Loops and Legs:

Methods and Results

O.L. Veretin

LEXI

LANDESEXZELLENZINITIATIVE HAMBURG

LEXI Particle Phenomenology Group at UHH/DESY Th

Pheno UHH | **High Energy QCD** | **LEXI** DESY TH

PEOPLE

Pheno UHH	High Energy QCD	LEXI DESY TH
S. Albino	J. Bartels	A. Ali
M. Butenschön	J. Kotanski	M. Diehl
M.Yu. Kalmykov	L. Lipatov	W. Khater
B.A. Kniehl	L. Motyka	S. Mishima
A.V. Kotikov	A. Prygarin	J.-H. Park
G. Moortgat-Pick		T. Underwood
O.V. Tarasov		G. Weiglein
O.L. Veretin		

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Pheno UHH

- fragmentation functions
- NRQCD:
  heavy quarkonia
  threshold production
- mixing renormalization
- unstable particles
- production of $W/Z/H$
- precision tests of SM
- collider phenomenology
- SUSY searches
- multiloop calculations

High Energy QCD

- higher twists in DIS
- BFKL equation
- AdS/CFT correspondence
- BDS Ansatz in $N = 4$ SUSY QCD

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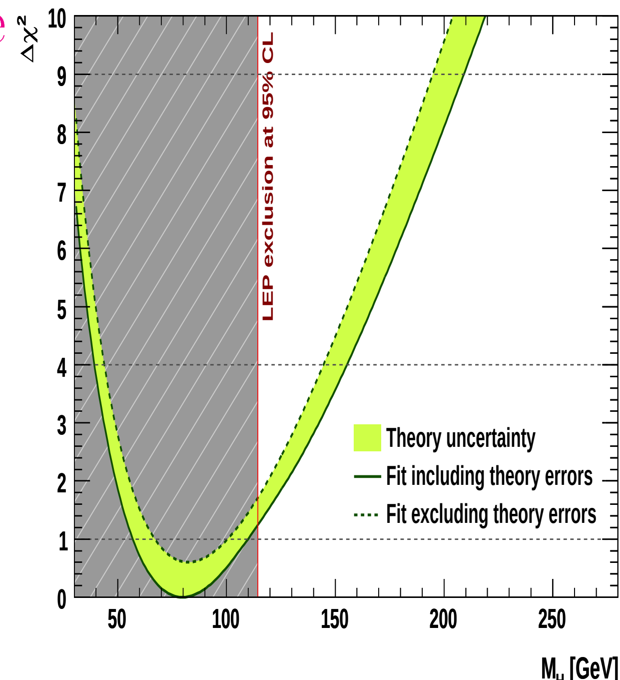
- collider phenomenology
- SUSYPERSYMMETRY
- electroweak symmetry breaking
- QCD in ep, e^+e^- and pp collisions
- exclusive processes in QCD
- factorization in QCD
- flavour physics
- extensions of the SM
- neutrino physics
- CP violation

introduction

- The past and current collider experiments show that the Standard Mode based on the gauge principle and the spontaneous symmetry breaking, the adequate theory up to few 100 GeV
- the only missing particle in the SM prediction is the Higgs boson
- though many problems remain that cannot be solved within the SI (hierarchy, unification, ...)
- radiative corrections are important and sizeable
  - constrains for m_t from RC
  - constraints for m_H
  - precision tests



introduction

- the Large Hadron Collider (LHC) starts to operate soon
  - goals:
    - searches for the Higgs boson
    - searches for the new physics
  - features:
    - new frontier in energy ($\sqrt{S} \sim 14$ TeV) and luminosity
    - radiation of quarks and gluons (jets) and chain decays $\Longrightarrow$ high-multiplicity events
- challenges for the theory
  - many legs (4,5,6,. . .)
  - many loops (1,2,3,. . .)
  - many scales

