Approaches to classical spins
and their gravitomagnetic interaction

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KMPB conference "From Classical Gravity to Quantum Amplitudes and Back:
post-Newtonian, post-Minkowskian, effective one-body, self-force, . . ."
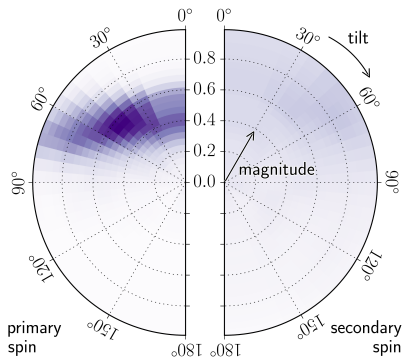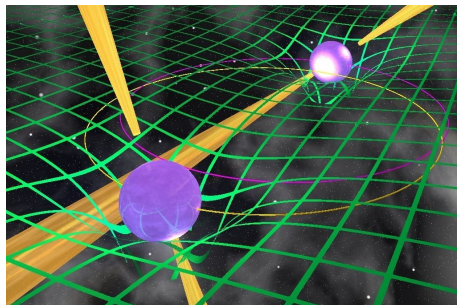Berlin, November 21th, 2019

Outline
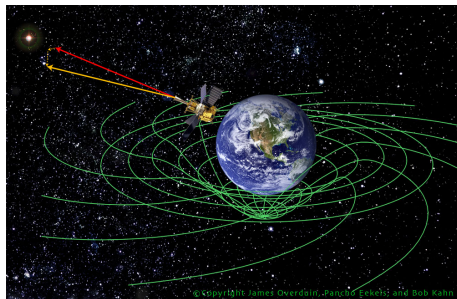
Outline

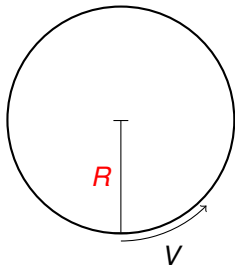# Introduction: Spin and Twisted Spacetime

angular momentum leads to
- gravito-magnetic fields
- dragging of reference frames

spin posterior probability for GW151226

Spinning bodies/particles have a minimal extension

- ring of radius R and mass m
- spin: $S = R\,m\,V$
- maximal velocity: $V \leq c$

$\Rightarrow$ minimal extension:

$$R = \frac{S}{mV} \geq \frac{S}{mc}$$

Center of a spinning body (special relativity)

fast & heavy

v ↑ Δ**x**

spin

slow & light
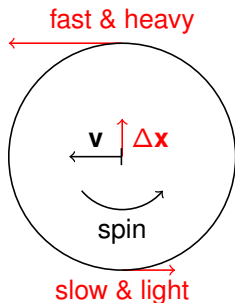
- Spin is a 4-tensor $S^{\mu\nu} = -S^{\nu\mu}$:
  - Spin is $S^{ij} = \epsilon^{ijk} S_k$.
  - Mass dipole related to S^{i0}.
- Different mass centers

Need spin supplementary condition:

- Pryce, Møller, Corinaldesi, Papapetrou: $\quad S^{\mu0} = 0$
- Frenkel, Mathisson, Pirani: $\quad S^{\mu\nu} \dot{x}_\nu = 0$
- Fokker, Tulczyjew, Dixon: $\quad S^{\mu\nu} p_\nu = 0$
- Pryce, Newton, Wigner: $\quad m S^{\mu0} + S^{\mu\nu} p_\nu = 0$

Can be viewed as a gauge freedom

Elementary particles

Wigner:
elementary particle $\rightarrow$ irreducible representation of the Poincaré group

Poincaré = Lorentz $\ltimes$ translations

construct irreducible representations:

- Lorentz group act on linear momentum p_μ
  $\rightarrow$ mass-shell
  e.g., for massive particles: $p_\mu p^\mu = m^2$

