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# Canonical Formalisms in General Relativity

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From Classical Gravity to Quantum Amplitudes and Back:  
post-Newtonian, post-Minkowskian, effective one-body problem, self-force, ...

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## Global View

- 1958 - 1959 (1962 tetrads, spin-1/2 field): DIRAC  
1978 Nelson/Teitelboim: spin-1/2 field
- 1959 - 1961: ARNOWITT/DESER/MISNER (ADM)  
1976 Deser/Isham: tetrads, spin-1/2 field
- 1963: SCHWINGER (tetrads, spin-0 & spin-1 fields)  
1963 Kibble: spin-1/2 field
- 1967: DeWitt energy surface-term from Hamiltonian
- 1974: Regge/Teitelboim Hamiltonian and surface terms

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# Canonical Field based Applications to PN/PM

- 1961: 1PN Kimura
- 1974: 2PN Ohta/Okamura/Kimura/Hiida
- 1985: 2.5 PN Schäfer
- 1985: 2PN Damour/S [EOB Buonanno/Damour 1999]
- 1997: 3.5PN Jaranowski/S
- 1998: 3PN Jaranowski/S (Routhian)
- 2001: 3PN Damour/Jaranowski/S (dimreg)
- 2008: 1PM Ledvinka/S/Bičák [PoMiN Matzner et col. 2018]
- 2008 - 2013: HO PN Spin dynamics Steinhoff, Herkt, Hartung, S
- 2009: Linear-in-Spin dynamics Steinhoff/S
- 2010 - 2011: LO dissipative Spin dynamics Steinhoff, Wang, Zeng, S
- 2014: 4PN Damour/Jaranowski/S [EOB@4PN DJS 2015]

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## The Dynamics of General Relativity

second order action

$$W = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(g, \partial g, \partial \partial g)$$

first order action

$$W = \int d^4x \sqrt{-g} g^{\mu\nu} R^\lambda_{\mu\lambda\nu}(\Gamma, \partial\Gamma) = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma, \partial\Gamma)$$

first order tetrad action

$$W = \int d^4x \det(e_\mu^a) e^{\mu a} e^{\nu b} R_{\mu\nu ab}(\omega, \partial\omega), \quad e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu}$$

$$e_\nu^a e^{\lambda b} R_{\mu\lambda ab} = R_{\mu\nu}, \quad \text{torsion-freeness: } R_{\mu\nu} = R_{\nu\mu} \quad (\delta W/\delta\omega = 0)$$

$$(16\pi G = 1, c = 1)$$

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(3+1)-slicing of spacetime, introduction of lapse  $N$  and shift  $N^i$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(Ndx^0)^2 + \gamma_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0)$$

$$Ndx^0 = -n_\mu dx^\mu, \quad n_\mu = -(N, 0, 0, 0)$$

$$N = (-g^{00})^{-1/2}, \quad N^i = \gamma^{ij} N_j, \quad N_i = g_{0i}, \quad \gamma_{ij} = g_{ij}$$

$$\pi^{ij} = \sqrt{-g}(\Gamma^0{}_{pq} - \gamma_{pq}\Gamma^0{}_{rs}\gamma^{rs})\gamma^{ip}\gamma^{jq}, \quad \pi_{ij} = \gamma_{ip}\gamma_{jq}\pi^{pq}, \quad \pi = \pi^{ij}\gamma_{ij}$$

$$\sqrt{-g} = N\sqrt{\gamma}, \quad \pi^{ij} = -\sqrt{\gamma}(K^{ij} - \gamma^{ij}K), \quad K_{ij} = -N\Gamma^0{}_{ij}$$

$$N = 1 + O(1/r), \quad N^i = O(1/r), \quad \gamma_{ij} = \delta_{ij} + O(1/r)$$

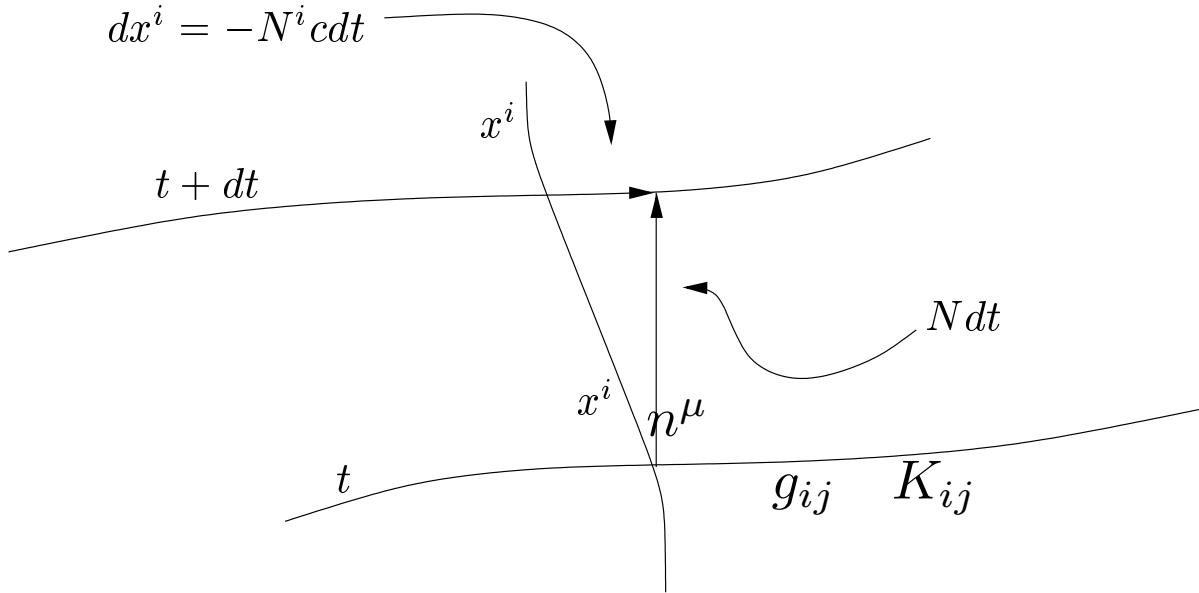
$$\partial_k \gamma_{ij} = O(1/r^2), \quad \pi^{ij} = O(1/r^2)$$

