Towards 5PN accuracy in Non Relativistic General Relativity

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S. Foffa, RS, submitted to PRD, arXiv:1907.02869

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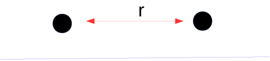
PN approximation to General Relativity

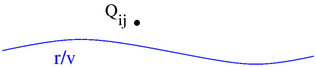
Small expansion parameter v, related to metric perturbation $v^2 \sim \frac{G_N M}{r}$

Near zone, $D \sim r$



Describe conservative dynamics

Far zone, $D \gtrsim \lambda = r/v$

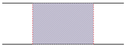


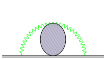
conservative + dissipative

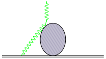
EFT framework pioneered by W. Goldberger and I. Rothstein, PRD '06

PN approximation for compact binary systems

	Near	Far
World-line	$-m_a \int d\tau = m_a \int dt \times$ $\left(\phi + A_i v^i + \sigma_{ij} v^i v^j + \ldots\right)$	$\int d^4x \left(E h_{00} + \frac{1}{2}\epsilon_{ijk} L^i h_{0j,k}\right.$ $\left.+ Q_{ij} E^{ij} \ldots\right)$
Bulk	$\frac{1}{16\pi G_N} \int d^4x \left[R - \frac{1}{2}\left(g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\right)^2\right]$	
5PN	G_N, G_N^2, G_N^3 ✓ [1] $G^4,\ G^5$ × G^6 ✓ RS et al. PRL (2019)	✓ Foffa& RS 1907.02869







5PN is the lowest order finite size effect are not forbidden effacement principle, but expected at $>$ 5PN order ($Love_{BH} = 0$)[2]

[1] PM: Duff ('73); Westpfahl & Goller, LNC ('79); Damour PRD ('18), Cheung et al. PRL ('18), Bern et al. PRL ('19)

[2] See Binnington & Poisson, Damour & Nagar PRD ('09); Kol & Smolkin JHEP ('12); Pani et al. PRD ('15)

Outline

1. Near zone

2. Far zone

Effective potential from integration over regions

Internal graviton momentum can be expanded upon the following scaling:

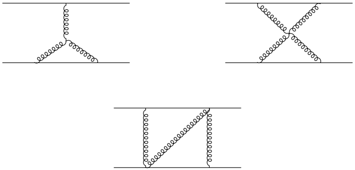
hard	(m, m)	quantum	×
soft	$(\lvert\vec{q}\rvert, \lvert\vec{q}\rvert)$	quantum	×
potential	$(v/r, 1/r)$	classical	✓
radiation	$(v/r, v/r)$	classical	next section

and then integrated over the full phase space

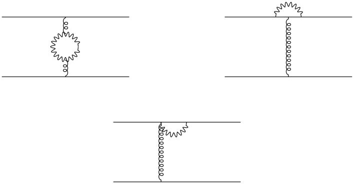
Only potential and radiation gravitons exchanged in classical processes: theory in terms of world lines selects diagrams that do not send source off-shell

Potential graviton $\rightarrow$ small change in energy wrt momentum, dominate classically

Ex. of classical connected diagrams



Ex. of quantum diagrams



Gravitational interactions do not create anti-BHs!

Graviton loop are negligible in astronomy!

From relativistic scattering amplitudes to 2-body potential

Equivalently, loops with massive particles contain a classical piece for $m \gg |\vec{q}|$:

$$V(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} A(\vec{q}, m_1, m_2, \vec{v}_1, \vec{v}_2, \ldots) e^{i\frac{\vec{q}\cdot\vec{r}}{\hbar}}$$

$$\supset \frac{1}{m^2 - \vec{q}^2} \quad \to \quad \underset{\textit{classical}}{\frac{1}{m^2}} \quad + \quad \underset{\textit{quantum}}{\frac{\vec{q}^2}{m^4}} \quad + \quad \ldots$$

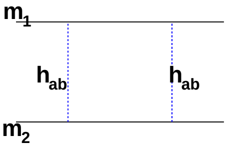
E.g.: $\frac{G_N}{q^2}, \frac{G_N^2}{|q|}, G_N^3 \log|q| \ldots$ are classical contributions

