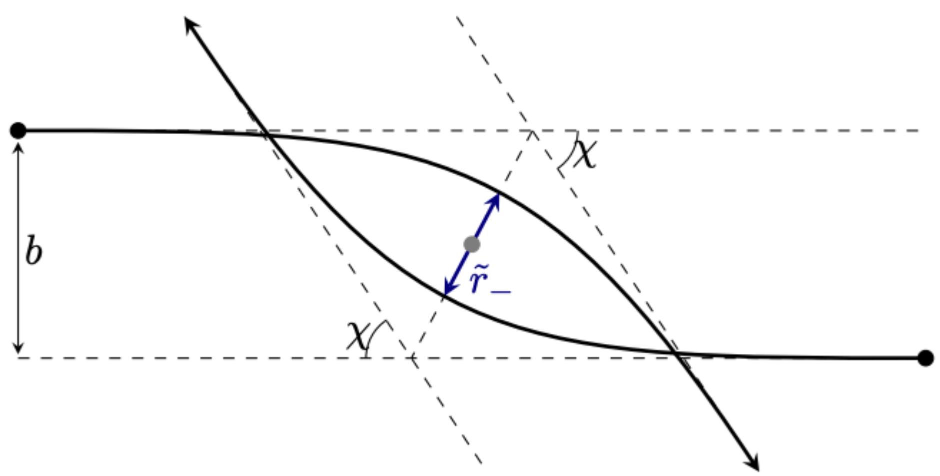
$$\left(\frac{\Delta\Phi}{2\pi}\right)_{\text{1-loop}} = -1 + \frac{1}{\sqrt{1-\frac{\widetilde{\mathcal{M}}_2G^2}{J^2}}} = \frac{\widetilde{\mathcal{M}}_2G^2}{2J^2} + \cdots = \frac{3}{4j^2}\left(\frac{5\gamma^2-1}{\Gamma}\right) + \cdots\,,$$

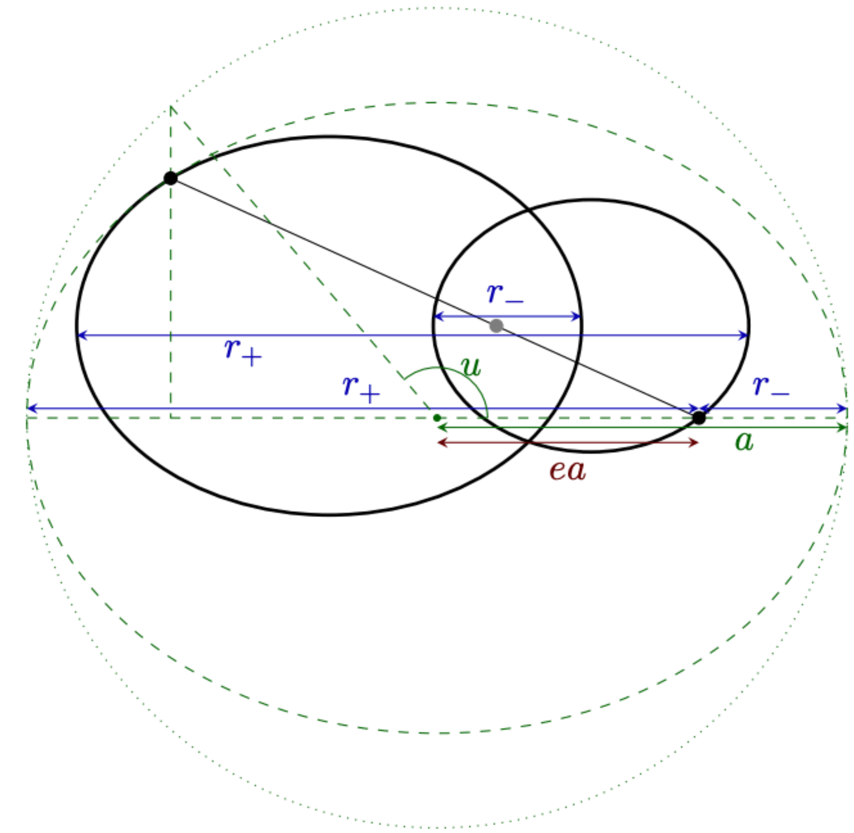
$1/j^2$ to all PN!

$$\begin{aligned}\left(\frac{\Delta\Phi}{2\pi}\right)_{\text{2-loop}} &= \frac{3}{j^2} + \frac{3(35-10\nu)}{4j^4} + \frac{3}{4j^2}\left(10 - 4\nu + \frac{194-184\nu+23\nu^2}{j^2}\right)\mathcal{E} \checkmark \\ &+ \frac{3}{4j^2}\left(5 - 5\nu + 4\nu^2 \checkmark + \frac{3535-6911\nu+3060\nu^2-375\nu^3}{10j^2}\right)\mathcal{E}^2 \\ &+ \frac{3}{4j^2}\left((5-4\nu)\nu^2 \checkmark + \frac{35910-126347\nu+125559\nu^2-59920\nu^3+7385\nu^4}{140j^2}\right)\mathcal{E}^3 \\ &+ \frac{3}{4j^2}\left((5-20\nu+16\nu^2)\frac{\nu^2}{4}\right)\mathcal{E}^4 + \cdots\,,\end{aligned}$$

5PN

Orbital Elements





$$r = \tilde{a}(\tilde{e}\cosh u - 1) \qquad \text{(Hyperbola)}$$

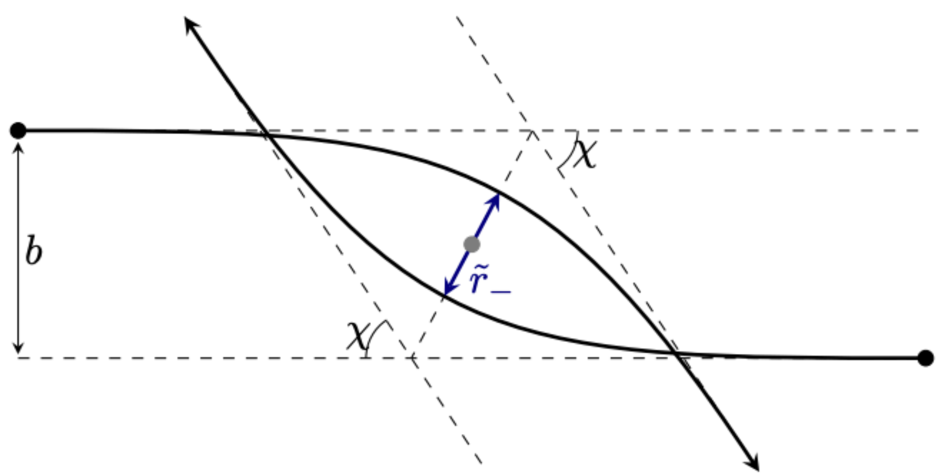
$$r = a(1 - e\cos u), \qquad \text{(Ellipse)}$$