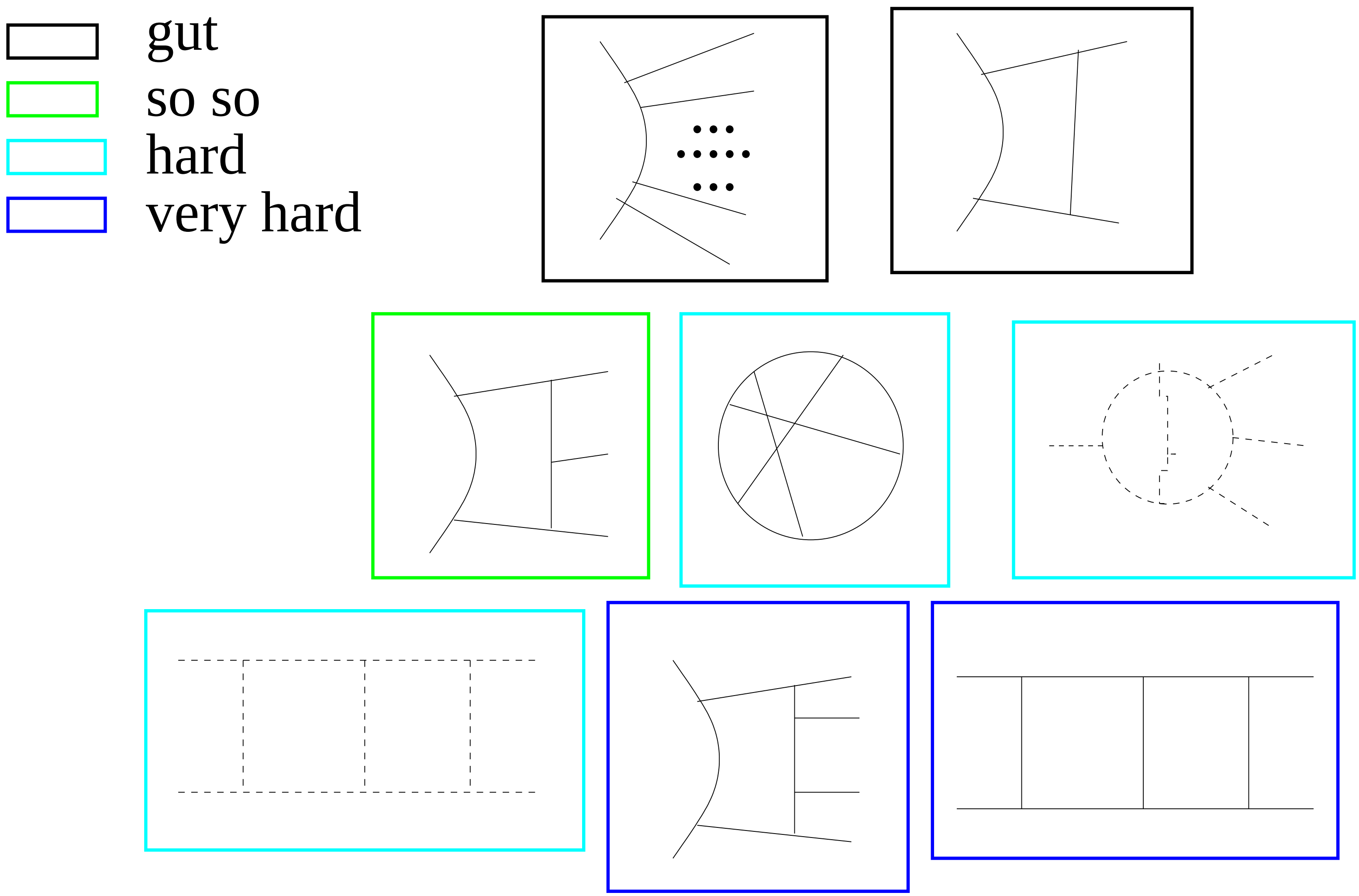
challenges and outlook for the theory

- LO suffers from large scale uncertainty
  $\Longrightarrow$ NLO corrections are important
  - QCD corrections are large (up to 100%!)
  - electroweak corrections are sizeable
- NLO calculations are required
  - Les Houches wishlist: $VVj, t\bar{t}jj, VVb\bar{b}, VVjj, Vjjj, VVV$
- QCD NNLO corrections for some processes
  - $pp \longrightarrow W^*W^*/t\bar{t}/H/W/Z$, $pp \longrightarrow Z/\gamma + j$
  - H decay etc.
- mixed EW/QCD NNLO
- resummations of large log's

problems

- number and complexity of diagrams grow very fast
- complicated structure of infrared and collinear singularities
  $\Longrightarrow$ needs special treatment
  - space slicing
  - dipole/antenna subtraction
- multidimensional numerical integrations
- numerical stability
- sufficient speed

how are we doing



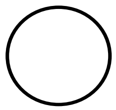
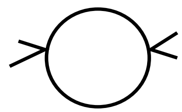
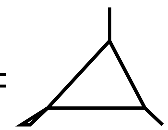
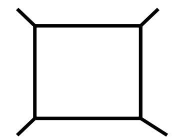
advanced methods

- integration by parts (IBP)
- differential equations
- difference equations
- functional equations
- sector decompositions
- Mellin–Barns representations
- unitarity methods
- ...

“Unitarity cut” methods (1-loop)

Any n-point 1 -loop amplitude can be decomposed into sum of scala
integrals, e.g.

$$M_4 = P_{4,0} D + \sum P_{3,j} C_j + \sum P_{2,jl} B_{jl} + \sum P_{1,jln} A_{jln}$$

where A=  B=  C=  D= 

and P's rational functions of kinematical invariants. The problem is to fin
P's.