Alternative view

Equivalently, loops with massive particles contain a classical piece for $m \gg \hbar|\vec{l}|$:

$$V(\vec{r}) = \hbar \int \frac{d^3 l}{(2\pi)^3} A(\vec{l}, m_1, m_2, \vec{v}_1, \vec{v}_2, \dots) e^{i\vec{l}\cdot\vec{r}}$$

$$\supset \frac{1}{m^2 - \hbar \vec{l}^2} \to \underset{classical}{\frac{1}{m^2}} + \underset{quantum}{\frac{\hbar^2 \vec{l}^2}{m^4}} + \dots$$



Quantum and classical

The gravity action

Useful ansatz:

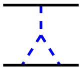
$$g_{\mu\nu} = e^{2\phi/m_{Pl}} \begin{pmatrix} -1 & A_j/m_{Pl} \\ A_i/m_{Pl} & e^{-c_d\phi}\left(\delta_{ij} + \sigma_{ij}/m_{Pl}\right) - A_i A_j/m_{Pl}^2 \end{pmatrix}$$

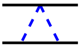
$$S_{pp} = \int dt\, e^{\phi/m_{Pl}} \sqrt{\left(1 - \frac{A_i v_i}{m_{Pl}}\right)^2 + e^{-c_d\phi/m_{Pl}}\left(v^2 + \frac{\sigma_{ij}}{m_{Pl}} v^i v^j\right)}$$

$$\begin{aligned} S_{EH} \;=\; & \int d^d x \sqrt{-\gamma}\left\{\frac{1}{4}\left[\left(\vec{\nabla}\sigma\right)^2 - 2\vec{\nabla}\sigma_{ij}{}^2\right] - c_d\left(\vec{\nabla}\phi\right)^2 + \right. \\ & + \frac{F_{ij}{}^2}{2} + \left(\vec{\nabla}\cdot\vec{A}\right)^2 + \dot{\sigma}^2 + \dot{\phi}^2 + \dot{A}^2 + \\ & \left. + \partial^2\phi^2\sigma^k + (\dot{\phi})^2\phi^k + \text{other interactions } \cancel{(\partial^2\phi)\phi^k}\right\} \end{aligned}$$

Near zone static sectors

1PN



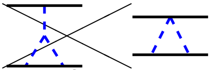


1 prime +

factorizable?

Near zone static sectors

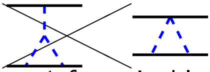
1PN



1 factorizable

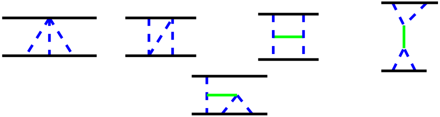
Near zone static sectors

1PN



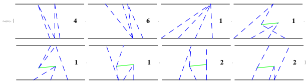
1 factorizable

2PN



2 factorizable + 3 prime

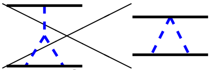
3PN



All factorizable

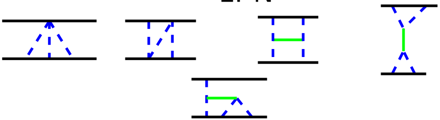
Near zone static sectors

1PN



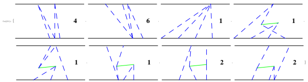
1 factorizable

2PN



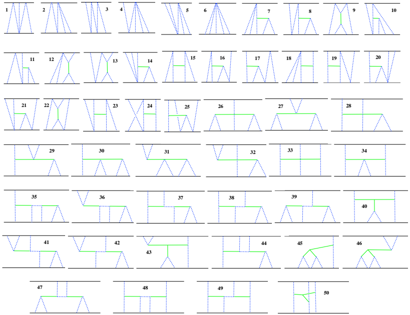
2 factorizable + 3 prime

3PN



All factorizable

4PN



25 factorizable +25 prime

Factorization theorem in the static sector

Theorem

At $(2n+1)$-PN order all static graphs are factorizable

Proof.

