Expansion by regions: an overview

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Experience tells us that the expansion at $t \to 0$ has the form

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The expansion is often called asymptotic, i.e. the remainder of expansion after keeping terms up to t^N is $o(t^N)$.

It is very useful to consider expansion at general ε,

$$G_\Gamma(x,\varepsilon) \sim \sum_{n=n_0}^{\infty} \sum_{k=0}^{h} \sum_{j=0}^{h} c'_{n,j,k}(\varepsilon) t^{n-j\varepsilon} \log^k t \quad .$$

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There are, however, two *general* strategies, *expansion by subgraphs* and *expansion by regions*, which provide a result in this form for any given Feynman integral, where coefficients are expressed either in graph-theoretical language, or in the language of polytopes associated with a given integral.

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- Integrate the integrand, expanded in this way in each region, over the *whole integration domain* of the loop momenta.

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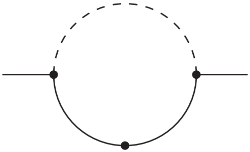
- Divide the space of the loop momenta into various regions and, in every region, expand the integrand in a series with respect to the parameters that are considered there small.
- Integrate the integrand, expanded in this way in each region, over the *whole integration domain* of the loop momenta.
- Set to zero any scaleless integral.

A simple example



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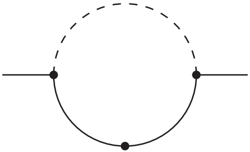

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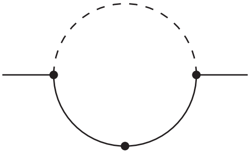

$$G(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2 (q - k)^2}$$

with $d = 4 - 2\varepsilon$ in the limit $m^2/q^2 \to 0$.

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which can then simply be expanded at $m/q \to 0$.

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$$G(q,m,\varepsilon) \sim \int_0^\infty \frac{k^{-1-\varepsilon}}{k+q} \mathrm{d}k + \frac{1}{q} \int_0^\infty \frac{k^{-\varepsilon}}{k+m} \mathrm{d}k + \dots$$

$$G \to G_s + G_l \equiv \int_0^{\Lambda} I(q, m, \varepsilon, k)\mathrm{d}k + \int_{\Lambda}^{\infty} I(q, m, \varepsilon, k)\mathrm{d}k$$

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Here one can change the order of integration and Taylor expansion.

Add and subtract the integral over $(0, \Lambda)$ which is by definition understood as the sum of integrals of the Taylor-expanded integrand:

$$G_l \sim \int_0^\infty \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k - \int_0^\Lambda \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k \ .$$

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$$G_s \sim \int_0^\infty \frac{k^{-\varepsilon}}{k+m} \mathcal{T}_k \frac{1}{k+q} \mathrm{d}k - \int_\Lambda^\infty \frac{k^{-\varepsilon}}{k+m} \mathcal{T}_k \frac{1}{k+q} \mathrm{d}k$$

'Additional' pieces:

$$-\int_0^\Lambda \frac{k^{-\varepsilon}}{k+q}\mathcal{T}_m \frac{1}{k+m}\mathrm{d}k = -\sum_{n=0}^{\infty}(-1)^n m^n \int_0^\Lambda \frac{k^{-\varepsilon-n-1}}{k+q}\mathrm{d}k$$

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We did not refer to the zero value of scaleless integrals

$$\int_0^{\infty} k^{\lambda}\mathrm{d}k = 0.$$

We arrive at the expansion $G \sim M_1 G + M_2 G$ with

$$M_1 G = \int_0^\infty \frac{k^{-\varepsilon}}{k+q} \mathcal{T}_m \frac{1}{k+m} \mathrm{d}k,$$
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The remainder can be described as

$$R^n G = (1 - M_1^n)(1 - M_2^n) G$$
$$= \int_0^\infty k^{-\varepsilon} \left[(1 - \mathcal{T}_m^n) \frac{1}{k+m} \right] \left[(1 - \mathcal{T}_k^n) \frac{1}{k+q} \right] \mathrm{d}k$$

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Set scaleless integrals in $M_1^n M_2^n$ to zero to obtain

$$G \sim M_1^n G + M_2^n G + R^n G$$

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$$M_1^n + M_2^n - M_1^n M_2^n = \sum_{j=0}^{n} (1 - M_2^{j-1}) M_1^{(j)} + \sum_{j=0}^{n} (1 - M_1^j) M_2^{(j)} \,.$$

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Then the first sum takes gives

$$\int_0^\infty k^{-\varepsilon} \left[(1 - \mathcal{T}_k^{j-1}) \frac{1}{k+q} \right] \mathcal{T}_m^{(j)} \frac{1}{k+m} \mathrm{d}k$$
$$\sim m^j \int_0^\infty k^{-\varepsilon - j - 1} \left[(1 - \mathcal{T}_k^{j-1}) \frac{1}{k+q} \right] \mathrm{d}k$$

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from $0 < -\mathrm{Re}\varepsilon < 1$ to $-j-1 < \mathrm{Re}\varepsilon < -j$.

