Scattering Amplitudes and the Spinor-Helicity Formalism

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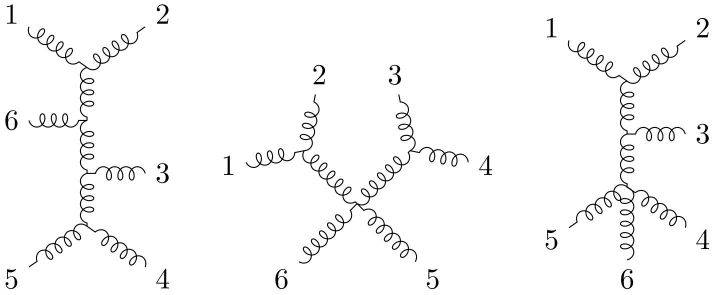
Introduction

- Scattering amplitudes are important physical observables
- In many cases (QCD, gravity), Feynman diagrams are an inefficient approach to compute scattering amplitudes

Comparison between SHF and Feynman diagrams

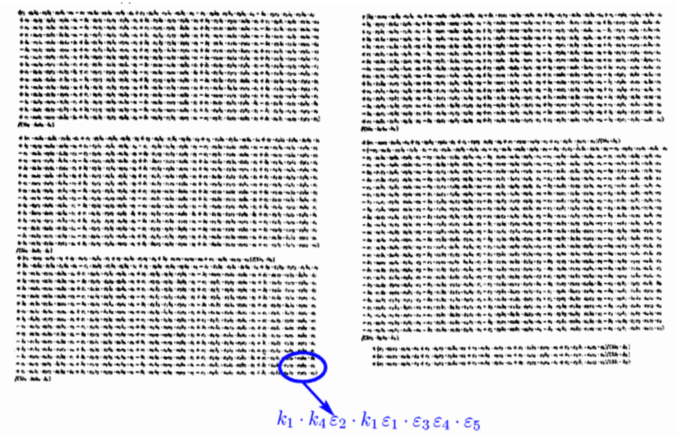
- Six-gluon tree amplitude



- 220 diagrams are needed

Comparison between Spinor-Helicity Formalism (SHF) and Feynman diagrams

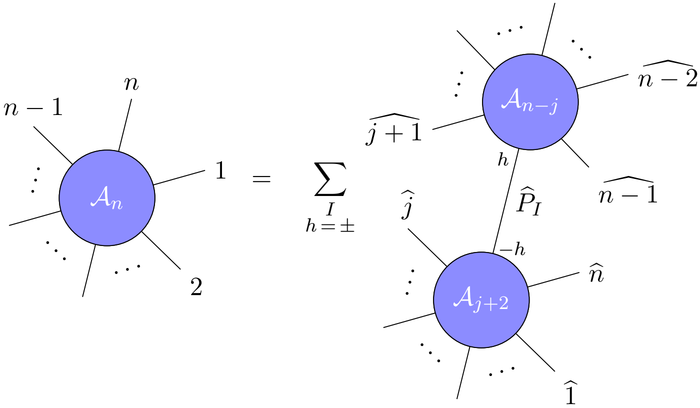
- Brute force approach



- Plus ∼100 pages

Comparison between SHF and Feynman diagrams

SHF exploits instead symmetries, dimensional analysis and recursion relations



Comparison between SHF and Feynman diagrams

- If all helicities are equal or all but one are equal, the amplitude vanishes
- The first non-vanishing result is remarkably simple

$$\mathcal{A}_6[1^-2^-3^+4^+5^+6^+] = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}$$

- And can be generalized to n gluons!

$$\mathcal{A}_n[1^+2^+\ldots i^-\ldots j^-\ldots n^+] = \frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}$$

$$\mathcal{A}_n[1^-2^-\ldots i^+\ldots j^+\ldots n^-] = \frac{[i\,j]^4}{[1\,2][2\,3]\ldots[n\,1]}$$

Massless spinor-helicity variables

- Map between 4-vectors and 2×2 matrices

$$p^\mu \mapsto p_{a\dot{b}} = p_\mu (\sigma^\mu)_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix}$$

- Square and angle spinors are defined by

$$p_{a\dot{b}} = -|p]_a \langle p|_{\dot{b}}$$

Massless spinor-helicity variables

- Angle and square spinors are solution to Dirac equation for $m = 0$

$$\bar{u}_+(p) = \begin{pmatrix} [p|^a & 0 \end{pmatrix}, \qquad \bar{u}_-(p) = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix},$$