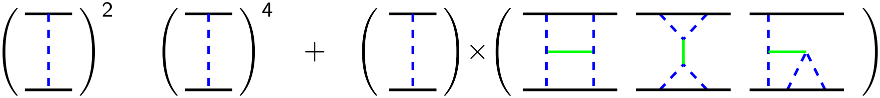
$$V \propto G^{d_M-1} m_1^{d_{m_1}} m_2^{d_{m_2}} = G_N^{d_M-1} m_1^{d_M} \left(\frac{m_2}{m_1}\right)^{d_{m_2}}$$

Only world-line $m_i\phi^n$ and bulk $\phi^2\sigma^k$ vertices matter
Prime diagrams must have all $n=1 \implies d_M = 2m$ since all internal static vertices have even number of ϕ. Then $V \propto G^{2m-1} \subset (2m-2)$-PN $\square$

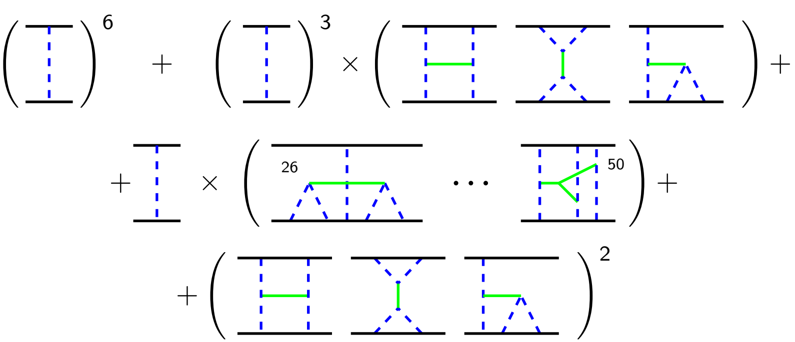
Crucial is the absence of $(\partial\phi)^2\phi^k$ bulk terms (but $\dot{\phi}^2\phi^k$ present)
At $(2n+1)$-PN order no integration needed, just multiplications!

How it works in practice:

1PN 3PN



5PN



- Finite- and rational-ness inherited from 4PN (non trivially!)
- Schwarzschild limit OK ($m_2 \ll m_1$)
- Result confirmed by explicit, indepedent calculation of ~ 100 diagrams in Blümlein et al arXiv:1902.11180
- Factorization does not hold for all diagrams at n-PN sectors $G^{1+n-j}v^{2j}$ $(0 < j \leq n)$ but for $\gtrsim 50\%$ of them.

$$V_{5PN\,static} = \frac{G_N^5 m_1^3 m_2}{r^5}\left(\frac{5}{16}m_1^2 + \frac{91}{6}m_1 m_2 + \frac{653}{6}m_2^2\right)$$

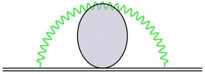
Outline

1 Near zone

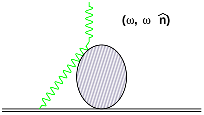
2 Far zone

Radiation scale: Including modes with $k \sim (\omega = v/r, \omega\hat{n})$

0 point



1point



$$\int d\omega \tilde{A}(\omega) = -\int dt V(t)$$

$$\int h_{ij}(\omega) T^{ij}$$

interactions among all possible modes
(radiative and longitudinal)

Double line represents the composite object

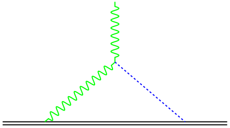
$$S = \frac{1}{2}\int d^4x \quad \left(E h_{00} + \epsilon_{ijk} L^i h_{0j,k} + Q_{ij} E^{ij} + \frac{1}{3} O_{ijk} E^{ij,k} - \frac{2}{3} J_{ij} B_{ij} + \ldots \right)$$

endowed with multipoles: v is the small parameter of the multipole expansion, but $G_N M\omega$ is the expansion parameter of perturbation theory.