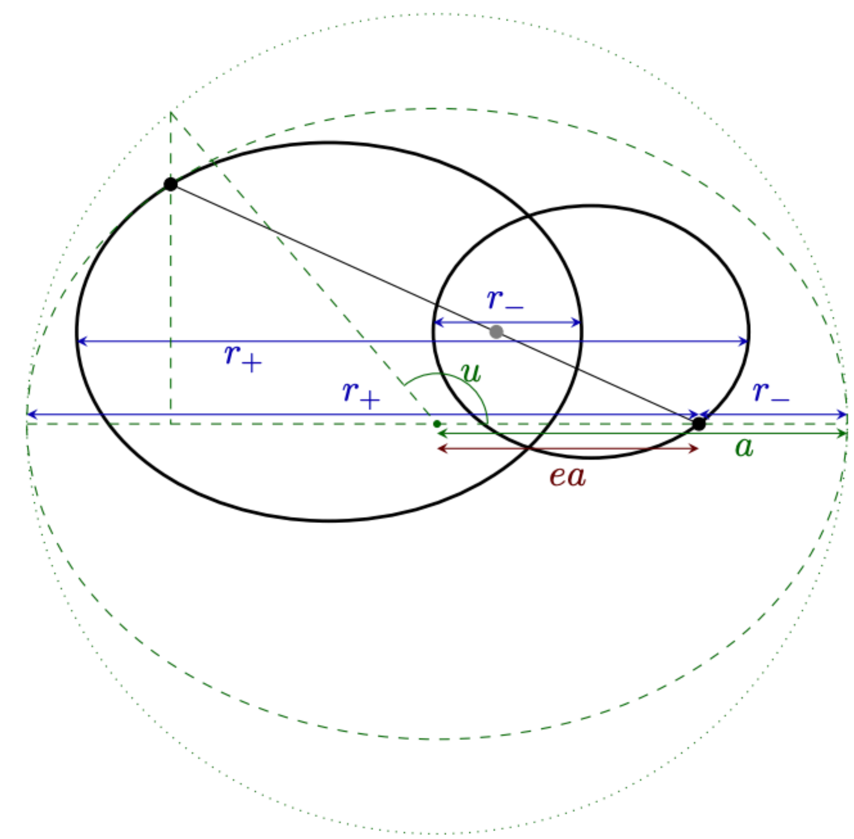
$$-\tilde{a} = \frac{\tilde{r}_+ + \tilde{r}_-}{2}, \quad \tilde{e} = \frac{\tilde{r}_+ - \tilde{r}_-}{\tilde{r}_+ + \tilde{r}_-},$$

$$a = \frac{r_+ + r_-}{2}, \quad e = \frac{r_+ - r_-}{r_+ + r_-}.$$

zeros of: $$r^2\left(1 + \sum_i f_i(\mathcal{E})\left(\frac{GM}{r}\right)^i\right) = b^2.$$

From Hyperbolas to Ellipses





$$r = \tilde{a}(\tilde{e}\cosh u - 1) \qquad \text{(Hyperbola)}$$

$$r = a(1 - e\cos u), \qquad \text{(Ellipse)}$$

$$r_-(J,E) = r_{\min}(ib, i\beta)\,.$$

$$\tilde{r}_- = b\exp\left[-\frac{1}{\pi}\int_b^\infty \frac{\chi(\tilde{b},E)\mathrm{d}\tilde{b}}{\sqrt{\tilde{b}^2 - b^2}}\right]\,.$$

$$r_-(b,\beta) = ib\prod_{n=1}^{\infty} e^{-\frac{(GM)^n \chi_b^{(n)}(i\beta)\Gamma\left(\frac{n}{2}\right)}{(ib)^n\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}}\,,$$

$$\tilde{r}_- = b\prod_{n=1}^{\infty} e^{-\frac{(GM)^n \chi_b^{(n)}(\beta)\Gamma\left(\frac{n}{2}\right)}{b^n\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}}\,.$$

$$r_+(b,\beta) = -ib\prod_{n=1}^{\infty} e^{-\frac{(GM)^n \chi_b^{(n)}(i\beta)\Gamma\left(\frac{n}{2}\right)}{(-ib)^n\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}} = r_-(-b,\beta)\,.$$

to Circular Orbits

Vanishing eccentricity

$$r_+ = r_-$$

$$\Leftrightarrow \prod_{n=1}^{\infty} \exp\left(\frac{1}{\sqrt{\pi}}\left(\frac{GM}{z}\right)^n \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\chi_b^{(n)}(-1+(-1)^n)\right) = -1$$

$$\Leftrightarrow \prod_{n=0}^{\infty} \exp\left(-\frac{2}{\sqrt{\pi}}\left(\frac{GM}{z}\right)^{2n+1} \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}\chi_b^{(2n+1)}\right) = -1$$

$$\Leftrightarrow -2\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{\pi}}\left(\frac{GM}{z}\right)^{2n+1} \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}\chi_b^{(2n+1)}\right) = i\pi + 2\pi i\mathbb{N}\,.$$

Example: one-loop theory

$$\operatorname{arcsinh}\left[\sqrt{\frac{G^2M^2f_1^2}{4(z^2-G^2M^2f_2)}}\right] = i\frac{\pi}{2} + i\pi\mathbb{I} \qquad j_{2\text{PM}}^2 = |\hat{p}_\infty|^2\left(\left(\frac{f_1}{2}\right)^2 - f_2\right) + \cdots\,,$$

$$x \equiv (GM\Omega_{\text{circ}})^{2/3} = \left(\frac{1}{\Gamma}\left(\frac{d\,j(\gamma)}{d\gamma}\right)^{-1}\right)^{2/3} = \epsilon + \frac{\epsilon^2}{12}(9+\nu) + \mathcal{O}(\epsilon^3)\,,$$

(first law)

Also follows from Sr=0

to Circular Orbits

Kalin Porto

Two-loops

$$j^2_{3\mathrm{PM}} = j^2_{2\mathrm{PM}} + |\hat{p}^2_\infty| \frac{f_1^2}{6}\left({}_2F_1\left(-\frac{2}{3}, -\frac{1}{3}; \frac{1}{2}; 27\mathcal{F}_3\right) - 1\right)$$
$$= j^2_{2\mathrm{PM}} + 3|\hat{p}^2_\infty| f_1^2 \sum_{m=0}^{\infty} \frac{4^{m+1}\Gamma(3m)}{(2(m+1))!\Gamma(m)} \mathcal{F}_3^{m+1}$$
$$= j^2_{2\mathrm{PM}} + |\hat{p}^2_\infty| \frac{f_3}{f_1}\left(2 + 4\mathcal{F}_3 + 32\mathcal{F}_3^2 + 384\mathcal{F}_3^3 + \cdots\right),$$

$$\mathcal{F}_n = f_n / f_1^n$$

$$\epsilon = -2\mathcal{E}$$

Orbital frequency/Binding energy to 3PM

$$\frac{x}{\epsilon} = 1 + \frac{\epsilon}{12}(9+\nu) + \frac{\epsilon^2}{2}\left(9 - \frac{17\nu}{4} + \frac{\nu^2}{9}\right) + \frac{5\epsilon^3}{48}\left(115 + 214\nu - \frac{191}{4}\nu^2 + \frac{7}{27}\nu^3\right)$$
$$+ \frac{\epsilon^4}{12}\left(1109 - \frac{11893\nu}{30} + \frac{10927\nu^2}{24} - \frac{10663\nu^3}{144} + \frac{25\nu^4}{162}\right) + \mathcal{O}(\epsilon^5).$$

$$\epsilon = x\left[1 - \frac{x}{12}(9+\nu) - \frac{x^2}{8}\left(27 - 19\nu + \frac{\nu^2}{3}\right) + \frac{x^3}{32}\left(\frac{535}{6} - \frac{5585\nu}{6} + 135\nu^2 - \frac{35\nu^3}{162}\right)\right.$$
$$\left. + \frac{x^4}{384}\left(-10171 + \frac{559993}{15}\nu - \frac{34027\nu^2}{3} + \frac{11354\nu^3}{9} + \frac{77\nu^4}{81}\right) + \mathcal{O}(x^5)\right].$$