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3+1 splitting of spacetime

$$n^\mu = (1, -N^i)/N$$

$$n_\mu = (-N, 0, 0, 0)$$



$$K_{ij} = -N\Gamma_{ij}^0 = -Ng^{0\mu}(g_{i\mu,j} + g_{j\mu,i} - g_{ij,\mu})/2$$

$$ds^2 = -(\textcolor{blue}{N} dt)^2 + \textcolor{blue}{g}_{ij}(dx^i + \textcolor{blue}{N}^i dt)(dx^j + \textcolor{blue}{N}^j dt)$$

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Einstein field equations in canonical form

$$\partial_t \gamma_{ij} = 2N\gamma^{-1/2} \left( \pi_{ij} - \frac{1}{2}\pi\gamma_{ij} \right) + \nabla_i N_j + \nabla_j N_i$$

$$\begin{aligned} \partial_t \pi^{ij} &= -N \gamma^{1/2} \left( R^{ij} - \frac{1}{2}\gamma^{ij}R \right) + \frac{1}{2}N\gamma^{-1/2}\gamma^{ij} \left( \pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2 \right) \\ &- 2N\gamma^{-1/2} \left( \pi^{im}\pi_m^j - \frac{1}{2}\pi\pi^{ij} \right) + \nabla_m(\pi^{ij}N^m) - (\nabla_m N^i)\pi^{mj} - (\nabla_m N^j)\pi^{mi} \\ &+ \sqrt{\gamma}(\nabla^i \nabla^j N - \gamma^{ij}\nabla^m \nabla_m N) + \frac{1}{2} \frac{\gamma^{ik}\gamma^{jl}p_k p_l}{(\gamma^{ij}p_i p_j + m^2 c^2)^{1/2}} N \delta \end{aligned}$$

$$\mathcal{H}(\partial_q \partial_p \gamma_{ij}, \partial_p \gamma_{ij}, \gamma_{ij}, \pi^{ij}, z^i, p_i) = 0 \quad (\text{Hamiltonian Constraint})$$

$$\mathcal{H}_k(\partial_p \gamma_{ij}, \gamma_{ij}, \partial_p \pi^{ij}, \pi^{ij}, z^i, p_i) = 0 \quad (\text{Momentum Constraints})$$

$$W=\int d^4x\left[\pi^{ij}\partial_t\gamma_{ij}-\textcolor{red}{N}(\mathcal{H}^{\mathrm{field}}+\mathcal{H}^{\mathrm{matter}})+\textcolor{red}{N}^{\textcolor{red}{i}}(\mathcal{H}_i^{\mathrm{field}}+\mathcal{H}_i^{\mathrm{matter}})\right]$$

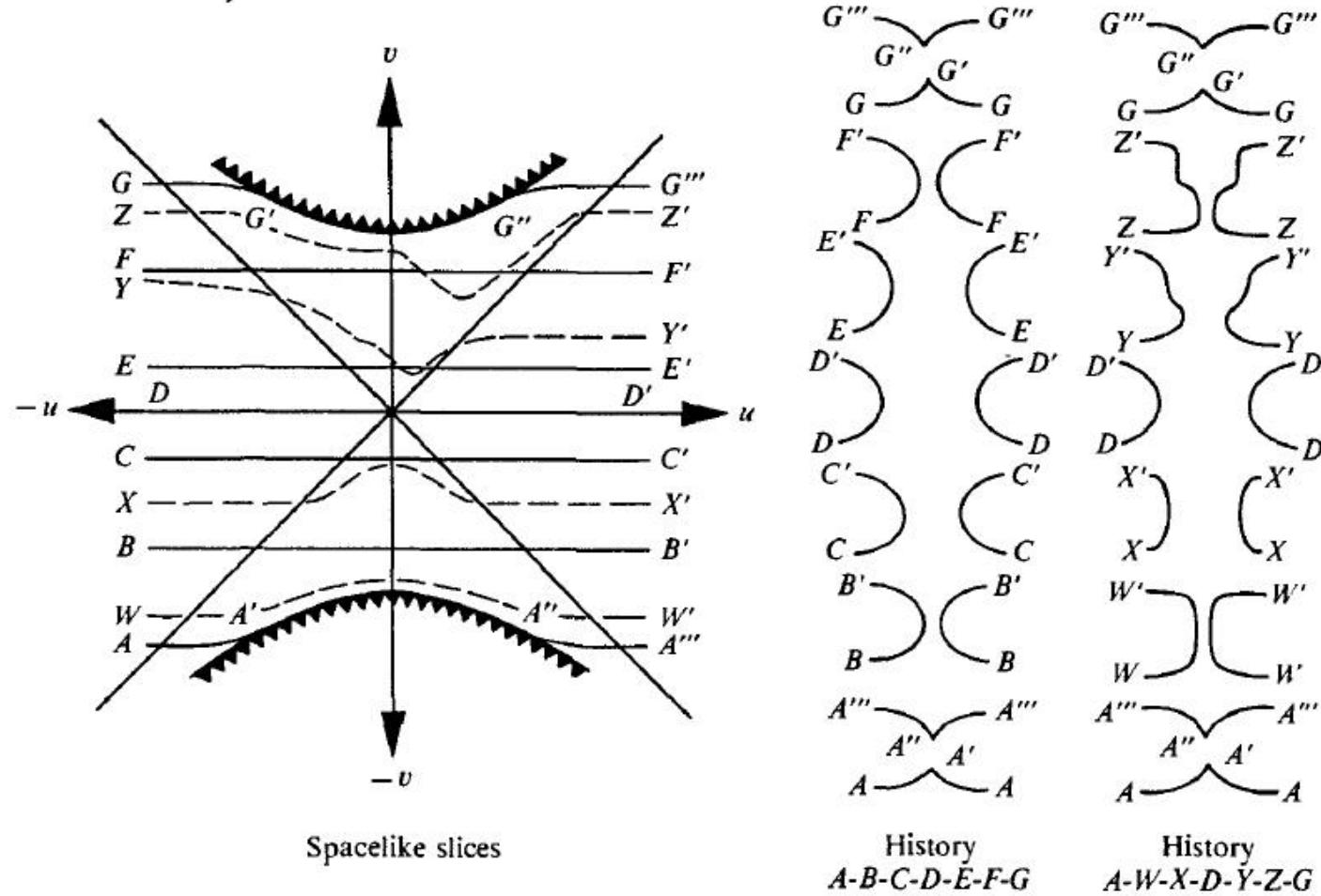
$$H=\int d^3x\left[\textcolor{red}{N}(\mathcal{H}^{\mathrm{field}}+\mathcal{H}^{\mathrm{matter}})-\textcolor{red}{N}^{\textcolor{red}{i}}(\mathcal{H}_i^{\mathrm{field}}+\mathcal{H}_i^{\mathrm{matter}})\right]$$