Hereditary terms

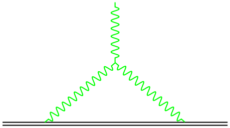
Wform depends on the history rather than source's state at retarded time (propagation inside the light-cone)

Tail:



Blanchet & Damour PRD (1988)

Memory:



Christodoulou PRL (1991)
(asymptotic states and IR property)
Blanchet & Damour PRD (1992)

$$Q_{ij}(t) = Q^{(0)}_{ij}(t) + 2G_N M \int_0^\infty d\tau\, \ddot{Q}_{ij}(t-\tau)\left[\log(\mu\tau) + \frac{11}{12}\right]$$

$$-\frac{2}{7}\int_0^\infty d\tau\, Q^{(3)}_{ik}(t-\tau)\, Q^{(3)}_{kj}(t-\tau)$$

Static curvature **GW**

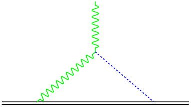
Hereditary terms

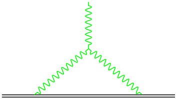
Wform depends on the history (propagation inside the light-cone)

Tail:

Memory:

1pt

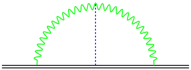


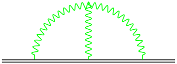


hereditary
↓

hereditary
↓

Self-energy





Static curvature GW

hereditary

Foffa & RS PRD ('13)

Galley & al. PRD ('15),Damour & al. PRD ('13)

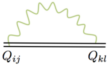
instantaneous

Foffa & RS 1907.02869

Conservative-dissipative interplay

Leading order power loss (i.e. quadrupole formula) from self-energy diagram:

$$S^{Q^2}_{eff\,2.5PN} = \int_{-\infty}^{\infty}\frac{\mathrm{d}k_0}{2\pi}\int_{\mathbf{k}}\Big(\qquad\qquad\qquad\Big)$$



$$\begin{aligned}
&= -2\pi G_N\int_{-\infty}^{\infty}\frac{\mathrm{d}k_0}{2\pi}\int_{\mathbf{k}}\frac{Q_{ij}(k_0)Q_{kl}(-k_0)}{\mathbf{k}^2-k_0^2}\left[\underbrace{-k_0^4\delta^{ik}\delta^{jl}}_{\sigma}+\underbrace{2k_0^2\delta^{ik}k^jk^l}_{A}-\underbrace{\frac{1}{2}k^ik^jk^kk^l}_{\phi}\right]\\
&= i\frac{G_N}{10}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_0}{2\pi}|k_0|k_0^4Q_{ij}(k_0)Q^{ij}(-k_0)\,,
\end{aligned}$$

Diagram purely imaginary, receives contribution from the $k_0 \sim |\vec{k}|$ region

Tail calculation

Tail self-energy diagram divergent ($\mathbf{q} \to 0, \mathbf{k} \to \infty$)

$$S^{tail}_{pole} = -64\pi^2 G_N^2 E \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{2\pi} \int_{\mathbf{kq}} \frac{k_0^2 Q_{ij}(k_0) Q_{kl}(-k_0)}{\mathbf{q}^2 \left[(\mathbf{k}+\mathbf{q})^2 - k_0^2\right] (\mathbf{k}^2 - k_0^2)} \times \left[\underbrace{-k_0^4 \delta^{ik} \delta^{jl}}_{\sigma} + \underbrace{2\mathbf{k}^2 \delta^{ik} k^j k^l}_{A} - \underbrace{\frac{1}{2} k^i k^k k^j k^l}_{\phi} \right] .$$

has the same structure as the self-energy diagram:

$$S^{tail}_{pole} = -64\pi^2 G_d^2 E \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{2\pi} (-k_0^2)^{d/2-1} \int_{\mathbf{k}} f\left(\frac{\mathbf{k}^2}{k_0^2}\right) \frac{Q_{ij}(k_0) Q_{kl}(-k_0)}{\mathbf{k}^2 - k_0^2} [\ldots]$$

$$f\left(\frac{\mathbf{k}^2}{k_0^2} \to 1\right) \simeq \frac{1}{\epsilon}, \; \frac{G_d^2}{\epsilon}(-k_0^2 - i0^+)^{\epsilon} \simeq G_N^2 \left(\frac{1}{\epsilon} + \underbrace{\log(k_0^2 \mu^2)}_{non-local\ tail} - \underbrace{i\pi}_{tail\ flux} \right)$$