to Circular Orbits

Power Counting

$$j^2 \sim \epsilon f_1^2 \Big(\underbrace{\underbrace{\underbrace{\underbrace{1}_{\text{1PM}} + \mathcal{O}(\mathcal{F}_2)}_{\text{2PM}} + \mathcal{O}(\mathcal{F}_3) + \mathcal{O}\left(\mathcal{F}_3^2\right) + \cdots}_{\text{3PM}} + \cdots + \mathcal{O}(\mathcal{F}_n) + \mathcal{O}\left(\mathcal{F}_n^2\right)}_{n\text{PM}} + \cdots + \cdots \Big) .$$

$$\mathcal{F}_n = f_n / f_1^n$$

$$\epsilon = -2\mathcal{E}$$

Orbital frequency/Binding energy to 3PM

$$\frac{x}{\epsilon} = 1 + \frac{\epsilon}{12}(9+\nu) + \frac{\epsilon^2}{2}\left(9 - \frac{17\nu}{4} + \frac{\nu^2}{9}\right) + \frac{5\epsilon^3}{48}\left(115 + 214\nu - \frac{191}{4}\nu^2 + \frac{7}{27}\nu^3\right) + \frac{\epsilon^4}{12}\left(1109 - \frac{11893\nu}{30} + \frac{10927\nu^2}{24} - \frac{10663\nu^3}{144} + \frac{25\nu^4}{162}\right) + \mathcal{O}(\epsilon^5) .$$

$$\epsilon = x\left[1 - \frac{x}{12}(9+\nu) - \frac{x^2}{8}\left(27 - 19\nu + \frac{\nu^2}{3}\right) + \frac{x^3}{32}\left(\frac{535}{6} - \frac{5585\nu}{6} + 135\nu^2 - \frac{35\nu^3}{162}\right) + \frac{x^4}{384}\left(-10171 + \frac{559993}{15}\nu - \frac{34027\nu^2}{3} + \frac{11354\nu^3}{9} + \frac{77\nu^4}{81}\right) + \mathcal{O}(x^5)\right] .$$

to Circular Orbits

Exact 1PM theory

$$x_{\rm 1PM} = \frac{(1-\gamma^2)}{\left(\Gamma(3\gamma - 2\gamma^3)\right)^{2/3}} \,. \qquad x_{\rm 1PM} = \epsilon\left(1 + \frac{1}{12}\left(-15+\nu\right)\epsilon + \frac{1}{72}(180+15\nu+4\nu^2)\epsilon^2 + \cdots\right) .$$

$$\epsilon = x\sum_{n=0}^{\infty}\cos\left(\frac{(n+1)\pi}{3}\right)\Gamma\left(\frac{5+2n}{6}\right)\Gamma\left(\frac{1+4n}{6}\right)\frac{(x\nu)^n}{\pi(n+1)!} + \cdots$$

Orbital frequency/Binding energy to 3PM

$$\frac{x}{\epsilon} = 1 + \frac{\epsilon}{12}(9+\nu) + \frac{\epsilon^2}{2}\left(9 - \frac{17\nu}{4} + \frac{\nu^2}{9}\right) + \frac{5\epsilon^3}{48}\left(115 + 214\nu - \frac{191}{4}\nu^2 + \frac{7}{27}\nu^3\right)$$
$$+ \frac{\epsilon^4}{12}\left(1109 - \frac{11893\nu}{30} + \frac{10927\nu^2}{24} - \frac{10663\nu^3}{144} + \frac{25\nu^4}{162}\right) + \mathcal{O}(\epsilon^5)\,.$$

$$\epsilon = x\left[1 - \frac{x}{12}(9+\nu) - \frac{x^2}{8}\left(27 - 19\nu + \frac{\nu^2}{3}\right) + \frac{x^3}{32}\left(\frac{535}{6} - \frac{5585\nu}{6} + 135\nu^2 - \frac{35\nu^3}{162}\right)\right.$$
$$\left. + \frac{x^4}{384}\left(-10171 + \frac{559993}{15}\nu - \frac{34027\nu^2}{3} + \frac{11354\nu^3}{9} + \frac{77\nu^4}{81}\right) + \mathcal{O}(x^5)\right] .$$

Kalin
Porto
(to appear)

Scattering Angle to Periastron Advanced

$$4\chi_j^{(4)} = \frac{3\pi\hat{p}_\infty^4}{4}\left(f_2^2 + 2f_1f_3 + 2f_4\right) = \frac{3\pi}{4M^4\mu^4}\left(\widetilde{\mathcal{M}}_2^2 + 2\widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_3 + 2p_\infty^2\widetilde{\mathcal{M}}_4\right)$$

$$\frac{\Delta\Phi}{2\pi} = \frac{\widetilde{\mathcal{M}}_2 G^2}{2J^2} + \frac{3(\widetilde{\mathcal{M}}_2^2 + 2\widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_3 + 2p_\infty^2\widetilde{\mathcal{M}}_4)G^4}{8J^4} + \mathcal{O}(G^6)\,,$$

This is true to all orders!

$$\Delta\Phi_j^{(2n)}(\mathcal{E}) = 4\,\chi_j^{(2n)}(\mathcal{E})$$

It can be shown through combinatorial expansions (to appear)

Kalin
Porto
(to appear)

Scattering Angle to Periastron Advanced

$$\frac{\chi+\pi}{2\pi} = -\frac{\partial S_r(J,\mathcal{E})}{\partial J} = \frac{1}{\pi}\int_{\tilde{r}_-(J,\mathcal{E})}^{\infty} \frac{J}{r^2\sqrt{\boldsymbol{p}^2(\mathcal{E},r) - J^2/r^2}}\,,$$

circle back!

$$\left(\frac{\chi(J,\mathcal{E})}{2\pi}+\frac{1}{2}\right) + \left(\frac{\chi(-J,\mathcal{E})}{2\pi}+\frac{1}{2}\right) = \frac{1}{\pi}\int_{\tilde{r}_-(J,\mathcal{E})}^{\tilde{r}_-(-J,\mathcal{E})} \frac{J}{r^2\sqrt{\boldsymbol{p}^2(\mathcal{E},r) - J^2/r^2}}\,.$$

After identifying the orbital elements:

$$\begin{aligned} 1+\frac{1}{2\pi}\left(\chi(J,\mathcal{E})+\chi(-J,\mathcal{E})\right) &= \frac{1}{\pi}\int_{r_-(J,\mathcal{E})}^{r_+(J,\mathcal{E})} \frac{J}{r^2\sqrt{\boldsymbol{p}^2(\mathcal{E},r) - J^2/r^2}} \\ &= 1+\frac{1}{2\pi}\Delta\Phi(J,\mathcal{E})\,, \end{aligned}$$

$$r_-(J,\mathcal{E}) = \tilde{r}_-(ib,\mathcal{E}<0) = r_-(J,\mathcal{E}<0)$$
$$r_+(J>0,\mathcal{E}) = r_-(-J,\mathcal{E})\,,$$

Reconstructing the Radial Action

$$i_r(j,\mathcal{E}) \equiv \frac{\mathcal{S}_r}{GM\mu} = \mathrm{sg}(\hat{p}_\infty)\chi_j^{(1)}(\mathcal{E}) - j\left(1 + \frac{2}{\pi}\sum_{n=1} \frac{\chi_j^{(2n)}(\mathcal{E})}{(1-2n)j^{2n}}\right)$$