$$\mathcal{H}^{\mathrm{field}}=-\gamma^{1/2}R+\gamma^{-1/2}\left(\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^2\right)\,,$$

$$\mathcal{H}_i^{\mathrm{field}}=2\gamma_{ij}\nabla_k\pi^{jk}\,,$$

$$\mathcal{H}^{\mathrm{matter}}=\sqrt{m^2+\gamma^{ij}p_ip_j}\,\,\delta(x^k-z^k),$$

$$\mathcal{H}_i^{\mathrm{matter}}=p_i\,\,\delta(x^k-z^k)$$



MTW: Gravitation (1973)

$$H=\int d^3x\left(\textcolor{red}{N}\mathcal{H}-\textcolor{red}{N}^i\mathcal{H}_i\right)$$

$$\mathcal{H} = \sqrt{\gamma}(T^{\mu\nu}-2G^{\mu\nu})n_\mu n_\nu = \sqrt{\gamma}N^2(T^{00}-2G^{00})$$

$$\mathcal{H}_i=-\sqrt{\gamma}(T_i^\mu-2G_i^\mu)n_\mu=\sqrt{\gamma}N(T_i^0-2G_i^0)$$

$$\partial_t \gamma_{ij} = \frac{\delta H}{\delta \pi^{ij}}, \qquad \partial_t \pi^{ij} = - \frac{\delta H}{\delta \gamma_{ij}}$$

$$0=\frac{\delta H}{\delta N},\qquad 0=\frac{\delta H}{\delta N^i}$$

$$10\,$$

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$$\mathcal{H}_D^{\text{field}} = \partial_i(\gamma^{-1/2} \partial_j(\gamma \gamma^{ij})) + B + \frac{1}{\gamma^{1/2}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right)$$

$$B = \frac{1}{4} \gamma^{1/2} \partial_k \gamma_{ij} \partial_n \gamma_{lm} [(\gamma^{il} \gamma^{jm} - \gamma^{ij} \gamma^{lm}) \gamma^{kn} + 2(\gamma^{ik} \gamma^{lm} - \gamma^{il} \gamma^{mk}) \gamma^{jn}]$$

$$(\gamma^{1/2} \mathcal{H}^{\text{field}})_S = \partial_i \partial_j q^{ij} + Q + (q^{ik} q^{jl} - q^{ij} q^{kl}) \Pi_{ij} \Pi_{kl}$$

$$Q = -\frac{1}{4} q^{mn} \partial_m q^{kl} \partial_n q_{kl} - \frac{1}{2} q_{ln} \partial_m q^{kl} \partial_k q^{mn} - \frac{1}{8} q^{kl} \partial_k \ln(q) \partial_l \ln(q)$$

$$q^{ij} = \gamma \gamma^{ij}, \quad q_{ij} = \frac{1}{\gamma} \gamma_{ij} \quad q = \gamma^2, \quad \Pi_{ij} = \frac{1}{\sqrt{\gamma}} K_{ij} = -\frac{1}{\gamma} \left( \pi_{ij} - \frac{1}{2} \delta_{ij} \pi \right)$$

$$(\mathcal{H}_i^{\rm field})_{\rm D/ADM}=\partial_j(2\pi^{jk}\gamma_{ki})-\pi^{kl}\partial_i\gamma_{kl}$$

$$H_{\rm S} = \int d^3x \left[ \frac{1}{\sqrt{\gamma}} \textcolor{red}{N} (\sqrt{\gamma} \mathcal{H}^{\rm field} + \sqrt{\gamma} \mathcal{H}^{\rm matter})_{\rm S} - \textcolor{red}{N}^{\textcolor{blue}{i}} (\mathcal{H}_i^{\rm field} + \mathcal{H}_i^{\rm matter}) \right]$$

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$N = 1$  at  $i^0$

$$H_{\text{ADM}} = \int d^3x (N\mathcal{H} - N^i\mathcal{H}_i) - \oint_{i^0} ds_i \partial_j (\delta_{ij}\gamma_{kk} - \gamma_{ij})$$