Tail flux = flux $\times 2\pi G_N M k_0$ Blanchet PRD '95, Goldberger & Ross PRD '10

$G_d = G_N \mu^{-\epsilon}$

Action for all non-local simple tails

From the flux formula:

$$F = \frac{1}{5}\dddot{Q}_{ij}^2 + \frac{1}{189}\ddddot{O}_{ijk}^2 + \frac{16}{45}\dddot{J}_{ij} + \dots$$

one can then fix the tail cotribution to the flux and the non-local contribution to the self-energy

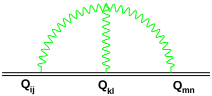
$$\begin{aligned}
\mathcal{S}_{log}^{tail} &= -G_N^2 E \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{2\pi} \log\left(\frac{k_0^2}{\mu^2}\right) \sum_{n\geq 2} k_0^{2(n+1)} c^n_{(I,J)} (I,J)^{\alpha_1\dots\alpha_n}(k_0)(I,J)_{\alpha_1\dots\alpha_n}(-k_0) \\
&= G_N^2 E \int_{-\infty}^{\infty} \mathrm{d}t \sum_{n\geq 2} c^n_{(I,J)} (I,J)^{(n+1)\alpha_1\dots\alpha_n}(t) \int_{-\infty}^{\infty} \mathrm{d}\tau \frac{1}{|\tau|} (I,J)^{(n+1)}_{\alpha_1\dots\alpha_n}(t+\tau)\,,
\end{aligned}$$

with

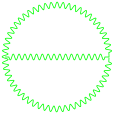
$$\begin{aligned}
c_I^{(n)} &= \frac{(n+1)(n+2)}{n(n-1)n!(2n+1)!!}\,, \\
c_J^{(n)} &= \frac{4n(n+2)}{(n-1)(n+1)!(2n+1)!!}\,.
\end{aligned}$$

as suggested in Damour, Jaranowski, Schäfer PRD '15

Memory integral



$$S^{memory} = -64\pi G_N \int \frac{k_0}{2\pi}\frac{q_0}{2\pi} Q_{ij} Q_{kl} Q_{mn} \int_{\mathbf{kq}} \frac{f_{ijklmn}(\mathbf{k},\mathbf{q},k_0,q_0)}{(\mathbf{q}^2-q_0^2)\left((\mathbf{k}+\mathbf{q})^2-(k_0+q_0)^2\right)(\mathbf{k}^2-k_0^2)}$$



However divergent 2-loop master integral cancel among different polarizations, leaving a finite, local contribution to the self-energy

Far zone self energy results at 5PN with finite terms

Real part $\to$ conservative dynamics (to be added to near zone results, starting 4PN order)
Imaginary part matches into flux formula $F \propto \dddot{Q}_{ij}^2 + \dots$
Divergent graphs regularized in dim. reg.:
divergence (and coeff. of logarithmic term) linked to imaginary part

$$
\begin{aligned}
S_{5PN\,tail} &= G_N^2 M \int \frac{dk_0}{2\pi}\left[-\frac{1}{5}\left(\frac{1}{\epsilon}+\log\left(k_0^2/\bar{\mu}^2\right)-i\pi+\frac{41}{30}\right)|Q_{ij}|^2\right.\\
&\quad -\frac{1}{189}\left(\frac{1}{\epsilon}+\log\left(k_0^2/\bar{\mu}^2\right)-i\pi+\frac{163}{35}\right)|O_{ijk}|^2\\
&\quad \left.-\frac{16}{45}\left(\frac{1}{\epsilon}+\log\left(k_0^2/\bar{\mu}^2\right)-i\pi-\frac{127}{60}\right)|J_{ij}|^2\right]\\
S_{5PN\,Ltail} &= \frac{8}{15}G_N^2\int dt\, \ddddot{Q}_{il}\dddot{Q}_{jl}\epsilon_{ijk}L_k\\
S_{5PN\,memory} &= G_N^2\int dt\left[-\frac{11}{14}\ddddot{Q}_{il}\ddddot{Q}_{jl}Q_{ij}-\frac{1}{5}\dddot{Q}_{il}\dddot{Q}_{jl}\ddot{Q}_{ij}\right]
\end{aligned}
$$