In the PM expansion (in terms of the amplitude too)

$$i_r(j,\mathcal{E}) = \frac{\hat{p}_\infty^2}{\sqrt{-\hat{p}_\infty^2}}\frac{f_1}{2} + \frac{j}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\left(\frac{\hat{p}_\infty}{j}\right)^{2n}\Gamma\left(n-\frac{1}{2}\right)\sum_{\sigma\in\mathcal{P}(2n)}\frac{1}{\Gamma\left(1+n-\Sigma^\ell\right)}\prod_\ell \frac{f_{\sigma_\ell}^{\sigma^\ell}}{\sigma^\ell!}$$

$$= \frac{1}{2\sqrt{-p_\infty^2}}\frac{\widetilde{\mathcal{M}}_1}{M\mu} + \frac{j}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\left(\frac{\Gamma\left(n-\frac{1}{2}\right)}{(\mu M j)^{2n}}\sum_{\sigma\in\mathcal{P}(2n)}\frac{p_\infty^{2(n-\Sigma^\ell)}}{\Gamma\left(1+n-\Sigma^\ell\right)}\prod_\ell \frac{\widetilde{\mathcal{M}}_{\sigma_\ell}^{\sigma^\ell}}{\sigma^\ell!}\right),$$

Up to three-loops

$$i_r(j,\mathcal{E}) = -j + \frac{\hat{p}_\infty^2}{\sqrt{-\hat{p}_\infty^2}}\frac{f_1}{2} + \frac{\hat{p}_\infty^2}{2j}f_2 + \frac{\hat{p}_\infty^4}{8j^3}\left(f_2^2 + 2f_1f_3 + 2f_4\right) + \cdots$$

$$= -j + \frac{1}{2\sqrt{-\hat{p}_\infty^2}}\frac{\widetilde{\mathcal{M}}_1}{M\mu^2} + \frac{1}{2j}\frac{\widetilde{\mathcal{M}}_2}{M^2\mu^2} + \frac{1}{8j^3}\frac{\left(\widetilde{\mathcal{M}}_2^2 + 2\widetilde{\mathcal{M}}_1\widetilde{\mathcal{M}}_3 + 2p_\infty^2\widetilde{\mathcal{M}}_4\right)}{M^4\mu^4} + \cdots$$

Gravitational Observables

$$x = x_{1\mathrm{PM}} \frac{\left(1 + \frac{2}{\pi}\sum_{n=1} \frac{\chi_j^{(2n)}(\gamma)}{j^{2n}}\right)^{2/3}}{\left(1 - x_{1\mathrm{PM}}^{3/2} \frac{2\Gamma}{\pi} \sum_{n=1} \frac{\partial_\gamma \chi_j^{(2n)}(\gamma)}{(1-2n) j^{2n-1}}\right)^{2/3}}, \qquad x_{1\mathrm{PM}} = \frac{(1-\gamma^2)}{(\Gamma(3\gamma - 2\gamma^3))^{2/3}}.$$

Radial Period

$$\frac{T_p}{2\pi} = GM \frac{\partial}{\partial \mathcal{E}} i_r(j, \mathcal{E}) = GM \left(\partial_{\mathcal{E}} \left(\mathrm{sg}(\hat{p}_\infty)\chi_j^{(1)}(\mathcal{E})\right) - \frac{2}{\pi}\sum_{n=1} \frac{\partial_{\mathcal{E}} \chi_j^{(2n)}(\mathcal{E})}{(1-2n) j^{2n-1}}\right)$$
$$= GE \left(\partial_\gamma \left(\mathrm{sg}(\hat{p}_\infty)\chi_j^{(1)}(\gamma)\right) - \frac{2}{\pi}\sum_{n=1} \frac{\partial_\gamma \chi_j^{(2n)}(\gamma)}{(1-2n) j^{2n-1}}\right),$$

We can also extract the redshift from first-law

$$\delta \mathcal{S}_r(J, \mathcal{E}, m_a) = -\left(1 + \frac{\Delta\Phi}{2\pi}\right)\delta J + \frac{\mu}{\Omega_r}\delta\mathcal{E} - \sum_a \frac{1}{\Omega_r}\left(\langle z_a\rangle - \frac{\partial E(\mathcal{E}, m_a)}{\partial m_a}\right)\delta m_a .$$

Gravitational Observables to 2-loops

$$\frac{GM\Omega_r^{(L=2)}}{\epsilon^{\frac{3}{2}}} = 1 - \frac{(15-\nu)}{8}\epsilon + \frac{555+30\nu+11\nu^2}{128}\epsilon^2 + \left(\frac{3(2\nu-5)}{2j} - \frac{194-184\nu+23\nu^2}{4j^3}\right)\epsilon^{\frac{3}{2}} + \left(\frac{15(17-9\nu+2\nu^2)}{8j} + \frac{21620-28592\nu+8765\nu^2-865\nu^3}{80j^3}\right)\epsilon^{\frac{5}{2}} + \cdots$$

$$\frac{GM\Omega_\phi^{(L=2)}}{\epsilon^{\frac{3}{2}}} = 1 + \frac{3}{j^2} - \frac{15(2\nu-7)}{4j^4} + \left(\frac{1}{8}(\nu-15) + \frac{15(\nu-5)}{8j^2} - \frac{3(1301-921\nu+102\nu^2)}{32j^4}\right)\epsilon + \left(\frac{3(2\nu-5)}{2j} + \frac{-284+220\nu-23\nu^2}{4j^3} + \frac{3(913-728\nu+106\nu^2)}{j^5}\right)\epsilon^{\frac{3}{2}} + \left(\frac{1}{128}(555+30\nu+11\nu^2) + \frac{3(895-150\nu+51\nu^2)}{128j^2} - \frac{3(-270085+251236\nu-70545\nu^2+7470\nu^3)}{2560j^4}\right)\epsilon^2 + \left(\frac{15(17-9\nu+2\nu^2)}{8j} + \frac{31520-34442\nu+10025\nu^2-865\nu^3}{80j^3}\right)\epsilon^{\frac{5}{2}} .$$

3PN match

3PN missmatch

Higher orders velocity

$$\langle z_2^{(L=2)}\rangle = 1 + \frac{1}{4}(2\nu-3\Delta-3)\epsilon + \left(-\frac{3(1+\Delta)}{j} + \frac{5((5\nu-14)(1+\Delta)-4\nu^2)}{4j^3}\right)\epsilon^{\frac{3}{2}} + \frac{1}{16}(3(10-\nu)(1+\Delta)+4\nu^2)\epsilon^2 + \left(-\frac{3(11\nu-35)(\Delta+1)-8\nu^2)}{8j} + \frac{(3378-3021\nu)(\Delta+1)+2165\nu^2+393\Delta\nu^2-388\nu^3}{32j^3}\right)\epsilon^{\frac{5}{2}}$$
$$\left(\frac{1}{32}(-(3\nu^2+130)(\Delta+1)+4\nu^3) - \frac{9(1+\Delta)(2\nu-5)}{2j^2} + \frac{3((738-633\nu)(\Delta+1)+196\nu^2+96\Delta\nu^2-4\nu^3)}{8j^4}\right)\epsilon^3 ,$$

Recall 1PM and 2PM carry over exact terms! (the same will happen once we know 4PM)

Kalin
Porto
(to appear)