(DeWitt 1967)

$$H_{\text{ADM}}^{\text{srf}} = - \oint_{i^0} ds_i \partial_j (\delta_{ij}\gamma_{kk} - \gamma_{ij}) = - \int d^3x \partial_i \partial_j (\delta_{ij}\gamma_{kk} - \gamma_{ij})$$

$$H_{\text{D}} = \int d^3x (N\mathcal{H} - N^i\mathcal{H}_i) - \oint_{i^0} ds_i \gamma^{-1/2} \partial_j (\gamma\gamma^{ij})$$

$$H_{\text{D}}^{\text{srf}} = - \oint_{i^0} ds_i \gamma^{-1/2} \partial_j (\gamma\gamma^{ij}) = - \int d^3x \partial_i (\gamma^{-1/2} \partial_j (\gamma\gamma^{ij}))$$

$$H_{\text{S}} = \int d^3x (\gamma^{-1/2} N(\sqrt{\gamma}\mathcal{H}) - N^i\mathcal{H}_i) - \oint_{i^0} ds_i \partial_j (\gamma\gamma^{ij})$$

$$H_{\text{S}}^{\text{srf}} = - \oint_{i^0} ds_i \partial_j (\gamma\gamma^{ij}) = - \int d^3x \partial_i \partial_j (\gamma\gamma^{ij})$$

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## Newtonian limits

$$H_{\text{ADM}}^{(\text{N})} = \int d^3x \left( \varrho \frac{v^2}{2} - \frac{1}{2} \varrho U \right)$$

$$H_{\text{D}}^{(\text{N})} = \int d^3x \left( \varrho \frac{v^2}{2} - \frac{1}{8\pi G} \partial_i U \partial_i U \right)$$

$$H_{\text{S}}^{(\text{N})} = \int d^3x \left( \varrho \frac{v^2}{2} + 3\varrho U - \frac{7}{8\pi G} \partial_i U \partial_i U \right) \quad (\text{cf. Landau/Lifshitz})$$

Test-Body limit (1 test particle and  $N$  massive particles)

$$H^{(\text{TB})} = \int d^3x (N \mathcal{H}^{\text{testmat}} - N^i \mathcal{H}_i^{\text{testmat}})$$

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## Decompositions

$$q^{ij} = q^{ijTT} + \frac{1}{2}(\partial_i q^j + \partial_j q^i) - \delta_{ij} \partial_k q^k + \partial_i \partial_j q$$

$$\Pi_{ij} = \Pi_{ij}^{TT} + \frac{1}{2}(\partial_i \Pi_j + \partial_j \Pi_i) - \delta_{ij} \partial_k \Pi_k + \partial_i \partial_j \Pi$$

on action level

$$\begin{aligned} \int d^4x \ \Pi_{ij} \partial_t q^{ij} &= \int d^4x \ [\Pi_{ij}^{TT} \partial_t q^{ijTT} + \Pi \partial_t \nabla^2 q] \\ &= \int d^4x \ [\Pi_{ij}^{TT} \partial_t q^{ijTT} + \partial_i \partial_j q^{ij} \partial_t (-\Pi)] \quad + \quad \text{TTD} \end{aligned}$$

Coordinate Condition (so by Schwinger)

$$-\Pi = t$$

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## Coordinate Conditions

Dirac:  $\gamma_{ij}\pi^{ij} \equiv \pi = 0, \quad \partial_j(\gamma^{1/3}\gamma^{ij}) = 0$

ADM:  $\pi^{ii} = 0, \quad 3\partial_j\gamma_{ij} - \partial_i\gamma_{jj} = 0 \quad (\text{2nd by ADM})$

Schwinger:  $\Pi_{ii} = 0, \quad 3\partial_j q^{ij} - \partial_i q^{jj} = 0 \quad (\text{others by Schwinger})$

more ADM:  $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ijTT}, \quad \gamma_{ij} = \psi\delta_{ij} + h_{ij}^{TT}$

$$\tilde{\pi}^{ij} = \partial_i V^j + \partial_j V^i - \frac{2}{3}\delta_{ij}\partial_k V^k$$

2 pairs  $(h_{ij}^{TT}, \pi^{ijTT})$  at each space point

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Dirac: pairs of canonical variables  $(\ln(\gamma^{1/3}), \pi)$  and  $(\tilde{\gamma}_{ij}, \tilde{\pi}^{ij})$

$$\begin{aligned}\tilde{\pi}^{ij} &= \gamma^{1/3}(\pi^{ij} - \frac{1}{3}\gamma^{ij}\pi), \quad \tilde{\gamma}_{ij} = \gamma^{-1/3}\gamma_{ij} \\ \tilde{\gamma}_{ij}\tilde{\pi}^{ij} &= 0, \quad \det(\tilde{\gamma}_{ij}) = 1\end{aligned}$$

Dirac's “maximal slicing” condition:  $\pi = 0$

5 pairs  $(\tilde{\gamma}_{ij}, \tilde{\pi}^{ij})$  at each space point (Regge/Teitelboim 1974)

$$\begin{aligned}\{F, G\} &= \tilde{\delta}_{ij}^{kl} \left( \frac{\delta F}{\delta \tilde{\gamma}_{ij}} \frac{\delta G}{\delta \tilde{\pi}^{kl}} - \frac{\delta G}{\delta \tilde{\gamma}_{ij}} \frac{\delta F}{\delta \tilde{\pi}^{kl}} \right) + \frac{1}{3}(\tilde{\pi}^{ij}\tilde{\gamma}^{kl} - \tilde{\pi}^{kl}\tilde{\gamma}^{ij}) \frac{\delta F}{\delta \tilde{\pi}^{ij}} \frac{\delta G}{\delta \tilde{\pi}^{kl}} \\ \tilde{\delta}_{ij}^{kl} &= \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k) - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}, \quad \tilde{\gamma}_{ij}\tilde{\gamma}^{jl} = \delta_i^l\end{aligned}$$

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$$H_{\text{D}}^{\text{red}} = - \int d^3x \Delta \gamma^{2/3} [z^i, p_i, \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}]$$