If one can evaluate all coefficients P's, the problem is solved!
$(A, B, C, D$ are known)

“Unitarity cut” methods (continue)

The basic idea:

- **evaluate all possible discontinuities**
  make “unitary cuts” (Cutkosky rule): $\frac{1}{k^2-m^2} \to 2\pi\delta^+(k^2-m^2)$
- **coefficients P’s are not cut, but M_4, A’s, B’s, C’s and D are**
- **evaluate (numerically) remaining integrals (no loops anymore!**
  integrals over *tree* amplitudes
- **get the relations between P’s**
- **having enough such relations solve them for P’s. Done.**

Advantages and disadvantages:

- $+$: **one can work directly with amplitudes (no diagrams!)**
- $-$: **amplitudes are obtained by Monte Carlo integration**

integration by parts (IBP)

$$\int udv = uv - \int vdu \qquad \longrightarrow \qquad \int d(uv) = uv|_{\text{boundaries}}$$

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in dimension d:

$$\int d^d k \, \frac{\partial J^\mu(k, \ldots)}{\partial k_\mu} = \oint J^\mu \, d\Sigma_\mu \longrightarrow 0$$

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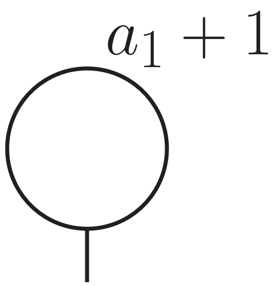
for L-loop integral, any r, a_j, q, D_j:

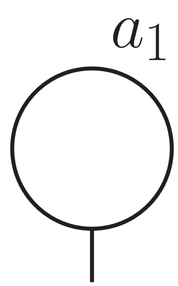
$$\int d^d k_1 \ldots d^d k_L \;\; \frac{\partial}{\partial k_\mu} \frac{q_\mu}{D_1^{a_1} D_2^{a_2} \ldots D_r^{a_r}} = 0$$

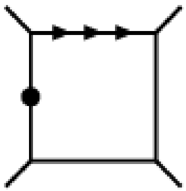
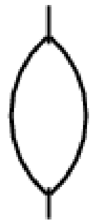
$\Longrightarrow$ algebraic relations among integrals with different sets of $(a_1, a_2, \ldots, a_r)$
and numerators

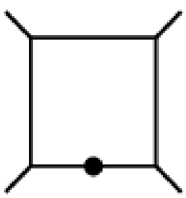
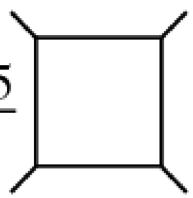
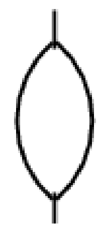
all together $L(L+E)$ relations

examples:

$2m^2 a_1$ 

$=$ $(d - 2a_1)$ 

 $= \frac{d(u+s) - 4s - 2t - 6u}{(d-3)(d-4)t^2}$ 

 $= -\frac{d-5}{s}$  $- \frac{4}{(d-3)(d-5)(d-6)\,s\,t\,(s+u)}$ 

$+ \frac{4}{(d-3)(d-5)(d-6)\,s\,(s+u)\,(s+t+u)}$ 

IBP consequences

- this works also for any number of loops, legs, scales ...
- these relations can be use to express some integrals in terms of others $\Longrightarrow$ reduction
- this is the powerfull tool
  - express any Feynman integral in terms of *master itegrals* M_j

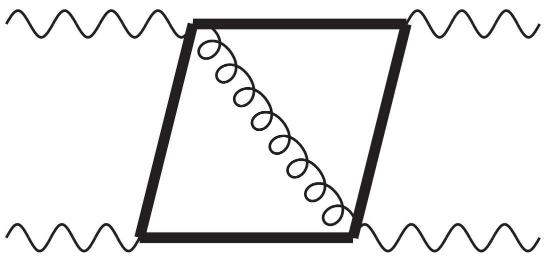
$$I = \sum_{j=1}^{n} R_j M_j,$$

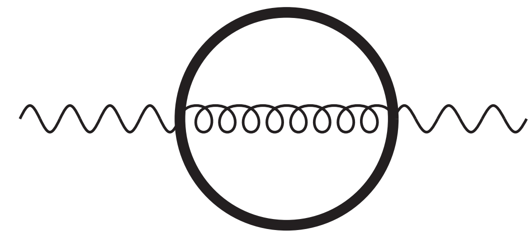
with rational functions $R_j(\text{dimension}, \text{external invariants})$

- IBP give us *difference* equation(s), that can be transformed further into *differential* equation(s)
- all corresponding equations are linear
- IBP should be taken in proper combinations

Differential equations

can be written in any invariant parameters (masses, kinematical invariants

$J(s,x) =$ 

$S(s,x) =$ 

Equation in $x = s/m^2$

$$x(1-x)J' + (1-x)J = \frac{x}{(1+x)^2}\left[4S + 2(1+4x+4x^2)S'\right]$$

Solution:

$$J(x) = \frac{C_1}{x} + \frac{1}{x}\int^x \frac{dx_1}{1-x_1}\mathrm{RHS}(x_1)$$

Differential equations

- all Feynman diagrams are subjects to linear differential equations
- IBP identities can be used to find DE's
- in addition to DE's one has to provide corresponding boundary conditions
- solutions
  - numerically
  - as (multiple) series of hypergeometric type