Scattering Angle to Periastron Advanced (with a twist)

$$\frac{\chi(J,\mathcal{E})+\chi(-J,\mathcal{E})}{2\pi}=\frac{\Delta\Phi(J,\mathcal{E})}{2\pi},$$

with J the total (canonical) angular momentum

$$\frac{\chi(\ell,a,\epsilon)}{2\pi}=$$

$$\left[\frac{1}{\pi}(-\epsilon)^{-\frac{1}{2}}-\frac{(\nu-15)}{8\pi}(-\epsilon)^{\frac{1}{2}}+\frac{35+30\nu+3\nu^2}{128\pi}(-\epsilon)^{\frac{3}{2}}\right]\frac{1}{\ell}$$
$$+\left[3+\frac{3(2\nu-5)}{4}\epsilon+\frac{3(5-5\nu+4\nu^2)}{16}\epsilon^2-\frac{7\tilde{a}_++\Delta\tilde{a}_-}{2\pi}\epsilon^{-\frac{1}{2}}+\frac{5\Delta(\nu-3)\tilde{a}_-+(23\nu-25)\tilde{a}_+}{16\pi}(-\epsilon)^{\frac{3}{2}}\right]\frac{1}{2\ell^2}$$
$$+\left[-\frac{7\tilde{a}_++\Delta\tilde{a}_-}{2}-\frac{(\nu-6)\Delta\tilde{a}_-+(7\nu-18)\tilde{a}_+}{2}\epsilon-\frac{3\left((15-14\nu+2\nu^2)\Delta\tilde{a}_-+(25-38\nu+14\nu^2)\tilde{a}_+\right)}{16}\epsilon^2\right.$$
$$\left.-\frac{2}{3\pi}(-\epsilon)^{-\frac{3}{2}}+\frac{33+\nu}{4\pi}(-\epsilon)^{-\frac{1}{2}}+\frac{3003-1090\nu-5\nu^2+128\tilde{a}_+^2}{64\pi}(-\epsilon)^{\frac{1}{2}}\right]\frac{1}{2\ell^3}$$
$$+\left[\frac{3(35+2\tilde{a}_+^2-10\nu)}{4}+\frac{10080-13952\nu+123\pi^2\nu+1440\nu^2}{128}\epsilon+\frac{624\Delta\tilde{a}_-\tilde{a}_++24(1-8\nu)\tilde{a}_-^2-24(12\nu-61)\tilde{a}_+^2}{128}\epsilon+\cdots\right]\frac{1}{2\ell^4}+\cdots.$$

Vines Steinhoff Buonanno to 3.5PN

1812.00956

$$\tilde{a}_\pm=\frac{a_1+a_2}{GM}$$

$$\frac{\Delta\Phi(\ell,a,\epsilon)}{2\pi}=$$

$$\left[3+\frac{3(2\nu-5)}{4}\epsilon+\frac{3(5-5\nu+4\nu^2)}{16}\epsilon^2\right]\frac{1}{\ell^2}$$
$$+\left[-\frac{7\tilde{a}_++\Delta\tilde{a}_-}{2}-\frac{(\nu-6)\Delta\tilde{a}_-+(7\nu-18)\tilde{a}_+}{2}\epsilon-\frac{3\left((15-14\nu+2\nu^2)\Delta\tilde{a}_-+(25-38\nu+14\nu^2)\tilde{a}_+\right)}{16}\epsilon^2\right]\frac{1}{\ell^3}$$
$$+\left[\frac{3(35+2\tilde{a}_+^2-10\nu)}{4}+\frac{10080-13952\nu+123\pi^2\nu+1440\nu^2}{128}\epsilon+\frac{624\Delta\tilde{a}_-\tilde{a}_++24(1-8\nu)\tilde{a}_-^2-24(12\nu-61)\tilde{a}_+^2}{128}\epsilon+\cdots\right]\frac{1}{\ell^4}+\cdots$$

Agrees with Tessmer Hartung Schäfer to 3.5PN

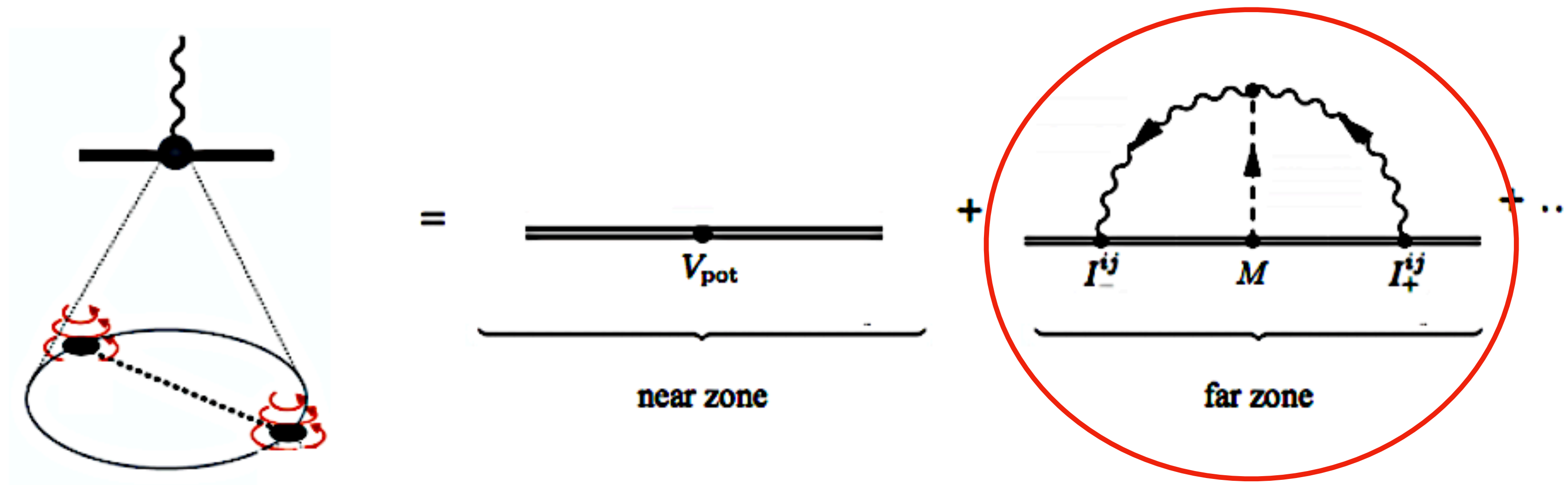
1207.6961

$$\ell=L/(GM\mu)$$

Radiation-Reaction

$$\mathbf{p}^2_{\rm sc}(r,E) = I_{(1)}(r,E) + I_{(2)}(r,E)\,,$$

In the EFT approach



$$\frac{41}{30} \sim \frac{5}{6}$$

+ 'Bethe Log'

$$\frac{2G_N^2 M}{5} I^{(3)ij} I^{(3)ij} \left(-\frac{1}{\epsilon_{\rm IR}} + 2\log(\mu r) + \cdots\right) + \left(\frac{1}{\epsilon_{\rm UV}} + 2\log(\Omega/\mu) + \cdots\right)$$

$$\mu \frac{d}{d\mu} V_{\rm ren}(\mu) = \frac{2G_N^2 M}{5} I^{ij(3)}(t) I^{ij(3)}(t)$$

$$E_{\log} = -2G_N^2 M \langle I^{ij(3)}(t) I^{ij(3)}(t)\rangle \log v$$

PHYSICAL REVIEW D **93,** 124010 (2016)

Tail effect in gravitational radiation reaction: Time nonlocality and renormalization group evolution

Chad R. Galley,[1] Adam K. Leibovich,[2] Rafael A. Porto,[3] and Andreas Ross[4]