$r \rightarrow \infty$ :

$$\partial_k \tilde{\gamma}_{ik} = O(1/r^{(2+\epsilon)}), \quad \partial_k \tilde{\gamma}_{ii} = O(1/r^{(2+\epsilon)})$$

$r \rightarrow \infty$ :

$$ds^2 = - \left( 1 - \frac{M}{8\pi r} \right) dt^2 + \left[ \left( 1 + \frac{M}{8\pi r} \right) \delta_{ij} + h_{ij}^{TT} \right] dx^i dx^j$$

Schwarzschild coordinates do not fit:

$$ds^2 = - \left( 1 - \frac{M}{8\pi r} \right) dt^2 + \left( \delta_{ij} + \frac{M}{8\pi r} \frac{x^i x^j}{r^2} \right) dx^i dx^j$$

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$$H_{\text{ADM}}^{\text{red}} = -2 \int d^3x \Delta \psi [z^i, p_i, h_{ij}^{TT}, \pi^{ijTT}]$$

$$\mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}} = 0 \quad \rightarrow \quad \psi$$

$$\mathcal{H}_i^{\text{field}} + \mathcal{H}_i^{\text{matter}} = 0 \quad \rightarrow \quad V^i$$

$$\partial_t h_{ij}^{TT} = \frac{\delta H_{\text{ADM}}^{\text{red}}}{\delta \pi^{ijTT}}, \quad \quad \partial_t \pi^{ijTT} = - \frac{\delta H_{\text{ADM}}^{\text{red}}}{\delta h_{ij}^{TT}}$$

$$\{F,G\}=\delta_{ij}^{\text{TT}kl}\left(\frac{\delta F}{\delta h_{ij}^{TT}}\frac{\delta G}{\delta \pi^{klTT}}-\frac{\delta G}{\delta h_{ij}^{TT}}\frac{\delta F}{\delta \pi^{klTT}}\right)$$

$$\dot{z}^i = \frac{\partial H_{\text{ADM}}^{\text{red}}}{\partial p_i}, \quad \quad \dot{p}_i = - \frac{\partial H_{\text{ADM}}^{\text{red}}}{\partial z^i}$$

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Transition to **Routhian**  $R$  to solve field equations and to get  
conservative Hamiltonian  $H_{\text{con}}$ :

$$R[z^i, p_i, h_{ij}^{TT}, \partial_t h_{ij}^{TT}] = H_{\text{ADM}}^{\text{red}} - \int d^3x \pi^{ijTT} \partial_t h_{ij}^{TT}$$

**Field Equations**

$$\frac{\delta \int R(t') dt'}{\delta h_{ij}^{TT}(t, x^i)} = 0$$

$$H_{\text{con}} = R [z, p, h^{TT}[z, p], \partial_t h^{TT}[z, p]]$$

**Equations of Motion**

$$\dot{z}^i(t) = \frac{\delta \int H_{\text{con}}(t') dt'}{\delta p_i(t)}, \quad \dot{p}_i(t) = -\frac{\delta \int H_{\text{con}}(t') dt'}{\delta z^i(t)}$$

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Global post-Newtonian expansion

$$\left(\Delta - \frac{1}{c^2} \partial_t^2\right) \hat{h}_{ij}^{TT} = \sum_{n=0}^{\infty} \frac{1}{c^{2n}} D_{ij(n)}^{TT}[x, z, p, \hat{h}^{TT}, \partial_t \hat{h}^{TT}]$$

$$h_{ij}^{TT} = \frac{16\pi G}{c^4} \hat{h}_{ij}^{TT}$$

Near-zone post-Newtonian expansion

$$\left(\Delta - \frac{1}{c^2} \partial_t^2\right)^{-1} = \left(1 + \sum_{n=0}^{\infty} \frac{1}{c^{2n}} \Delta^{-n} \partial_t^{2n}\right) \Delta^{-1} \delta(t - t')$$

$$\delta_{ij}^{\text{TT}kl} = \frac{1}{2}(P_{il}P_{jk} + P_{ik}P_{jl} - P_{kl}P_{ij}), \quad P_{ij} = \delta_{ij} - \Delta^{-1} \partial_i \partial_j$$

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## Leading order field equation for tails

$$(\Delta - \partial_{ct}^2) \hat{h}_{ij}^{TT\text{tail}} = \delta_{ij}^{\text{TT}kl} \sum_a \frac{m_a}{4\pi c^2 r_a} \partial_{ct}^2 \hat{h}_{kl}^{TT}$$

## Tail Green's function

$$(\Delta - \partial_t^2) G(x^i, x'^i, t - t') = \frac{1}{r} \frac{\partial}{\partial t} G_{\text{ret}}(x^i - x'^i, t - t')$$

$$G(x^i, x'^i, t - t') = -\frac{1}{2R} \delta(t - t' - R) \ln \frac{r + r' + R}{r + r' - R} + \frac{\Theta(t - t' - r - r')}{(t - t')^2 - R^2}$$

$$R = |x^i - x'^i|, r = |x^i|, r' = |x'^i|$$

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## ADM-formalism for spinning classical objects

$$W = \int d^4x \mathcal{L}_M$$

$$\mathcal{L}_M = \int d\tau \left[ \left( p_\mu - \frac{1}{2} S_{ab} \omega_\mu^{ab} \right) \frac{dz^\mu}{d\tau} + \frac{1}{2} S_{ab} \frac{\delta\theta^{ab}}{d\tau} \right] \delta_{(4)}$$

matter variables  $z^\mu, p_\mu, S_{ab}, \Lambda^{Cb}$

$$\delta\theta^{ab} = \Lambda_C^a d\Lambda^{Cb} = -\delta\theta^{ba}$$

$$\Lambda^{Aa} \Lambda^{Bb} \eta_{AB} = \eta^{ab} \text{ and } \Lambda_{Aa} \Lambda_{Bb} \eta^{ab} = \eta_{AB}$$

$$\eta_{AB} = \text{diag}(-1, 1, 1, 1) = \eta^{ab}$$

$$\delta_{(4)} = \delta(x^\nu - z^\nu(\tau)), \quad \int d^4x \delta_{(4)} = 1$$

Ricci rotation coefficients  $\omega_\mu^{ab}$

$$\omega_{\mu\alpha\beta} = e_{a\alpha} e_{b\beta} \omega_\mu^{ab} = -\Gamma_{\beta\alpha\mu} + \partial_\mu e_\alpha^c e_{c\beta},$$

$$\Gamma_{\beta\alpha\mu} = \frac{1}{2} (\partial_\mu g_{\beta\alpha} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\alpha\mu}), \quad g_{\mu\nu} = e_{a\mu} e_{b\nu} \eta^{ab}$$