Hypergeometric functions

$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$

$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sin x$

$\sum_{n=0}^{\infty} c_n x^n$, where $\frac{c_{n+1}}{n}$ is rational in n

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$\sum_{n=0}^{\infty} c_n x^n,$ where $\frac{c_{n+1}}{n}$ is rational in n

$$\sum^{\infty} c_{n_1,n_2,\ldots,n_k}\; x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k},$$

where x_j depend on invariants (masses, scalar products of momenta) and c depend on space-time dimension $d = 4 - 2\varepsilon$

For practical purposes we need:

- expansion in ε (limit $d \to 4$)
- analytical continuation
- asymptotics
- fast numerical evaluation

Functional equations

Feynman integrals $I_{k,r}$ (r refers to a number of lines and k numerate
integrals with different topologies and powers of propagators) satisf
generalized recurrence relations:

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where Q_j, R_k are polynomials in masses, scalar products of extern
momenta, dimension of space-time d, powers of propagators ν_l.

General method for obtainig functional equations:

Remove integrals with n external legs (lines) by choosing scalar invariant
masses, d and ν_j i.e. satisfy conditions:

$$Q_j = 0$$

and keep integrals with lesser number of external legs (lines), i.e: $R_k \neq 0$

Functional equations for box integrals

Functional equations for box-integrals:

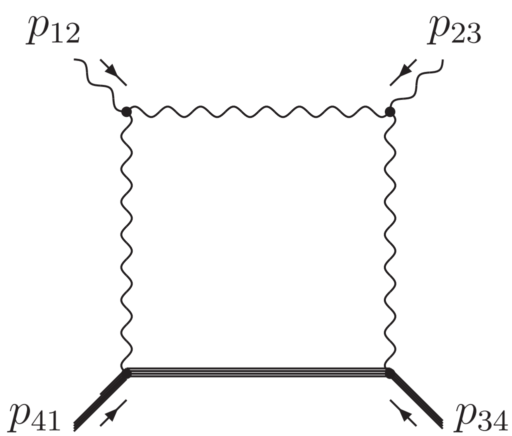
$$I_4^{(d)}(m_n^2, m_j^2, m_k^2, m_l^2;\ s_{nj}, s_{jk}, s_{kl}, s_{nl};\ s_{jl}, s_{nk})$$
$$= \int \frac{d^dq}{i\pi^{d/2}} \frac{1}{[(q-p_n)^2 - m_n^2][(q-p_j)^2 - m_j^2][(q-p_k)^2 - m_k^2][(q-p_l)^2 - m_l^2]}.$$

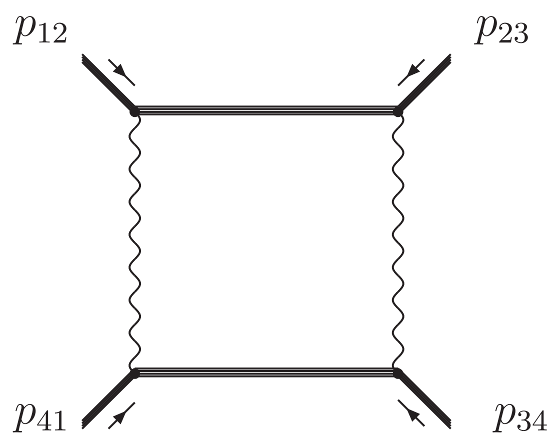
Integrals for Bhabha scattering and heavy-quark production:

$$B(s,t) = I_4^{(d)}(0, m^2, 0, m^2;\ m^2, m^2, m^2, m^2;\ s, t),$$
$$D_2(s,t) = I_4^{(d)}(0, 0, 0, m^2;\ 0, 0, m^2, m^2;\ t, s).$$

$s = (p_{12} + p_{41})^2$

$t = (p_{12} + p_{23})^2$





Functional equations for $B(s,t)$ and $D_2(t,s)$

Integrals $D_2(s,t)$, $B(s,t)$ satisfy the following functional equation:

$$D_2(s,t) = \frac{t+m^2}{2t}\, B\left(\frac{(t+m^2)^2}{t}, s\right) + \frac{t-m^2}{2t}\, I_4^{(d)}\left(0,0,0,0;\ 0,0,0,0;\ \frac{(t-m^2)^2}{t}, s\right).$$

- Functional equations allows one to reduce complicated integrals wit many variables to simpler integrals
- Functional equations can be used for analytic continuation of complicate integrals