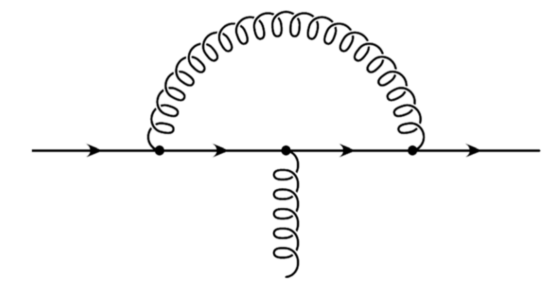
Radiation-Reaction

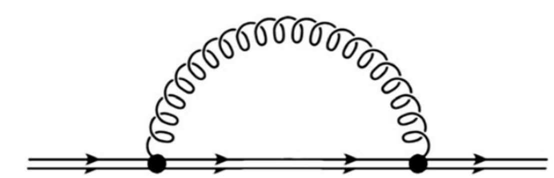
PHYSICAL REVIEW D **96**, 024063 (2017)

Lamb shift and the gravitational binding energy for binary black holes

Rafael A. Porto

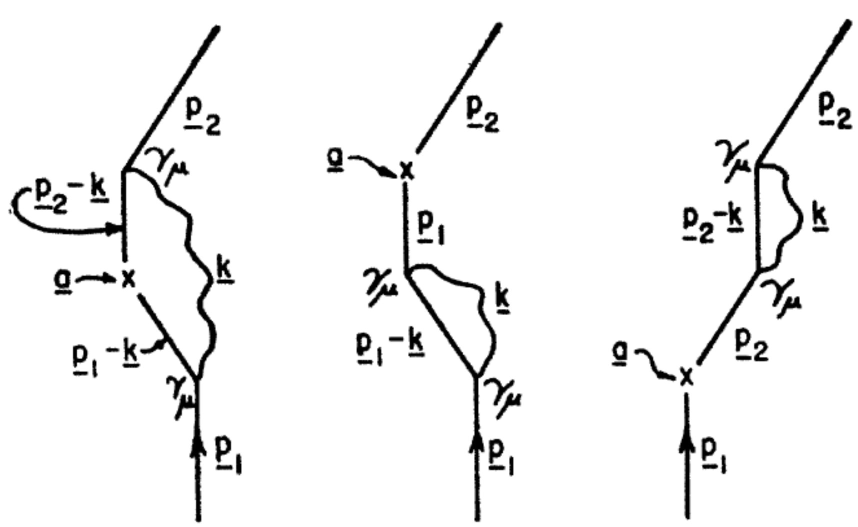
Computation in NRQED





$$
\begin{aligned}
\delta E_{n,\ell} &= (\delta E_{n,\ell})_{US} + (\delta E_{n,\ell})_{cv} + \cdots \\
&= \frac{2\alpha_e}{3\pi}\left[\frac{5}{6}e^2 \frac{|\psi_{n,\ell}(\boldsymbol{x}=0)|^2}{2m_e^2} - \sum_{m\neq n,\ell}\left\langle n,\ell\left|\frac{\boldsymbol{p}}{m_e}\right|m,\ell\right\rangle^2 (E_m - E_n)\log\frac{2|E_n - E_m|}{m_e}\right] + \\
&\quad + \frac{4\alpha_e^2}{3m_e^2}\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right)|\psi_{n,\ell}(\boldsymbol{x}=0)|^2 .
\end{aligned}
$$

$$(p^0_{US}, \boldsymbol{p}_{US}) \sim (m_e v^2, m_e v^2)$$



a. Eq.12 b. Eq.13 c. Eq.14

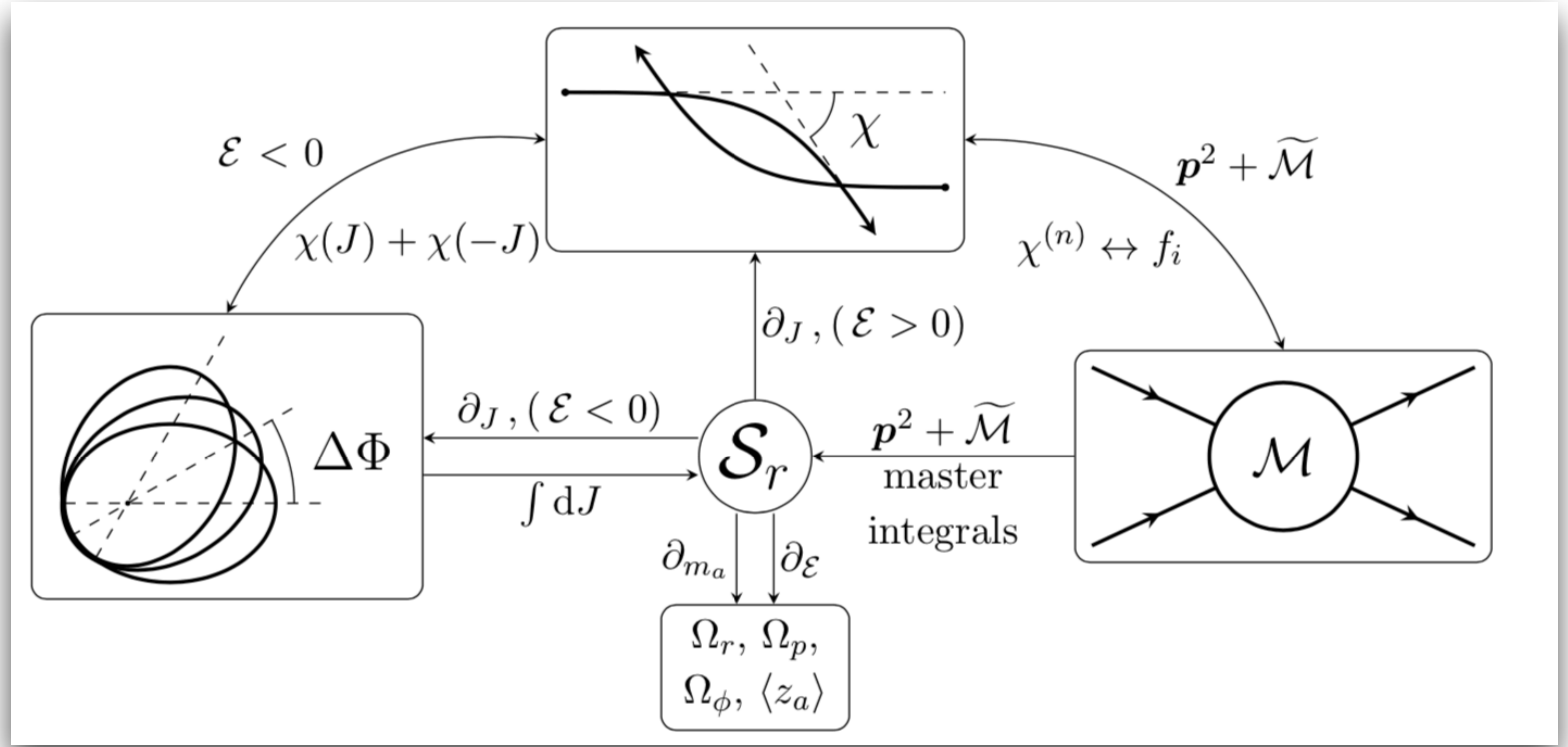
Space-Time Approach to Quantum Electrodynamics

R. P. Feynman
Department of Physics, Cornell University, Ithaca, New York
(Received May 9, 1949)

Lamb shift as interpreted in more detail in B.[13]