- Imaginary part of self-energy diagrams linked to n-PN flux formula (trivial) and to divergent (and log) part of real part of tails
- Real part combines with near zone dynamics at $(n+4)$-PN, its divergence fixed by flux formula at n-PN
- Log-term is non-instantaneous (but causal) adding to the conservative dynamics
- Log-term becomes instantaneous on circular orbits, contribution to $E(x)$ agrees with 5PN log computed in Le Tiec et al. PRD (2012) and Bini & Damour PRD (2014)
- Finite terms are physical and computable independently of divergences

In-in vs. in-out

- We work with Feynman propagator within the time symmetric in-out framework, which returns *conservative* dynamics and *averaged* emission.
- The in-in formalism is appropriate for time-asymmetric problems, needed to compute *back-reaction* force and *instantaneous* emission, see Galley & Tiglio PRD (2009) and Galley, Leibovich, Porto and Ross (2016)

Conclusions

- 5PN is not so ugly as it seems, qualitatively different from lowest order (finite size effects not forbidden by effacement principle)
- Static sector solved at 5PN (with trivial computations), method helping out for non-static sectors
- Relationship established among
  1. nPN flux
  2. $(n+1.5)$PN tail flux terms
  3. $(n+4)$PN conservative logs
  4. $(n+4)$PN spurious poles
- Uniquely defined finite terms from (non-)hereditary derived at 5PN

Spare slides

Post-Newtonian vs. post-Minkowskian

Post-Minkowskian expansion parameter is $G_N M/r$, vs PN expansion

$$\mathcal{L} = -Mc^2 + \frac{\mu v^2}{2} + \frac{GM\mu}{r} + \frac{1}{c^2}[\ldots] + \frac{1}{c^4}[\ldots]$$

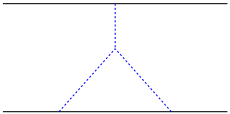
Terms known so far

		N	1PN	2PN	3PN	4PN	5PN	...
0PM	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM					$1/r^4$	v^2/r^4	v^4/r^4	...
5PM						$1/r^5$	v^2/r^5	...
...							$1/r^6$	...

3PM recently computed (!) by Z. Bern et al. PRL (2019)
What's next? Amplitude program with modern methods: generalized unitarity, gravity as a double copy of gauge theory...

Method of regions and far-near zone interplay

Trouble can arise when using method of region in Feyman diagrams:



- In the full theory:

$$
\begin{aligned}
V \quad \supset & \quad \int dt_{1,2,2'} d^4 p \, e^{ip_\mu (x_1^\mu(t_1) - x_2^\mu(t_2))} \frac{p^\alpha p^\beta}{p^2} \int d^4 k \frac{e^{ik^\mu (x_2(t_2) - x_2(t_2'))}}{(p-k)^2 k^2} \\
= & \quad \int dt_{1,2,2'} d^4 p e^{ip_\mu (x_1^\mu(t_1) - x_2^\mu(t_2))} \frac{p^\alpha p^\beta}{p^2} \Delta(p^\mu (x_2(t_2) - x_2(t_2')))
\end{aligned}
$$

which is both IR and UV finite

- Method of regions

$$
\int dt \, d^3 p \, e^{i\vec{p}(\vec{x}_1 - \vec{x}_2)} \frac{p^i p^j}{|\mathbf{p}^2|} \int d^3 k \frac{1}{|\mathbf{k}|^2 |\mathbf{p} - \mathbf{k}|^2} \left(1 + \ldots + \frac{\omega^6}{|\mathbf{k}|^6} + \ldots \right)
$$

$$
= \int dt \, d^3 p e^{i\vec{p}\cdot\vec{x}_{12}} \frac{p^i p^j}{|\mathbf{p}|^3} \left\{ 1 + \ldots + \frac{1}{|\mathbf{p}|^6} \left[(\vec{p}\cdot\vec{v}_1)^3 (\vec{p}\cdot\vec{v}_2)^3 + \ldots + \vec{p}\cdot\dot{\vec{a}}_1 \vec{p}\cdot\dot{\vec{a}}_2 \right] \right\}
$$