- pick representative on mass-shell
  e.g., $p_\mu = k_\mu \equiv (m, 0, 0, 0)$

- little group: Lorentz transformations which leave k_μ invariant
  for massive particles: rotations! $\Rightarrow$ label: $S = 0, \frac{1}{2}, 1, ...$

all irreducible representations of the Poincaré group are given by picking a
mass-shell and an irreducible representation of the little group

e.g., massive elementary particles are characterized by: m, $S = 0, \frac{1}{2}, 1, ...$
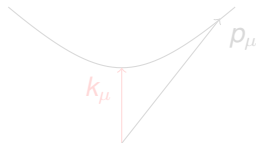
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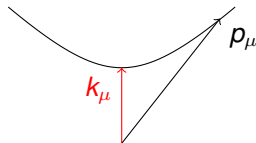
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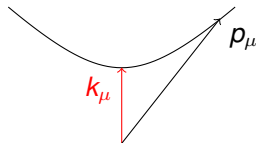e.g., massive elementary particles are characterized by: m, $S = 0, \frac{1}{2}, 1, ...$

Outline

pick configuration space: [Hanson, Regge, Ann. Phys. **87** (1974) 498–566]

$(\Lambda_A{}^\mu, x^\mu) \sim$ Poincaré group, $\Lambda_A{}^\mu \Lambda_B{}^\nu \eta_{\mu\nu} = \eta_{AB}$
$\Rightarrow$ group action on $(\Lambda_A{}^\mu, x^\mu)$ is group multiplication

boost $\Lambda_A{}^\mu \to \tilde{\Lambda}_A{}^\mu$, such that time direction $\tilde{\Lambda}_0{}^\mu = (1, 0, 0, 0)$ in rest-frame

$\tilde{\Lambda}_0{}^\mu$ is invariant under little group
$\tilde{\Lambda}_i{}^\mu$ carries little group SO(3) index $i = 1, 2, 3$

$\Lambda_0{}^\mu$ is redundant/gauge!
  $\to$ what is the generator of gauge trafos $\mathcal{C}_\mu$?
  $\to$ leave physical degrees of freedom $\tilde{\Lambda}_i{}^\mu$ invariant:

$$\{\mathcal{C}_\mu, \tilde{\Lambda}_i{}^\nu\} = 0 \qquad \Leftarrow \qquad \mathcal{C}_\mu = S_{\mu\nu}\left[\frac{p^\nu}{p} + \Lambda_0{}^\nu\right]$$

Configuration space (special relativistic)

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Spin gauge constraint vs spin supplementary condition

generator of gauge transformations:
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gauge transformations affect position:
$$\Delta x^\mu = \{\epsilon^\alpha \mathcal{C}_\alpha, x^\mu\} \neq 0$$

gauge generator $\mathcal{C}_\mu$ $\leftrightarrow$ first-class constraint

$\Rightarrow$ $\mathcal{C}_\mu = 0$ $\Rightarrow$ $\{\mathcal{C}_\mu, \mathcal{C}_\nu\} = 0$

fast & heavy

v Δ**x**

spin

slow & light

gauge fixing $\Rightarrow$ spin supplementary condition

$\Lambda_0{}^\mu = \dfrac{p^\mu}{p}$ $\Rightarrow$ $S_{\mu\nu}p^\nu = 0$ Fokker, Tulczyjew, Dixon (covariant)

$\Lambda_0{}^\mu = \delta_0^\mu$ $\Rightarrow$ $S_{\mu\nu}(p^\nu + p\delta_0^\nu) = 0$ Pryce, Newton, Wigner (canonical)

$\Lambda_0{}^\mu = \dfrac{2p^0\delta_0^\mu - p^\mu}{p}$ $\Rightarrow$ $S_{\mu 0} = 0$ Pryce, Møller, Corinaldesi, Papapetrou

Frenkel, Mathisson, Pirani: $S_{\mu\nu}\dot{X}^\nu = 0$

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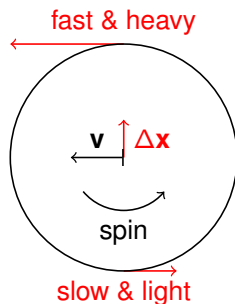$\Lambda_0{}^\mu = \dfrac{p^\mu}{p}$ $\Rightarrow$ $S_{\mu\nu}p^\nu = 0$ — Fokker, Tulczyjew, Dixon (covariant)

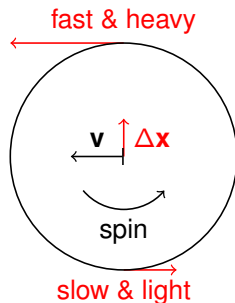$\Lambda_0{}^\mu = \delta_0^\mu$ $\Rightarrow$ $S_{\mu\nu}(p^\nu + p\delta_0^\nu) = 0$ — Pryce, Newton, Wigner (canonical)