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}$$

- Product of spinors

$$\langle p\, q\rangle = \langle p|_{\dot{a}} |q\rangle^{\dot{a}}, \qquad [p\, q] = [p|^a |q]_a, \qquad [p\, q\rangle = 0$$

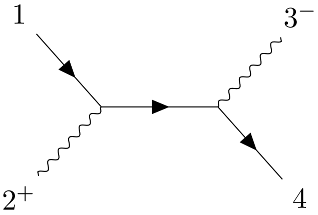
Massive spinor-helicity variables

- In the massive case $(I = 1, 2)$

$$p_{a\dot{b}} = -|p^I]_a \langle p_I|_{\dot{b}} = -|\mathbf{p}]_a \langle \mathbf{p}|_{\dot{b}}$$

- A massive state with spin S is a symmetric tensor with rank $2S$
- Three-point amplitudes can be computed for all external legs

Example: Compton scattering



$$\mathcal{A}^S[\mathbf{1}2^+3^-\mathbf{4}] = \begin{cases} \frac{(\langle 3|p_1|2])^2}{(s-m^2)(u-m^2)}\,, & S=0 \\ \frac{\langle 3|p_1|2](\langle \mathbf{1}\,3\rangle[\mathbf{4}\,2]+[\mathbf{1}\,2]\langle \mathbf{4}\,3\rangle)}{(s-m^2)(u-m^2)}\,, & S=1/2 \\ \frac{(\langle \mathbf{1}\,3\rangle[\mathbf{4}\,2]+[\mathbf{1}\,2]\langle \mathbf{4}\,3\rangle)^2}{(s-m^2)(u-m^2)}\,, & S=1 \end{cases}$$

Proposal of a UV completion of gravity

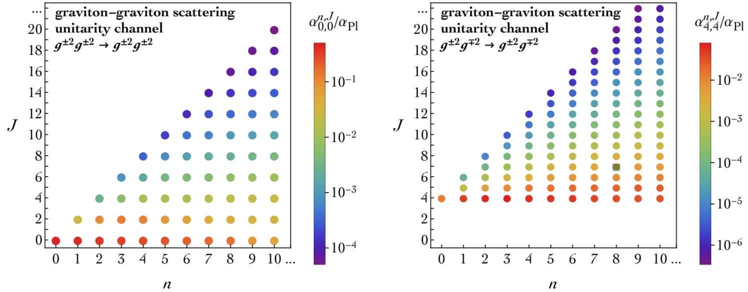
- Gravity is a not renormalizable theory
- We expect general relativity to be an EFT of "something else"
- A proposed, top-down solution is string theory

A bottom-up approach to UV completion

- Recently, SHF was used to develop a UV completion of gravity
- GR scattering amplitudes are UV-completed introducing massive resonances
- The corresponding amplitudes are multiplied either by Veneziano or Virasoro-Shapiro amplitudes

A bottom-up approach to UV completion

Massive resonances have quantized mass ($M_n^2 = (n+1)M^2$, $n = 0, 1, 2, \ldots$) and increasing spin



Open questions

- Are there other constraints on the free parameters of the UV completion?
- Is the construction valid up to the Planck mass?
- Is the solution unique?

Suggested references

H. Elvang and Y. t. Huang, "Scattering Amplitudes," arXiv:1308.1697 [hep-th].

N. Arkani-Hamed, T. C. Huang and Y. t. Huang, "Scattering Amplitudes For All Masses and Spins," arXiv:1709.04891 [hep-th].

R. Alonso and A. Urbano, "On amplitudes, resonances and the ultraviolet completion of gravity," arXiv:1906.11687 [hep-ph].