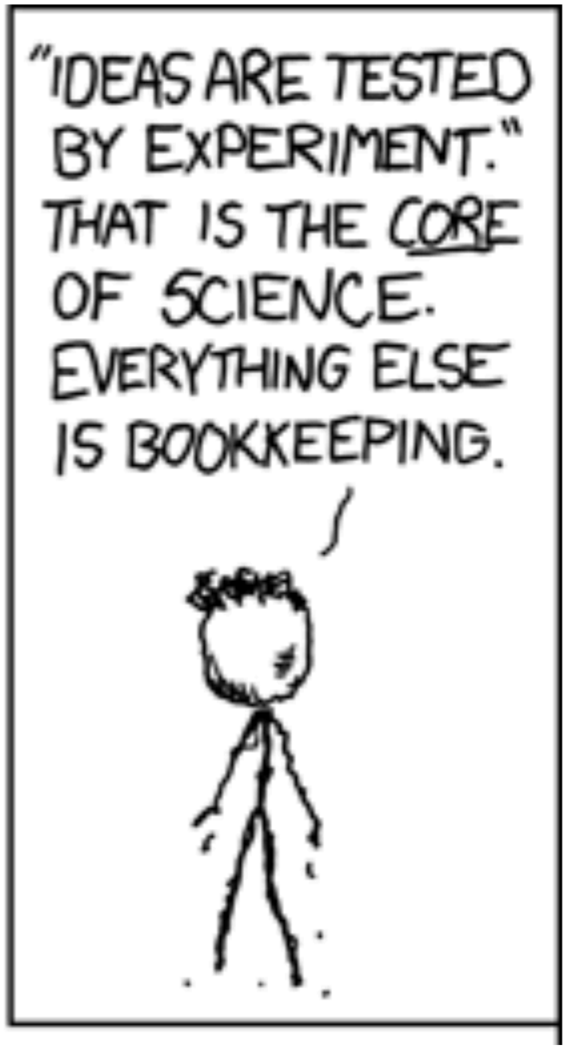
[13] That the result given in B in Eq. (19) was in error was repeatedly pointed out to the author, in private communication, by V. F. Weisskopf and J. B. French, as their calculation, completed simultaneously with the author's early in 1948, gave a different result. French has finally shown that although the expression for the radiationless scattering B, Eq. (18) or (24) above is correct, it was incorrectly joined onto Bethe's non-relativistic result. He shows that the relation $\ln 2k_{\max} - 1 = \ln\lambda_{\min}$ used by the author should have been $\ln 2k_{\max} - 5/6 = \ln\lambda_{\min}$. This results in adding a term $-(1/6)$ to the logarithm in B, Eq. (19) so that the result now agrees with that of J. B. French and V. F. Weisskopf,

Conclusions



Boundary-to-Bound (B2B) dictionary mapping scattering angle/amplitudes to dynamical invariants for bound orbits — to all PM orders

Extra Slides



"New directions in science are launched by new tools much more often than by new concepts. The effect of a concept-driven revolution is to explain old things in a new way. The effect of a tool-driven revolution is to discover new things that have to be explained"

Freeman Dyson, "Imagined Worlds"

Kalin
Porto

Impetus Formula

$$H_{\rm eff}|\psi_{\boldsymbol p}(p_\infty)\rangle = \left(\boldsymbol p^2 + V_{\rm eff}\right)|\psi_{\boldsymbol p}(p_\infty)\rangle = p_\infty^2(E)|\psi_{\boldsymbol p}(p_\infty)\rangle\,,$$

$$V_{\rm eff} = -\sum_i P_i(E)\frac{G^i}{r^i}\,.$$

Born Approximation:

$$\langle \boldsymbol p+\boldsymbol q|V_{\rm eff}|\boldsymbol p\rangle + \cdots = \frac{1}{\rm Vol}\sum_i \int {\rm d}^3\boldsymbol r\left(P_i(E)\frac{G^i}{r^i}\right)e^{i\boldsymbol q\cdot\boldsymbol r} + \cdots\,.$$

$$V_{\rm eff}(\boldsymbol k,\boldsymbol k',E) = -\sum_{n=1}^{\infty}P_n(E)\frac{G^n}{|\boldsymbol k'-\boldsymbol k|^{d-n}}\frac{(4\pi)^{d/2}\Gamma[d-n/2]}{2^n\Gamma[n/2]}\,. \qquad G_0(\boldsymbol k,E) = \frac{1}{H_0-E+i\epsilon} = \frac{1}{\boldsymbol k^2-E+i\epsilon}\,.$$

Iterations are purely super-classical IR-divergent:

$$\int {\rm d}^d\boldsymbol l\,\frac{f^{(\alpha\beta\gamma)}(\boldsymbol l,\boldsymbol p,\boldsymbol q)}{|\boldsymbol l|^\alpha|\boldsymbol l+\boldsymbol q|^\beta(2\boldsymbol l\cdot\boldsymbol p+\boldsymbol l^2)^\gamma}\,, \qquad \gamma = 1$$

No Recoil Approximation

$$2M\widetilde{\mathcal{M}}_{\text{no-rec}}(r,\mathcal{E}_0) \to 2M\left(\mathbf{p}^2_{\text{Sch}}(r,\mathcal{E}_0) - \mu^2(\mathcal{E}_0^2 - 1)\right) \; ,$$

$$\hat{\mathbf{p}}^2_{\text{Sch}} = \frac{\left(1+\frac{GM}{2r}\right)^6}{\left(1-\frac{GM}{2r}\right)^2}\mathcal{E}_0^2 - \left(1+\frac{GM}{2r}\right)^4 , \qquad \mathcal{E}_0 = \frac{Mp_1^0}{\mu M} \to \frac{p_1\cdot p_2}{m_1 m_2} = \gamma \, .$$

After boosting to the center of mass

$$\widetilde{\mathcal{M}}_{\text{no-rec}}(r,E) = \frac{1}{2E}\left(2M\widetilde{\mathcal{M}}_{\text{no-rec}}(r,\mathcal{E}_0\to\gamma)\right) \; ,$$

No Recoil Approximation

$$2M\widetilde{\mathcal{M}}_{\text{no-rec}}(r,\mathcal{E}_0) \to 2M\left(\mathbf{p}^2_{\text{Sch}}(r,\mathcal{E}_0) - \mu^2(\mathcal{E}_0^2-1)\right) ,$$

$$\hat{\mathbf{p}}^2_{\text{Sch}} = \frac{\left(1+\frac{GM}{2r}\right)^6}{\left(1-\frac{GM}{2r}\right)^2}\mathcal{E}_0^2 - \left(1+\frac{GM}{2r}\right)^4 , \qquad \mathcal{E}_0 = \frac{Mp_1^0}{\mu M} \to \frac{p_1\cdot p_2}{m_1 m_2} = \gamma\,.$$

After boosting to the center of mass

$$\widetilde{\mathcal{M}}_{\text{no-rec}}(r,E) = \frac{1}{2E}\left(2M\widetilde{\mathcal{M}}_{\text{no-rec}}(r,\mathcal{E}_0\to\gamma)\right) ,$$

$$\mathbf{p}^2_{\text{no-rec}} = p_\infty^2 + \widetilde{\mathcal{M}}_{\text{no-rec}}(r,E) = p_\infty^2 + \frac{1}{\Gamma}\Delta\mathbf{p}^2_{\text{Sch}}(r,\mathcal{E}_0\to\gamma)\,,$$

$$f_1 = 2\chi_b^{(1)} = 2\Gamma\frac{2\gamma^2-1}{\gamma^2-1}, \quad f_2 = \frac{4}{\pi}\chi_b^{(2)} = \frac{3}{2}\Gamma\frac{5\gamma^2-1}{\gamma^2-1},$$

$$\frac{1}{\Gamma}f_1^{\text{no-rec}}(E) = 2\frac{2\gamma^2-1}{\gamma^2-1},$$

$$\frac{1}{\Gamma}f_2^{\text{no-rec}}(E) = \frac{3}{2}\frac{5\gamma^2-1}{\gamma^2-1},$$

$$\frac{1}{\Gamma}f_3^{\text{no-rec}}(E) = \frac{1}{2}\frac{18\gamma^2-1}{\gamma^2-1},$$