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Action principle and coupling to gravity
ADM-type form, see talk by Gerhard Schäfer

$$S = \int \underbrace{-p_\mu dx^\mu - \tfrac{1}{2} S_{\mu\nu} \Lambda^{A\mu} d\Lambda_A{}^\nu}_{p\,dq \sim \text{canonical form}} - \underbrace{\left[\lambda(p^2 - m^2) + \chi^\mu \mathcal{C}_\mu\right]}_{\text{Dirac Hamitonian } H_D} d\tau$$

use gauge invariant variables in $H_D(\tilde{x}^\mu, \tilde{S}_{\mu\nu}, \dots)$: $\{\mathcal{C}_\alpha, \tilde{x}^\mu\} = 0 = \{\mathcal{C}_\alpha, \tilde{S}_{\mu\nu}\}$

rewrite canonical form in terms of $\tilde{x}^\mu$ before coupling to gravity:

$$\Rightarrow \qquad S = \int d\tau \left\{ -p_\mu \frac{D}{d\tau} \underbrace{\left[\tilde{x}^\mu - S^{\mu\nu} \frac{p_\nu}{p^2}\right]}_{x^\mu} - \frac{1}{2} S_{\mu\nu} \underbrace{\Lambda^{A\mu} \frac{D\Lambda_A{}^\nu}{d\tau}}_{\Omega^{\mu\nu}} - H_D \right\}$$

this action does not lead to the Mathisson-Papapetrou-Dixon equations!

parallel transport worldline to x^μ, redefine linear momentum
  $\rightarrow$ new action leads to the Mathisson-Papapetrou-Dixon equations
    [Vines, Kunst, Steinhoff, Hinderer, PRD **93** (2016) 103008]
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Equations of motion

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$$\frac{Dp_\mu}{d\tau} = 0 \ -\frac{1}{2}R_{\mu\rho\beta\alpha}u^\rho S^{\beta\alpha} \ -\frac{1}{6}\nabla_\mu R_{\nu\rho\beta\alpha}J^{\nu\rho\beta\alpha} \ -\frac{1}{2}\nabla_\mu F_{\alpha\beta}D^{\alpha\beta} + ...$$

$$\frac{DS^{\mu\nu}}{d\tau} = 2p^{[\mu}u^{\nu]} \ +\frac{4}{3}R^{[\mu}{}_{\rho\alpha\beta}J^{\nu]\rho\alpha\beta} \ + 2D^{\alpha[\mu}F^{\nu]}{}_\alpha + ...$$

$$J^{\mu\nu\alpha\beta} \propto \frac{\partial H_D}{\partial R_{\mu\nu\alpha\beta}}, \quad D^{\mu\nu} \propto \frac{\partial H_D}{\partial F_{\mu\nu}}$$

- Geodesic equation: momentum p_μ
- Mathisson (1937), Papapetrou (1951):   spin / dipole $S^{\mu\nu}$
- Dixon ($\sim$1974):   (see talk by Justin Vines)   quadrupole $J^{\mu\nu\alpha\beta}$, ...

Traditional approach: $T^{\mu\nu}{}_{;\nu} = 0 \ \rightsquigarrow \ $ EOM
That is, EOM for p_μ and $S^{\mu\nu}$ follow from generic principles!

Action approach: assume generic covariant action
Simple, but more restrictive. Still the resulting EOM have the same form!

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Action approach: \hspace{0.5cm} assume generic covariant action
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Outline

Canonical structure

$$S = \int \underbrace{-p_\mu dx^\mu - \tfrac{1}{2} S_{\mu\nu} \Lambda^{A\mu} D \Lambda_A{}^\nu}_{B \,=\, \text{canonical form}} - H_D \, d\tau$$

Equations of motion for phase space coordinates $q = (p_\mu, x^\mu, S_{\mu\nu}, \Lambda_A{}^\mu)$:

$$\dot{q}^n = M^{nm} \frac{\partial H_D}{\partial q^m} = \{H_D, q^n\},$$

where $\quad M = (dB)^{-1}, \quad \{X, Y\} = M^{mn} \dfrac{\partial X}{\partial q^n} \dfrac{\partial Y}{\partial q^m}$