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## Matter constraints

$$\mathcal{L}_C = \int d\tau \left[ \lambda_1^a p^b S_{ab} + \lambda_2^{[i]} \Lambda^{[i]a} p_a - \frac{\lambda_3}{2} (p^2 + m^2) \right] \delta_{(4)}$$

$p_\mu p^\mu + m^2 = p_a p^a + m^2 = 0$  **Mass-shell Condition**

$p^b S_{ab} = 0$  **Spin Supplementary Condition (SSC)**: in the rest frame the spin tensor contains the 3-dim. spin  $S_{(i)(j)}$  only (i.e., there the mass dipole part  $S_{(0)(i)}$  vanishes)

The **Conjugate Constraint**  $\Lambda^{[i]a} p_a = 0$  ensures that  $\Lambda^{Ca}$  is a pure 3-dim. rotation matrix in the rest frame (no Lorentz boosts)

The complete Lagrangian density is the sum

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M + \mathcal{L}_C$$

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The total action

$$W[e_{a\mu}, z^\mu, p_\mu, \Lambda^{Ca}, S_{ab}, \lambda_1^a, \lambda_{2[i]}, \lambda_3] = \int d^4x \mathcal{L}$$

Variation of action  $\delta W = 0$  leads to the equations of motion

$$\begin{aligned} \frac{DS_{ab}}{D\tau} &= 0, & \frac{D\Lambda^{Ca}}{D\tau} &= 0 \\ u^\mu \equiv \frac{dz^\mu}{d\tau} &= \lambda_3 p^\mu, & \frac{Dp_\mu}{D\tau} &= -\frac{1}{2} R_{\mu\rho ab} u^\rho S^{ab} \\ \sqrt{-g} T^{\mu\nu} &= e_a^\mu \frac{\delta(\mathcal{L}_M + \mathcal{L}_C)}{\delta e_{a\nu}} = \int d\tau \left[ \lambda_3 p^\mu p^\nu \delta_{(4)} + D_\alpha \left( u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right) \right] \end{aligned}$$

Preservation of the constraints in time requires  $\lambda_1^a$  to be proportional to  $p^a$  and  $\lambda_{2[i]}$  to be zero, so that  $\lambda_1^a$  and  $\lambda_{2[i]}$  drop out of the matter EOM and of the stress-energy tensor. The Lagrangian multiplier  $\lambda_3 = \lambda_3(\tau)$  represents the reparametrization invariance of the action (notice  $\lambda_3 = \sqrt{-u^2}/m$ ).  $D_\mu T^{\mu\nu} = 0$  and  $S_{ab} S^{ab} = \text{const.}$

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A (3+1)-split with respect to a hypersurface orthogonal timelike unit 4-vector  $n_\mu$  with components  $n_\mu = (-N, 0, 0, 0)$  or  $n^\mu = (1, -N^i)/N$ , most naturally fits to a fully reduced canonical formulation of gravity. The three matter constraints can be solved as

$$np \equiv n^\mu p_\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$$

$$nS_i \equiv n^\mu S_{\mu i} = \frac{p_k \gamma^{kj} S_{ji}}{np} = \gamma_{ij} n S^j$$

$$\Lambda^{[j](0)} = \Lambda^{[j](i)} \frac{p(i)}{p^{(0)}} , \quad \Lambda^{[0]a} = -\frac{p^a}{m}$$

in terms of  $p_i$ ,  $S_{(i)(j)}$ , and  $\lambda^{[i](k)}$ , so one can drop  $\mathcal{L}_C$  from now on.

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A split of the Ricci rotation coefficients results in:

$$(1) \quad \omega_{kij} = -\Gamma_{jik} + \partial_k e_i^a e_{aj}$$

$$(2) \quad n^\mu \omega_{k\mu i} = K_{ki} - \gamma_{ij} \frac{\partial_k N^j}{N} + \frac{e_{ai}}{N} (\partial_k e_0^a - \partial_k e_l^a N^l)$$

$$(3) \quad \omega_{0ij} = NK_{ij} - \nabla_i N_j + \partial_t e_i^a e_{aj}$$

$$(4) \quad n^\mu \omega_{0\mu i} = K_{ij} N^j - \nabla_i N - \gamma_{ij} \frac{\partial_t N^j}{N} + \frac{e_{ai}}{N} (\partial_t e_0^a - \partial_t e_l^a N^l)$$

With time gauge  $e_{(0)}^\mu = n^\mu$  (Dirac 1962, Schwinger 1963), it holds

$$e_i^{(0)} = 0 = e_{(i)}^0, \quad e_0^{(0)} = N = 1/e_{(0)}^0$$

$$N^i = -N e_{(0)}^i, \quad e_0^{(i)} = N^j e_j^{(i)}$$

$$\gamma_{ij} = e_i^{(m)} e_{(m)j}, \quad \gamma^{ij} = e_{(m)}^i e^{(m)j}$$

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The matter action in the covariant SSC  $p^b S_{ab} = 0$  turns into

$$\mathcal{L}_M = \mathcal{L}_{MK} + \mathcal{L}_{MC} + \mathcal{L}_{GK} + (\text{TD}) ,$$

where (TD) denotes an irrelevant total divergence.