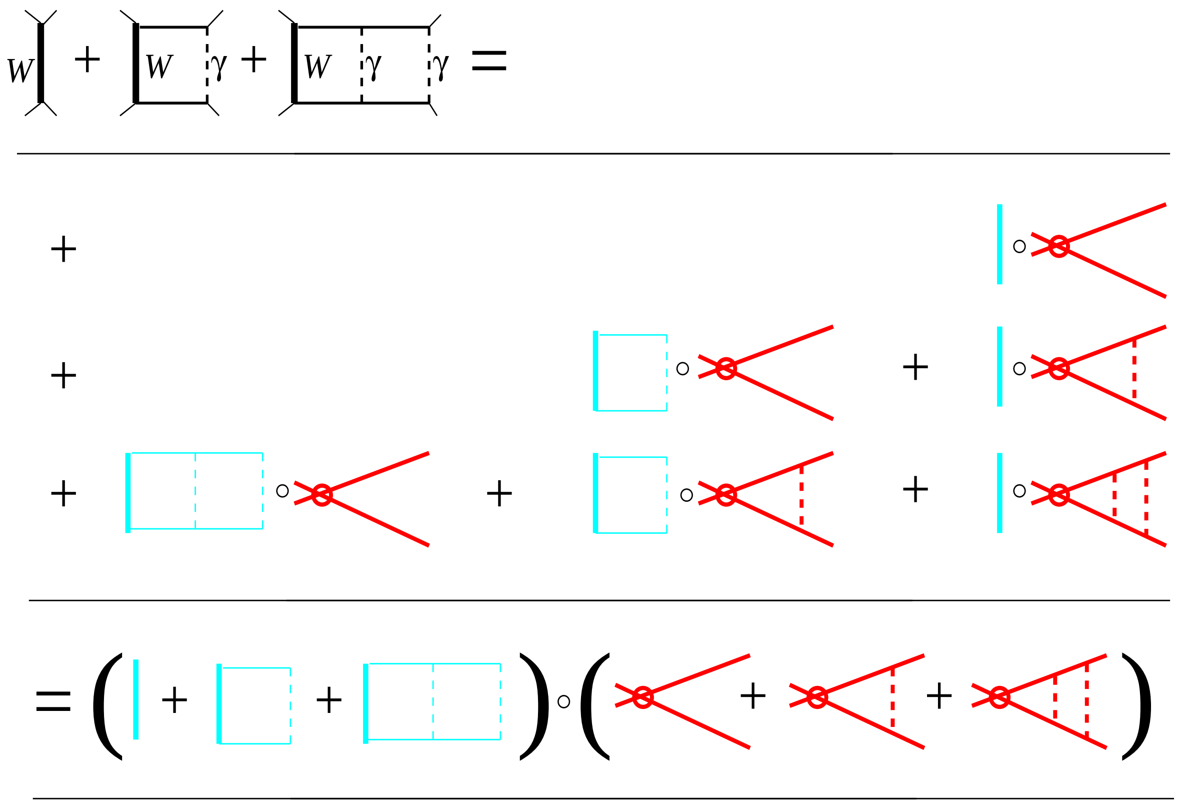
Asymptotic expansioins

- in the SM the spectrum of parameters is very rich
- it is offen possible to find large (small) in the process parameter
- use asymptotic expansion methods to reduce number of parameters $\Longrightarrow$ from n-scale problem to $(n-1)$-scale problem
  - large mass $\longrightarrow \frac{1}{m_t}, \frac{1}{m_H}$
  - close masses $\frac{m_Z^2 - m_W^2}{m_Z^2} = \sin^2\theta_W \sim 0.25,\ \frac{m_H^2 - m_t^2}{m_t^2}$
  - large (small) kinematical invariant, e.g Sudakov limit $Q^2 \to \infty$ etc.
- the above procedure can be repeated (nested expansions)

Rules of asymptotic expansioins

Separation of "soft" and "hard" dynamics

$$F \overset{\text{as}}{\sim} \sum_{H} H \circ S$$



Calculations of master integrals (MI) without direct calcualtions

$$
\begin{aligned}
MI \quad \sim \quad & \sum_{n=1} C_n \, x^n \, \{ F_0(n) + [\ln x \, F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n)] \\
& + [\ln^2 x \, F_{2,1}(n) + \frac{1}{\varepsilon} \ln x \, F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \zeta(2) \, F_{2,4}(n)] \\
& + \ldots,
\end{aligned}
$$

where $x = q^2/m^2$ and

$$
F_{N,k}(n) \;\sim\; \frac{S_{a,\ldots}}{n^b}, \qquad (a + b = M - N),
$$

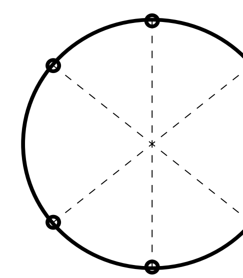
Using the above ansatz and an information about the most singular part of MI (or some simpler diagram with "similar topology") it is possinble usually to reconsruct the complete MI result **without direct calculations**.