Tradeoff: simple M vs. simple Hamiltonian vs. #DOF

4D phase space

$$S = \int \underbrace{-p_\mu dx^\mu - \frac{1}{2}S_{\mu\nu}\Lambda^{A\mu}D\Lambda_A{}^\nu}_{B \,=\, \text{canonical form}} -H_D \, d\tau$$

Use spin in local Lorentz frame, $e_{a\mu}e_b{}^\mu = \eta_{ab}$:

$$B = B_n(q) \, dq^n \quad = -\underbrace{\left(p_\mu + \frac{1}{2}\omega_\mu{}^{ab}S_{ab}\right)}_{P_\mu} dx^\mu - \tfrac{1}{2}S_{ab}\Lambda^{Aa}d\Lambda_A{}^b$$

Poisson brackets: (all other zero)

$$\{x^\mu, P_\nu\} = \delta^\mu_\nu, \quad \{\Lambda_A{}^c, S_{ab}\} = 2\Lambda_A{}^d\eta_{d[a}\delta^c_{b]}, \quad \{S_{ab}, S_{cd}\} = S_{ac}\eta_{bd} - ...$$

What to do with λ, ξ^μ in H_D?
E.g. gauge fixing $\lambda = \text{const}$, $\xi^\mu = 0 \quad \Rightarrow \quad H_D \propto g^{\mu\nu}p_\mu p_\nu$

$$S = \int -\underbrace{\left(p_\mu + \frac{1}{2}\omega_\mu{}^{ab}S_{ab}\right)}_{P_\mu}dx^\mu - \frac{1}{2}S_{ab}\Lambda^{Aa}d\Lambda_A{}^b - H_D\,d\tau$$

make gauge choices $\tau = t = x^0$ and $\Lambda_0{}^a = \delta_0^a \Rightarrow \Lambda_A{}^0 = \delta_A^0$
and solve constaints ($H_D = 0$):

$$S = \int \underbrace{-P_i dx^i - \frac{1}{2}S_{ij}\Lambda^{ki}d\Lambda_k{}^j}_{\text{new canonical form}} -(H = P_0)\,dt$$

Here i, j, k, ℓ run through $1, 2, 3$. $H \equiv P_0$ from $g^{\mu\nu}p_\mu p_\nu = m^2$.

Poisson brackets: (all other zero)

$$\{x^i, P_j\} = \delta_j^i, \quad \{\Lambda_\ell{}^k, S_{ij}\} = -2\Lambda_\ell{}^m\delta_{m[i}\delta_{j]}^k, \quad \{S_{ij}, S_{k\ell}\} = -S_{ik}\delta_{j\ell} - ...$$

Literature

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. . .

Outline

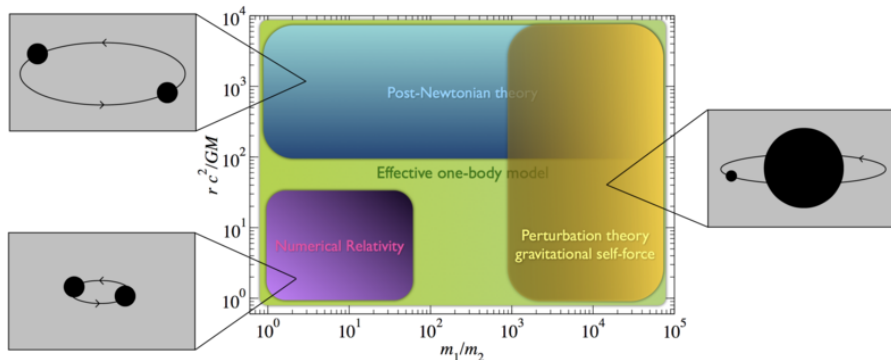# Approaches to the relativistic binary problem

image credit: A. Buonanno, B.S. Sathyaprakash

Spin and Gravitomagnetism

Interaction with gravito-magnetic field $A_i \approx -g_{i0}$:

$$\frac{1}{2} S_{\mu\nu} \Lambda_A{}^\mu D \Lambda^{A\nu} \rightsquigarrow \frac{1}{2} S^{ij} \partial_i A_j$$