Kinetic Matter part:

$$\begin{aligned} \mathcal{L}_{MK} = & \left[ p_i + K_{ij} n S^j + A^{kl} e_{(j)k} \partial_i e_l^{(j)} \right. \\ & \left. - \left( \frac{1}{2} S_{kj} + \frac{p_{(k} n S_{j)}}{np} \right) \Gamma^{kj}_i \right] \dot{z}^i \delta + \frac{n S^i}{2np} \dot{p}_i \delta \\ & + \left[ S_{(i)(j)} + \frac{n S_{(i)} p_{(j)} - n S_{(j)} p_{(i)}}{np} \right] \frac{\Lambda_{[k]}^{(i)} \dot{\Lambda}^{[k](j)}}{2} \delta \\ g_{ik} g_{jl} A^{kl} = & \frac{1}{2} S_{ij} + \frac{n S_i p_j}{2np} . \end{aligned}$$

---

Matter part to the Constraints:

$$\mathcal{L}_{MC} = -N\mathcal{H}^{\text{matter}} + N^i \mathcal{H}_i^{\text{matter}}$$

$$\mathcal{H}^{\text{matter}} = -np\delta - K^{ij} \frac{p_i n S_j}{np} \delta - \nabla_k (n S^k \delta)$$

$$\mathcal{H}_i^{\text{matter}} = (p_i + K_{ij} n S^j) \delta + \nabla_m \left( \frac{1}{2} \gamma^{mk} S_{ik} \delta + \delta_i^{(k} \gamma^{l)m} \frac{p_k n S_l}{np} \delta \right)$$

Gravitational field Kinetic part:

$$\mathcal{L}_{GK} = A^{ij} e_{(k)i} \partial_t e_j^{(k)} \delta$$

---

Now we proceed to [Newton-Wigner variables](#)  $\hat{z}^i$ ,  $P_i$ ,  $\hat{S}_{(i)(j)}$ , and  $\hat{\lambda}^{[i](j)}$ , which turn the Kinetic Matter part  $\mathcal{L}_{MK}$  into canonical form.

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i n S_j}{m - np} + \frac{p_j n S_i}{m - np}$$

$$\lambda^{[i](j)} = \hat{\lambda}^{[i](k)} \left( \delta_{kj} + \frac{p_{(k)} p^{(j)}}{m(m - np)} \right)$$

$$p_i = P_i - K_{ij} n S^j - \hat{A}^{kl} e_{(j)k} e_{l,i}^{(j)} + \left( \frac{1}{2} S_{kj} + \frac{p_{(k)} n S_j}{np} \right) \Gamma^{kj}_i$$

$$\gamma_{ik} \gamma_{jl} \hat{A}^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{mp_{(i)} n S_j}{np(m - np)}$$

$$\hat{S}_{(i)(j)} \hat{S}_{(i)(j)} = 2s^2 = \text{const},$$

$$\delta \hat{\theta}^{(i)(j)} = \hat{\lambda}_{[k]}^{(i)} d\hat{\lambda}^{[k](j)} = -\delta \hat{\theta}^{(j)(i)}, \quad \hat{\lambda}_{[k]}^{(i)} \hat{\lambda}^{[k](j)} = \delta_{ij}$$

---


$$\mathcal{L}_{GK} + \mathcal{L}_{MK} = \hat{\mathcal{L}}_{GK} + \hat{\mathcal{L}}_{MK} + (\text{TD})$$

$$\begin{aligned}\hat{\mathcal{L}}_{MK} &= P_i \dot{\hat{z}}^i \hat{\delta} + \frac{1}{2} \hat{S}_{(i)(j)} \dot{\hat{\theta}}^{(i)(j)} \hat{\delta} \\ \hat{\mathcal{L}}_{GK} &= \hat{A}^{ij} e_{(k)i} \partial_t e_j^{(k)} \hat{\delta} \\ \hat{\delta} &= \delta(x^i - \hat{z}^i(t))\end{aligned}$$

The canonical momentum conjugate to  $e_{(k)j}$  is given by

$$\bar{\pi}^{(k)j} = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial(\partial_t e_{(k)j})} = e_i^{(k)} \pi^{ij} + e_i^{(k)} \frac{1}{2} \hat{A}^{ij} \hat{\delta}$$

$$\pi^{ij} = \sqrt{\gamma} (\gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{jl}) K_{kl}$$

Legendre transformation leads to

$$\hat{\mathcal{L}}_{GK} + \mathcal{L}_G = 2\bar{\pi}^{(k)j} \partial_t e_{(k)j} - \mathcal{E}_{i,i} + \mathcal{L}_{GC} + (\text{TD})$$

---

The explicit form of the non-irrelevant total divergence  $\mathcal{E}_{i,i}$  emerges as  $\mathcal{E}_i = \gamma_{ij,j} - \gamma_{jj,i}$  for asymptotically flat spacetimes. The total energy reads  $E = \oint d^2 s_i \mathcal{E}_i$ .

The gravitational constraint part takes the form

$$\mathcal{L}_{GC} = -N\mathcal{H}^{\text{field}} + N^i\mathcal{H}_i^{\text{field}}$$

with

$$\begin{aligned}\mathcal{H}^{\text{field}} &= -\frac{1}{\sqrt{\gamma}} \left[ \gamma R + \frac{1}{2} (\gamma_{ij}\pi^{ij})^2 - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl} \right] \\ \mathcal{H}_i^{\text{field}} &= 2\gamma_{ij}\nabla_k\pi^{jk}\end{aligned}$$

Due to the symmetry of  $\pi^{ij}$ , not all components of  $\bar{\pi}^{(k)j}$  are independent variables (i.e., the Legendre map is not invertible).