running α_s

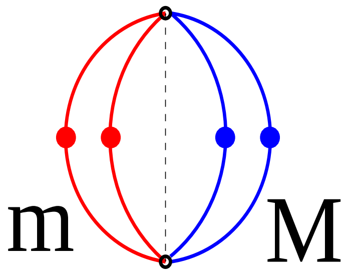
$$\mu^2 \frac{\alpha_s(\mu)}{d\mu^2} = \beta(\alpha)\alpha(\mu), \qquad \beta = \alpha_s^2\beta_0 + \alpha_s^3\beta_1 + \alpha_s^4\beta_2 + \alpha_s^5\beta_3 + \dots$$

- $\beta_{0,1,2,3}(n_f)$ are known analytically in $\overline{\text{MS}}$ scheme
- for $m_Q \gg \mu$ heavy does not automaticaly decouple (Appelquist Carrazone)
- α_s is not an observable
- effective theory with $n_f - 1$ massless and 1 massive quark
- matching $\alpha_s^{(n_f-1)}(m_Q) = \alpha_s^{(n_f)}(m_Q)$ holds to NLO
- matching coefficient $\xi(\mu) = \alpha_s^{(n_f-1)}(m_Q)/\alpha_s^{(n_f)}(m_Q)$
- recent 4-loop calcualation of ξ

  Schröder, Steinhauser, Chetyrkin, Kühn
- one bubble X could be computed numerically $\longrightarrow$



use of “large” mass expansion



$J(x)$ is a function of $x = \frac{m^2}{M^2}$

$$J(x) \sim (-8+88\varepsilon-16\varepsilon\zeta_3+\ldots)+x(-\frac{8}{27}+\frac{212}{27}\varepsilon-\frac{16}{3}\varepsilon\zeta_3+\ldots)+x^2(-\frac{8}{125}+\frac{135479}{33750}\varepsilon-\frac{16}{5}\varepsilon\zeta_3+\ldots)+O(x$$

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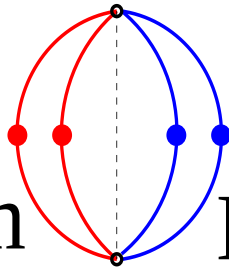

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$\longrightarrow$ fit $\longrightarrow$

$$J(1) \sim \sum_{n=1}^{\infty}\left\{+\frac{-8}{(2n-1)^3}+\varepsilon\left[\frac{16\zeta(3)-2\sum_{j=1}^{2n-1}\frac{1}{j^3}+16\sum_{j=1}^{n-1}\frac{1}{j^3}}{2n-1}+\ldots\right]+\varepsilon^2\left[\text{2-fold sums}\right]+\ldots\right\}$$

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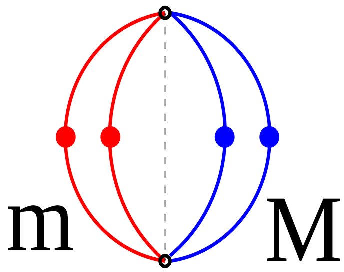

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harmonic sums

$$S_a(n) = \sum_{j=1}^{n} \frac{1}{j^a}$$

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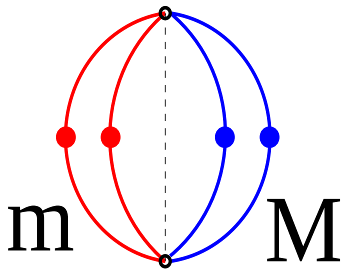

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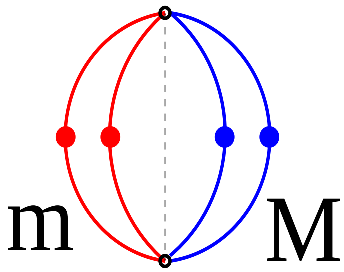

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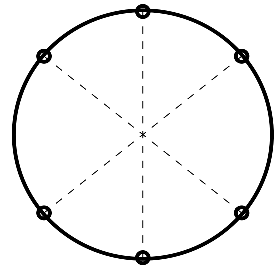
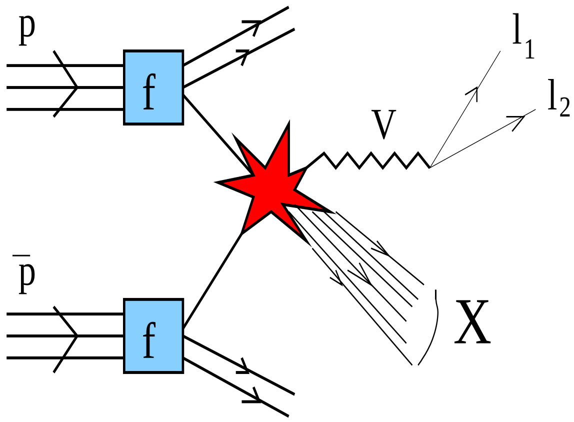
result for X



$$X = -318\zeta(4)\ln 2 + \frac{837}{2}\zeta(5) - 16\left(2\zeta(2) - \frac{1}{5}\ln^2 2\right)\ln^3 2 + 384\mathrm{Li}_5\left(\frac{1}{2}\right)$$

$$\frac{\alpha_s^{(n-1)}}{\alpha_s^{(n)}} = 1 + \alpha_s^{(n)} A_1 + (\alpha_s^{(n)})^2 A_2 + (\alpha_s^{(n)})^3 A_3 + (\alpha_s^{(n)})^4 A_4$$