$\rightarrow$ universal for all objects!
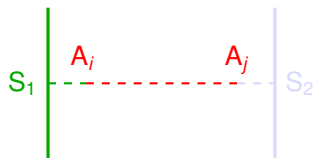

see talk by Michele Levi

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L_{S_1 S_2} &= \frac{1}{2} S_1^{ki} \langle \partial_k A_i \, \partial_\ell A_j \rangle \, \frac{1}{2} S_2^{\ell j} \qquad \text{[ignoring time integrals and } \delta(t_1 - t_2) \text{ factors]} \\
&= \frac{1}{2} S_1^{ki} \frac{1}{2} S_2^{\ell j} \, \delta_{ij}(-16\pi G) \frac{\partial}{\partial x_1^k \partial x_2^\ell} \int \frac{dk}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2} \\
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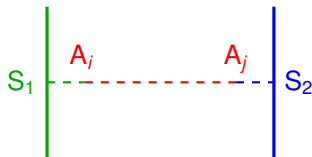$\rightarrow$ universal for all objects!

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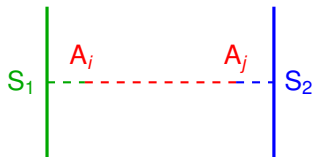
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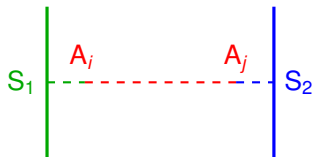$\rightarrow$ universal for all objects!

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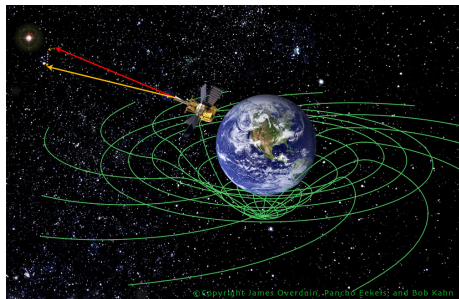
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Frame dragging

angular momentum/spin leads to
- gravito-magnetic effects
- dragging of reference frames

Orbital angular momentum $L_{ij} = 2x_{[i}p_{j]}$ generates rotations of the orbit

$$\{x^k, L_{ij}\} = -x^i\delta_{jk} + x^j\delta_{ik}, \quad \{p_k, L_{ij}\} = -p_i\delta_{jk} + p_j\delta_{ik}$$

Spin generates rotations of the body-fixed frame $\Lambda_\ell{}^k$:

$$\{\Lambda_{\ell k}, S_{ij}\} = -\Lambda_{\ell i}\delta_{jk} + \Lambda_{\ell j}\delta_{ik}$$

$\Rightarrow$ spin interactions in the Hamiltonian rotate the body-fixed frame over time!

Results for the post-Newtonian potential

conservative part of the motion of the binary

post-Newtonian (PN) approximation: expansion around $\frac{1}{c} \to 0$ (Newton)

order	c^0	c^{-1}	c^{-2}	c^{-3}	c^{-4}	c^{-5}	c^{-6}	c^{-7}	c^{-8}
	N		1PN		2PN		3PN		4PN
non spin	✔		✔		✔		✔		✔
spin-orbit				✔		✔		✔	
Spin^2					✔		✔		✔
Spin^3								✔	
Spin^4									✔
⋮									⋱

Work by many people ("just" for the spin sector): Barker, Blanchet, Bohé, Buonanno, O'Connell, Damour, D'Eath, Faye, Hartle, Hartung, Hergt, Jaranowski, Marsat, Levi, Ohashi, Owen, Perrodin, Poisson, Porter, Porto, Rothstein, Schäfer, Steinhoff, Tagoshi, Thorne, Tulczyjew, Vaidya

Code for the spin part using EFT: M. Levi, JS, CQG **34** (2017), 244001

dynamical tides:
orbital motion can excite
oscillation modes

gravitomagnetism:
$\rightarrow$ frame dragging effect
$\sim$ Zeeman effect

also: redshift effect
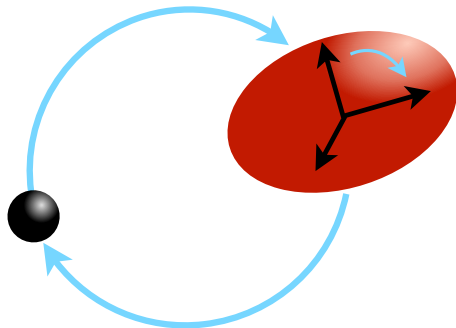
frame of the neutron star is dragged
in the direction of the orbital motion