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## Additional constraint

$$\bar{\pi}^{[ij]} = \frac{1}{2} \hat{A}^{[ij]} \hat{\delta}$$

Constraint eliminated by going to the **spatial symmetric gauge**  
 $e_{(i)j} = e_{ij} = e_{ji}$ ,  $e^{(i)j} = e^{ij} = e^{ji}$  (Kibble 1963).

Triad is fixed as matrix square-root  $e_{ij}e_{jk} = \gamma_{ik}$  or,

$$e_{ij} = \left( \sqrt{(\gamma_{kl})} \right)_{ij}$$

Therefore, we can define a quantity  $B_{ij}^{kl}$  as

$$e_{k[i} \partial_\mu e_{j]k} = B_{ij}^{kl} \partial_\mu \gamma_{kl}$$

or, in explicit form,

$$2B_{ij}^{kl} = e_{mi} \frac{\partial e_{mj}}{\partial \gamma_{kl}} - e_{mj} \frac{\partial e_{mi}}{\partial \gamma_{kl}}$$

---

It holds  $B_{ij}^{kl}\delta_{kl} = 0$ . Furthermore,

$$e_{(k)i}\partial_\mu e_j^{(k)} = B_{ij}^{kl}\partial_\mu\gamma_{kl} + \frac{1}{2}\partial_\mu\gamma_{ij}$$

which gets applied as

$$\bar{\pi}^{(k)j}\partial_t e_{(k)j} = \frac{1}{2}\pi_{\text{can}}^{ij}\partial_t\gamma_{ij}$$

with the new [canonical symmetric field momentum](#)

$$\pi_{\text{can}}^{ij} = \pi^{ij} + \frac{1}{2}\hat{A}^{(ij)}\hat{\delta} + B_{kl}^{ij}\hat{A}^{[kl]}\hat{\delta}$$

---

The gravitational constraints arising from the variations  $\delta N$  and  $\delta N^i$ ,

$$\mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}} = 0$$

$$\mathcal{H}_i^{\text{field}} + \mathcal{H}_i^{\text{matter}} = 0$$

are eliminated by also imposing the gauge conditions

$$3\gamma_{ij,j} - \gamma_{jj,i} = 0, \quad \pi_{\text{can}}^{ii} = 0$$

which allow for the decompositions

$$\gamma_{ij} = \Psi\delta_{ij} + h_{ij}^{\text{TT}}$$

$$\pi_{\text{can}}^{ij} = \tilde{\pi}_{\text{can}}^{ij} + \pi_{\text{can}}^{ij\text{TT}}$$

where  $h_{ij}^{\text{TT}}$  and  $\pi_{\text{can}}^{ij\text{TT}}$  are transverse traceless, e.g.,  $h_{ii}^{\text{TT}} = \partial_j h_{ij}^{\text{TT}} = 0$ , and  $\tilde{\pi}_{\text{can}}^{ij}$  is  $\tilde{\pi}_{\text{can}}^{ij} = \partial_j V_{\text{can}}^i + \partial_i V_{\text{can}}^j - \frac{2}{3}\delta_{ij}\partial_k V_{\text{can}}^k$

---

The gravitational constraints can now be solved for  $\Psi$  and  $\tilde{\pi}_{\text{can}}^{ij}$ , leaving  $h_{ij}^{\text{TT}}$  and  $\pi_{\text{can}}^{ij\text{TT}}$  as the final degrees of freedom of the gravitational field. Notice that our gauge condition  $\pi_{\text{can}}^{ii} = 0$  deviates from the original ADM one  $\pi^{ii} = 0$  by spin corrections at 5PN.

The action reads,

$$W = \int d^4x \pi_{\text{can}}^{ij\text{TT}} \partial_t h_{ij}^{\text{TT}} + \int dt \left[ P_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \dot{\hat{\theta}}^{(i)(j)} - E \right]$$

and is in fully reduced canonical form. The dynamics is completely described by the ADM energy  $E$ , which turns into the volume integral

$$E = -2 \int d^3x \Delta \Psi[\hat{z}^i, P_i, \hat{S}_{(i)(j)}, h_{ij}^{\text{TT}}, \pi_{\text{can}}^{ij\text{TT}}]$$

and is the total Hamiltonian ( $E = H$ ) once it is expressed in terms of the canonical variables.

---


$$\hat{S}_{(i)} = \tfrac{1}{2} \epsilon_{ijk} \hat{S}_{(j)(k)}$$

$$\begin{aligned} \{\hat{z}^i, P_j\} &= \delta_{ij}, \quad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} = \epsilon_{ijk} \hat{S}_{(k)}, \\ \{h_{ij}^{\text{TT}}(t, x^m), \pi_{\text{can}}^{kl\text{TT}}(t, x'^m)\} &= \delta_{ij}^{\text{TT}kl} \delta(x^m - x'^m), \\ &\quad \text{zero otherwise} \end{aligned}$$

$$\begin{aligned} P_i^{\text{tot}} &= \sum_a P_{ai} - \int d^3x \pi_{\text{can}}^{kl\text{TT}} \partial_i h_{kl}^{\text{TT}} \\ J_{ij}^{\text{tot}} &= \sum_a (\hat{z}_a^i P_{aj} - \hat{z}_a^j P_{ai} + \hat{S}_{a(i)(j)}) \\ &\quad - 2 \int d^3x (\pi_{\text{can}}^{ik\text{TT}} h_{kj}^{\text{TT}} - \pi_{\text{can}}^{jk\text{TT}} h_{ki}^{\text{TT}}) \\ &\quad - \int d^3x (x^i \pi_{\text{can}}^{kl\text{TT}} \partial_j h_{kl}^{\text{TT}} - x^j \pi_{\text{can}}^{kl\text{TT}} \partial_i h_{kl}^{\text{TT}}) \end{aligned}$$

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