Exact 2-body dynamics at 2PM

No Recoil Approximation

$$2M\widetilde{\mathcal{M}}_{\text{no-rec}}(r,\mathcal{E}_0) \to 2M\left(\mathbf{p}^2_{\text{Sch}}(r,\mathcal{E}_0) - \mu^2(\mathcal{E}_0^2-1)\right) ,$$

$$\hat{\mathbf{p}}^2_{\text{Sch}} = \frac{\left(1+\frac{GM}{2r}\right)^6}{\left(1-\frac{GM}{2r}\right)^2}\mathcal{E}_0^2 - \left(1+\frac{GM}{2r}\right)^4 , \qquad \mathcal{E}_0 = \frac{Mp_1^0}{\mu M} \to \frac{p_1\cdot p_2}{m_1 m_2} = \gamma .$$

After boosting to the center of mass

$$\widetilde{\mathcal{M}}_{\text{no-rec}}(r,E) = \frac{1}{2E}\left(2M\widetilde{\mathcal{M}}_{\text{no-rec}}(r,\mathcal{E}_0\to\gamma)\right) ,$$

$$\mathbf{p}^2_{\text{no-rec}} = p^2_\infty + \widetilde{\mathcal{M}}_{\text{no-rec}}(r,E) = p^2_\infty + \frac{1}{\Gamma}\Delta\mathbf{p}^2_{\text{Sch}}(r,\mathcal{E}_0\to\gamma) ,$$

$$f_3(\gamma) = \frac{r^3}{2Ep^2_\infty M^3}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\mathcal{M}_3(\mathbf{q},\mathbf{p}^2=p^2_\infty(E))e^{-i\mathbf{q}\cdot\mathbf{r}} \tag{5.64}$$

$$= -\frac{\Gamma}{6(\gamma^2-1)}\left(3-54\gamma^2+\nu\left(-6+206\gamma+108\gamma^2+4\gamma^3-\frac{18\Gamma(1-2\gamma^2)(1-5\gamma^2)}{(1+\Gamma)(1+\gamma)}\right)\right.$$

$$\left.-48\nu(3+12\gamma^2-4\gamma^4)\frac{\operatorname{arcsinh}\sqrt{\frac{\gamma-1}{2}}}{\sqrt{\gamma^2-1}}\right),$$

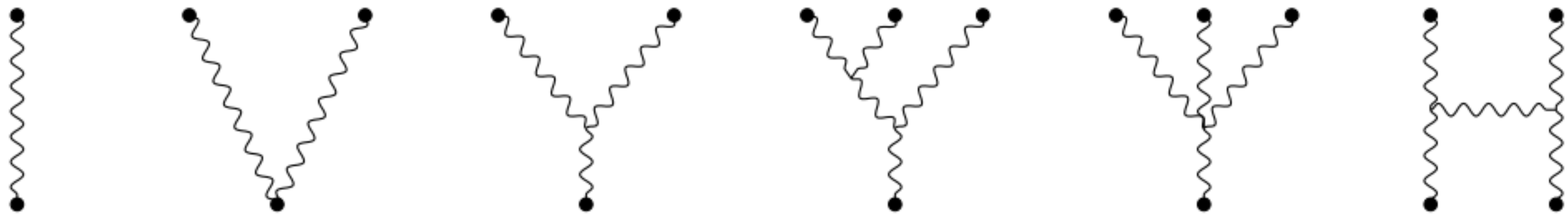
$$\frac{1}{\Gamma}f_1^{\text{no-rec}}(E) = 2\frac{2\gamma^2-1}{\gamma^2-1} ,$$

$$\frac{1}{\Gamma}f_2^{\text{no-rec}}(E) = \frac{3}{2}\frac{5\gamma^2-1}{\gamma^2-1} ,$$

$$\frac{1}{\Gamma}f_3^{\text{no-rec}}(E) = \frac{1}{2}\frac{18\gamma^2-1}{\gamma^2-1} ,$$

No Recoil Approximation

Also in the EFT approach



(a) (b) (c) (d) (e) (f)

$$2p_\infty(\gamma)\sin\frac{\chi_{1\mathrm{pt}}(\gamma)}{2} = 2p_\infty^{\mathrm{test}}(\mathcal{E}_0\to\gamma)\sin\frac{\chi_{\mathrm{test}}(\mathcal{E}_0\to\gamma)}{2}\,,$$

$$2p_\infty^{\mathrm{test}}(\mathcal{E}_0\to\gamma)/(2p_\infty(\gamma)) = \Gamma\,.$$

$$\chi_{1\mathrm{pt}}(\gamma) + \mathcal{O}(\chi_{1\mathrm{pt}}^3) = \Gamma\,\chi_{\mathrm{test}}(\mathcal{E}_0\to\gamma) + \mathcal{O}(\chi_{\mathrm{test}}^3)\,,$$

$$\frac{1}{\Gamma} f_1^{\text{no-rec}}(E) = 2\frac{2\gamma^2-1}{\gamma^2-1}\,,$$

$$\frac{1}{\Gamma} f_2^{\text{no-rec}}(E) = \frac{3}{2}\frac{5\gamma^2-1}{\gamma^2-1}\,,$$

$$\frac{1}{\Gamma} f_3^{\text{no-rec}}(E) = \frac{1}{2}\frac{18\gamma^2-1}{\gamma^2-1}\,,$$

Kalin
Porto
(to appear)

Black Holes as Elementary Particles

Impetus for spin^2k with canonical position/momentum

$$\boldsymbol{P}^2(R, \mathcal{E}, a_1, a_2) = p_\infty^2 + \widetilde{\mathcal{M}}^{S^{2k}}(\boldsymbol{R}, p_\infty, a_1, a_2)\,.$$

One-loop amplitude for one spinning body $(m_1 m_2 = \mu M)$

$$\mathcal{M}^S_{\rm 1PM}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{a}) = 8\pi \frac{G\mu^2 M^2}{\boldsymbol{q}^2}\gamma^2 \sum_{\pm}(1 \pm v)^2 e^{\pm i\boldsymbol{q}\cdot\boldsymbol{a}}\,.$$

At 1PM order (to all orders in velocity)

$$\widetilde{\mathcal{M}}^{S^{2k}}_{\rm 1PM}(\boldsymbol{r}, \mathcal{E}, \boldsymbol{a}) = \frac{2\mu^2(2\gamma^2 - 1)}{\Gamma}\sum_{\ell}^{\rm even}\frac{1}{\ell!}\left((i\boldsymbol{a})\cdot\boldsymbol{\nabla}\right)^{\ell}\frac{GM}{R}$$
$$= \frac{2\mu^2(2\gamma^2 - 1)}{\Gamma}\cos\left(\boldsymbol{a}\cdot\boldsymbol{\nabla}\right)\frac{GM}{R} = \frac{2\mu^2(2\gamma^2 - 1)}{\Gamma}\frac{GMr}{r^2 + a^2\cos^2\theta}\,.$$

Vines
Steinhoff

Non-Relativistic Limit

$$\mathcal{E} = \frac{\boldsymbol{P}^2}{2\mu} + \mu\phi + \cdots\,, \qquad \textbf{Kerr!} \qquad \phi = -\frac{GMr}{r^2 + a^2\cos^2\theta}\,.$$