Dynamic tides and tidal spin

$$S = \int \underbrace{-p_\mu \, dx^\mu - P_{\mu\nu} DQ^{\mu\nu}}_{B} - \underbrace{\left\{ \lambda \left[p^2 - (m + H_t)^2 \right] + ... \right\}}_{H_D} d\tau$$

with the tidal (harmonic oscillator) Hamiltonian:

$$H_t = \frac{1}{2m_Q} P_{\mu\nu} P^{\mu\nu} + \frac{m_Q \omega_0^2}{2} Q_{\mu\nu} Q^{\mu\nu} + \frac{1}{2} E_{\mu\nu} Q^{\mu\nu} = ...$$

The tidal spin $S_Q^{\mu\nu} = 4Q^{\rho[\mu} P^{\nu]}{}_\rho$ shows up in the canonical form:

$$B = - \underbrace{\left(p_\mu + \frac{1}{2} \omega_{\mu ab} S_Q^{ab} \right)}_{P_\mu} dx^\mu - P_{ab} \, dQ^{ab}$$

Equations of motion

Vary the action:

$$\frac{Dp_\mu}{d\tau} = \frac{1}{2} S_Q^{\alpha\beta} R_{\alpha\beta\rho\mu} \dot{x}^\rho - \frac{1}{6} \nabla_\mu R_{\alpha\rho\beta\sigma} \left[J_Q^{\alpha\rho\beta\sigma} = -\frac{3}{2m^3\lambda} p^{[\alpha} Q^{\rho][\beta} p^{\sigma]} \right]$$

$$\frac{1}{2\lambda m} \frac{DP_{\mu\nu}}{d\tau} = -m_Q \omega_0^2 Q_{\mu\nu} - \frac{1}{2} E_{\mu\nu}$$

$$\frac{1}{2\lambda m} \frac{DQ^{\mu\nu}}{d\tau} = \frac{P^{\mu\nu}}{m_Q}$$

Spin EOM is not fundamental, the EOMs for $Q^{\mu\nu}$ and $P_{\mu\nu}$ are!

Using $S_Q^{\mu\nu} = 4Q^{\rho[\mu} P^{\nu]}{}_\rho$:

$$\frac{DS_Q^{\mu\nu}}{d\tau} = 2p^{[\mu} \dot{x}^{\nu]} + \frac{4}{3} R_{\alpha\beta\rho}{}^{[\mu} J_Q^{\nu]\rho\beta\alpha}$$

Agreement with the Mathisson-Papapetrou-Dixon equations of motion!

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Poisson brackets

$$B = -\underbrace{\left(p_\mu + \frac{1}{2}\omega_{\mu ab}S_Q^{ab}\right)}_{P_\mu} dx^\mu - P_{ab}\,dQ^{ab}$$

4D Poisson brackets:

$$\{x^\mu, P_\nu\} = \delta_\nu^\mu, \quad \{Q^{ab}, P_{cd}\} = 2\eta_c^{(a}\eta_d^{b)}$$

Derived brackets:

$$\{S_Q^{ab}, S_Q^{cd}\} = \eta^{ac}S_Q^{bd} - ...$$

$$\{Q^{ab}, S_Q^{cd}\} = \eta^{ad}Q^{cb} - ..., \quad \{P^{ab}, S_Q^{cd}\} = \eta^{ad}P^{cb} - ...$$

3D Poisson brackets similar.

The tidal spin S_Q^{ij} generates rotations of Q^{ij} and P_{ij} $\quad \rightarrow \quad$ frame dragging

Conclusions

Spin presents its own challenges:

- definition of center
- supplementary conditions

ADM worldline action with spin-gauge symmetry very useful:

- connecting various incarnations of action principles
  and canonical formalisms
- close connection to quantum fields
  e.g., $\lambda(g^{\mu\nu}p_\mu p_\nu + m^2)$ vs. $\sqrt{-g}(g^{\mu\nu}\partial_\nu\phi\partial_\nu\phi + m^2\phi^2)$
- useful for double-copy constructions, see talk by Jan Plefka

Many things omitted and to do. . .
eikonal limit of the action with spinning fields, spinor helicity, twistor actions. . .

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