W- and Z-boson production



- one of the main processes at LHC
- luminosity measerement
- detector calibration
- seach for new physics $\longrightarrow Z'$

For the high p_T:
QCD NNLO and mixed EW/QCD corrections are known (exep
QED/QCD in Z-production)

Missing contributions for mixed EW/QCD evaluation

- 2-loop virtual contribution
- collinear limits of 1-loop amplitudes

W- and Z-boson production

- expansion in parameter q^2/m^2
- summation of the series using ansatz

Results for the formfactors of V-boson

$$F_R = i\frac{e}{s}(1 + C_F\frac{\alpha_s}{4\pi}f^{(0,1)})[g_R + \frac{\alpha}{4\pi s^2}g_R^3\rho_{\rm A} + C_F\frac{\alpha_s}{4\pi}\frac{\alpha}{4\pi s^2}g_R^3\phi_{\rm A}]\,,$$

$$F_L = i\frac{e}{s}(1 + C_F\frac{\alpha_s}{4\pi}f^{(0,1)})[g_L + \frac{\alpha}{4\pi s^2}(g_L^3\rho_{\rm A} + \frac{g_L}{2}\rho_{\rm A} + c\frac{I_3}{2}\rho_{\rm NA})$$
$$+ C_F\frac{\alpha_s}{4\pi}\frac{\alpha}{4\pi s^2}(g_L^3\phi_A + \frac{g_L}{2}\phi_{\rm A} + c\frac{I_3}{2}\phi_{\rm NA})]\,,$$

$$\phi_{\rm A} = 14 + 72\zeta_2 l_2 - 64\zeta_2 l_2^2 - \frac{16}{3}l_2^4 + 22\zeta_2 - 28\zeta_3 + 16\zeta_4 - 128{\rm Li}_4(\tfrac{1}{2})$$
$$+i\pi(85 + 32l_2 + 24l_2^2 - \frac{32}{3}l_2^3 + 14\zeta_2 - 120\zeta_3)$$

$l_2 = \ln 2$

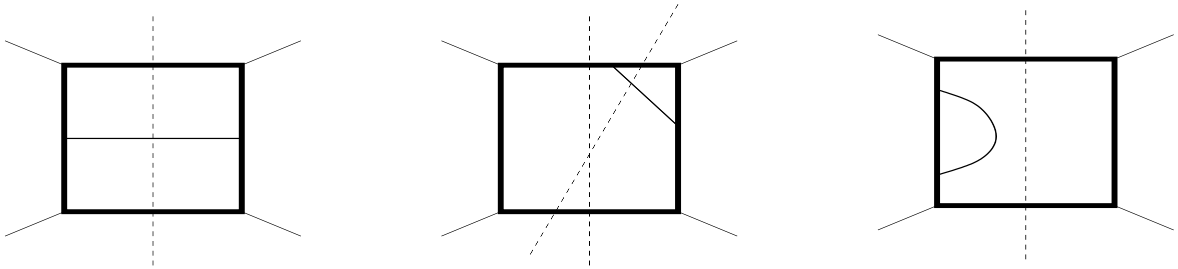
$\bar{t}t$ production (inclusive)

	LHC ($S = 14$TeV)	Tevatron ($S = 1.96$TeV)	
NLO QCD EW MSSM	$\sim 50\%$ $\sim 0.5\%$ $\sim 5\%$	$\sim 25\%$ $\sim 1\%$ $\sim 5\%$	Nason, Bennaker et al. Benakker et al. Berge et al.
NNLO QCD est. th. err.	? $\sim 14\%$	? $\sim 12\%$	 Moch, Uwer
est. exp. err.	$\sim 5-10\%$	$\sim 10\%$	

$\Longrightarrow$ NNLO corrections are necessary

- some partial NNLO results are available in the literature
- recent *analytical* NLO result Czakon, Mitov
- also for ILC Kniehl, Kotikov, Merebashvili, OV

$t\bar{t}$ production



- velocity of heavy quark as kinematical variable $\beta = \sqrt{1 - 4m^2/s}$
- expansion in β or
- differetial equation up to degree 7
- result in terms of $H_{a,b,c}(\beta)$ + 3 new functions

Test of NRQCD at NLO

Nonrelativistic QCD (NRQCD) is the rigouros effective field theory, base
on factorization of soft and hard scales
(scale hierarchy: $Mv^2, Mv \ll \Lambda_{\text{QCD}} \ll M$).