Example of massless spinor-helicity variables

If

$$p^\mu = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

massless spinor-helicity variables are

$$|p\rangle^{\dot{a}} = \sqrt{2E}\begin{pmatrix} c_{\theta/2} \\ s_{\theta/2}e^{i\phi} \end{pmatrix}, \quad [p|^{a} = \sqrt{2E}\begin{pmatrix} c_{\theta/2} \\ s_{\theta/2}e^{-i\phi} \end{pmatrix}^T$$

$$|p]_a = \sqrt{2E}\begin{pmatrix} -s_{\theta/2}e^{-i\phi} \\ c_{\theta/2} \end{pmatrix}, \ \langle p|_{\dot{a}} = \sqrt{2E}\begin{pmatrix} -s_{\theta/2}e^{i\phi} \\ c_{\theta/2} \end{pmatrix}^T$$

Spinor-helicity variables in four-particle scattering

Consider the scattering $1\,2 \to 3\,4$ in CM reference frame

$$
\begin{aligned}
p_1^\mu &= E(1,0,0,1)\,, \qquad & -p_3^\mu &= E(1,\sin\theta,0,\cos\theta) \\
p_2^\mu &= E(1,0,0,-1)\,, \qquad & -p_4^\mu &= E(1,-\sin\theta,0,-\cos\theta)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\langle 1\,2\rangle &= \sqrt{s}\,, & \langle 3\,4\rangle &= \sqrt{s} \\
\langle 1\,3\rangle &= \sqrt{s}\sin\tfrac{\theta}{2}\,, & \langle 1\,4\rangle &= \sqrt{s}\cos\tfrac{\theta}{2} \\
\langle 2\,3\rangle &= -\sqrt{s}\cos\tfrac{\theta}{2}\,, & \langle 2\,4\rangle &= -\sqrt{s}\sin\tfrac{\theta}{2}
\end{aligned}
$$

Introduction of massive resonances

- Graviton mediated scattering amplitude for distinguishable scalars

$$\mathcal{A}_{\phi\phi\varphi\varphi} = \frac{8\pi}{M_{\text{Pl}}^2}\left(\frac{tu}{s} - as\right)$$

- In the decomposition

$$\mathcal{A}_{\phi\phi\varphi\varphi}(\theta) = \sum_J a_J(s) P_J(\cos\theta)$$

$a_J(s) \propto s$

- One is forced to introduce a massive resonance with $J > 2$. But then another coefficient $a_J(s)$ with bad high energy behaviour appears
- One is forced to introduce an infinite number of resonances

Tower of resonances

Perturbative unitarity, causality and locality imply the presence of *two* tower of resonances, with masses

$$M_n^2 = nM^2 \qquad \hat{M}_n^2 = \hat{M}_0^2 + n\hat{M}^2$$

for $n = 1, 2, \ldots$

Modified gravity amplitudes

The general form of a scattering amplitude is

$$\mathcal{A}[1^h 2^{-h} 3^{-h'} 4^{h'}] = \mathcal{A}^{\text{GR}}[1^h 2^{-h} 3^{-h'} 4^{h'}] \times \begin{cases} \mathcal{A}_{\text{VZ}}^{\eta,\gamma_0} \\ \mathcal{A}_{\text{VS}}^{\gamma_0} \end{cases}$$

with

$$\mathcal{A}_{\text{VZ}}^{\eta,\gamma_0}(s,t) = \frac{\Gamma(1-\tilde{s})\Gamma(1+\eta\gamma_0-\eta\tilde{t})}{\Gamma(1+\eta\gamma_0-\eta\tilde{t}-\tilde{s})}$$

$$\mathcal{A}_{\text{VS}}^{\gamma_0}(s,t,u) = \frac{\Gamma(1+2\gamma_0)\Gamma(1+\gamma_0-\tilde{u})\Gamma(1+\gamma_0-\tilde{t})\Gamma(1-\tilde{s})}{\Gamma(1+\tilde{u}+\gamma_0)\Gamma(1+\tilde{t}+\gamma_0)\Gamma(1+\tilde{s}+2\gamma_0)}$$

$$\tilde{s} = \frac{s}{M^2}\,, \qquad \tilde{t} = \frac{t}{M^2}\,, \qquad \eta = \frac{M^2}{\hat{M}^2}\,, \qquad \gamma_0 = \frac{\hat{M}_0^2}{M^2}$$

Modified gravity amplitudes

All four-particle tree level amplitudes

$\mathcal{A}^{\mathrm{UV}}$	Scalar	Fermion	Vector	Graviton
Scalar	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}^{\gamma_0}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}^{\gamma_0}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$
Fermion		$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VZ}}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VZ}}^{\eta,\gamma_0}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$
Vector			$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VZ}}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$
Graviton				$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$

Unitarity constraints impose

$$\gamma_0 < \frac{3}{2}, \qquad \eta = 1$$