- description of heavy quarkonia
- production of heavy quarks near threshold

Heavy quarkonia (charmonia $c\bar{c}$ and bottomonia $b\bar{b}$) are bound states
heavy quark and antiquark.
Can be produced in different states

- color singlet (S-wave)
- color octet (P-wave)

J/ψ Production with NRQCD

Factorization theorem: $\sigma_{J/\psi} = \sum_n \sigma_{c\bar{c}[n]} \cdot \langle O^{J/\psi}[n] \rangle$

- n: Every possible Fock state, including color-octet states.
- $\sigma_{c\bar{c}[n]}$: Production rate of $c\bar{c}[n]$, calculated in perturbative QCD.
- $\langle O^{J/psi}[n] \rangle$: Long distance matrix elements (ME): describe $c\bar{c}[n] \to J/psi$, universal, extracted from experiment.

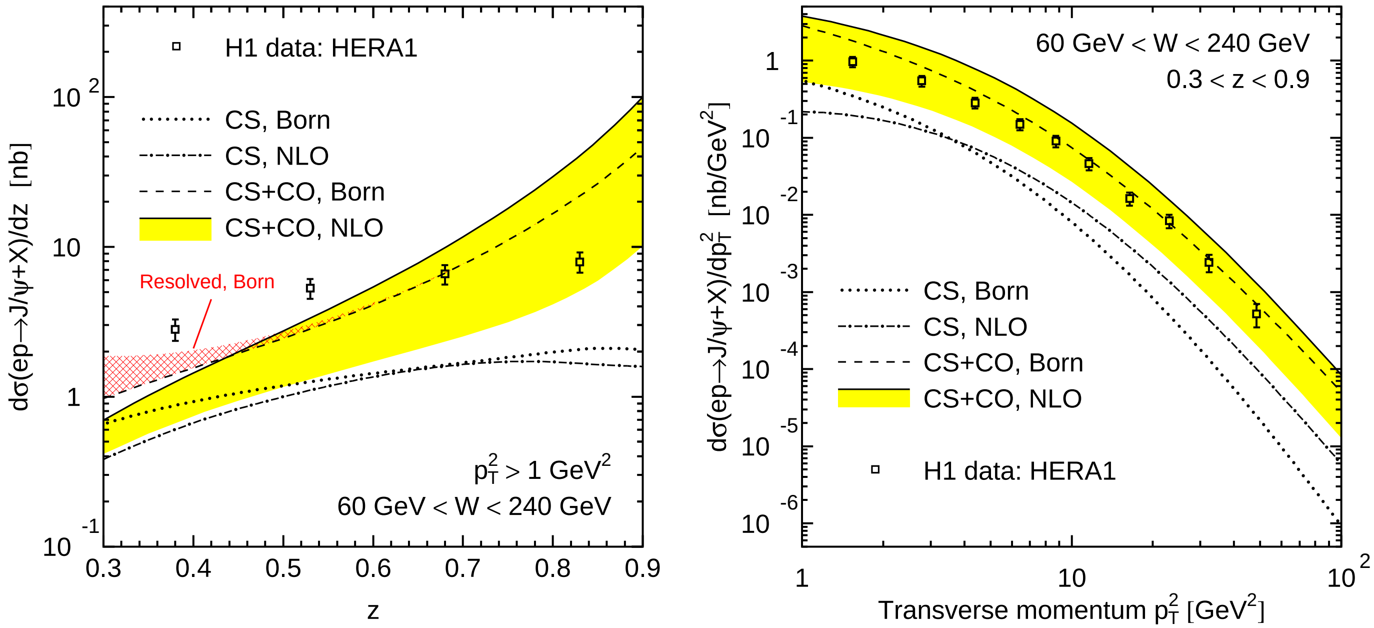
Scaling rules: MEs scale with relative velocity v ($v^2 \approx 0.2$):

scaling	v^3	v^7	v^{11}
n	${}^3S_1^{[1]}$	${}^1S_0^{[8]}$, ${}^3S_1^{[8]}$, ${}^3P_{0/1/2}^{[8]}$	...

- Double expansion in v and α_s.
- Leading term in v ($n = {}^3S_1^{[1]}$) equals color-singlet model.

Numerical Evaluation and Final Results

Butenschön, Knie



- Color-octet MEs from leading order Tevatron fit
- NLO Tevatron fit $\Longrightarrow$ Decrease of CO MEs: Yellow bands

Outlook

- the activity of the phenomenology group includes both
  - QCD and electroweak physics
  - SM and beyond SM
  - phenomenological and mathematical aspects
- NLO and NNLO calculations is a real challenge for a theory and requi special advanced methods of calculation
- Feynman diagrams can be considered generally as multip hypergeometric functions, which requires
  - methods of reduction, i.e. express general functions in terms of onl few (basis) ones
  - method of expansion in ε

Outlook

- general task: find the full set of (special) functions so that **all orde**
  ϵ-expansion is expressible in terms of them
- algorithms for the fast and stable numerical evaluation of these functio
  should be constructed
- automatization: general purpose programs exist only for LO ord
  approximation and the creation of such programs for NLO is challengi
  task