This reminds the analytic continuation of the distribution x_+^λ from $\mathrm{Re}\lambda > -1$ to the whole complex plane [I.M. Gelfand '55], i.e. for integrals

$$\int_0^\infty x^\lambda \phi(x)\mathrm{d}x$$

Similarly, the second sum takes the form

$$\int_0^\infty k^{-\varepsilon} \left[(1 - \mathcal{T}_m^j) \frac{1}{k+m} \right] \mathcal{T}_k^{(j)} \frac{1}{k+q} \mathrm{d}k$$
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$$\sim \int_0^\infty k^{-\varepsilon-j}\left[(1-\mathcal{T}_m^j)\frac{1}{k+m}\right]\mathrm{d}k\ ,$$

and this is the analytic continuation of the integral

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from $0 < -\mathrm{Re}\varepsilon < 1$ to $j < \mathrm{Re}\varepsilon < j+1$.

This means that we can represent the terms of expansion described by the operator $M_1 + M_2 - M_1 M_2$ also in an equivalent way using just the sum of the operators $M_1 + M_2$, with the prescription that each resulting integral is evaluated in its own domain of convergence and then the result obtained is analytically continued to a given domain.

Jantzen [B. Jantzen'11] provided detailed explanations, using one- and two-loop examples, of how this strategy works by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained.

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Expansion by subgraphs [K.G. Chetyrkin'88, S. Gorishny'89], for example, in the off-shell large-momentum limit, i.e. where a momentum Q is considered large and momenta q_i as well as the masses m_j are small,

Jantzen [B. Jantzen'11] provided detailed explanations, using one- and two-loop examples, of how this strategy works by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained.

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$$G_\Gamma \sim \sum_\gamma G_{\Gamma/\gamma} \circ \mathcal{T}_{q_\gamma, m_\gamma} G_\gamma$$

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For the threshold limit $y = m^2 - q^2/4 \to 0$, one has

$$\begin{aligned}
&\text{(hard)}, && k_0 \sim \sqrt{q^2}\,,\ \vec{k} \sim \sqrt{q^2}\,,\\
&\text{(soft)}, && k_0 \sim \sqrt{y}\,,\ \vec{k} \sim \sqrt{y}\,,\\
&\text{(potential)}, && k_0 \sim y/\sqrt{q^2}\,,\ \vec{k} \sim \sqrt{y}\,,\\
&\text{(ultrasoft)}, && k_0 \sim y/\sqrt{q^2}\,,\ \vec{k} \sim y/\sqrt{q^2}\,.
\end{aligned}$$

where $q = (q_0, \vec{0})$.

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and U and V are two basic functions
(Symanzik polynomials, or graph polynomials).

One can consider quite general limits for a Feynman integral which depends on external momenta q_i and masses and is a scalar function of kinematic invariants and squares of masses, s_i, and assume that each s_i has certain scaling ρ^{κ_i} where ρ is a small parameter.

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A region $\to$ scaling, i.e. $x_i \to \rho^{r_i} x_i$ where ρ is a small parameter connected with a given limit.

A systematical procedure to find regions based on geometry of polytopes and implemented as a public computer code `asy.m` [A. Pak & A.V. Smirnov'10] which is now included in the code `FIESTA` [A.V. Smirnov'09-16]

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Numerous applications have shown that the code `asy.m` works consistently even in cases where the function F is not positive – see, e.g.
[J.M. Henn, K. Melnikov & V.S.'14; F. Caola, J.M. Henn, K. Melnikov & V.S.'14]

Generalizations of this procedure to some cases where terms of the function F are negative
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An example: one-loop diagram with two massive lines in the threshold limit $y = m^2 - q^2/4 \to 0$

$$F(q^2, y) = \mathrm{i}\pi^{d/2}\,\Gamma(\varepsilon) \times \int_0^\infty \int_0^\infty \frac{(\alpha_1+\alpha_2)^{2\varepsilon-2}\,\delta(\alpha_1+\alpha_2-1)\;\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{\left[\frac{q^2}{4}(\alpha_1-\alpha_2)^2 + y(\alpha_1+\alpha_2)^2 - \mathrm{i}0\right]^\varepsilon}$$

The code `asy.m` in its first version revealed only the contribution of the hard region, i.e. $\alpha_i \sim y^0$.

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$$i\pi^{d/2}\,\frac{\Gamma(\varepsilon)}{2}\int_0^\infty\int_0^\infty \frac{(\alpha_1+\alpha_2)^{2\varepsilon-2}\,\delta\left(\alpha_1+\alpha_2-1\right)\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{\left[\frac{q^2}{4}\alpha_2^2+y(\alpha_1+\alpha_2)^2-\mathrm{i}0\right]^{\varepsilon}}\,.$$

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Two regions: $(0,0)$ and $(0,1/2)$. The second one, with $\alpha_1 \sim y^0, \alpha_2 \sim \sqrt{y}$ gives

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A recent suggestion [T. Semenova, A. Smirnov & V.S.'19]:

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Feynman parametric representation can be obtained from it by inserting $1 = \int \delta(\sum_i x_i - \eta)\mathrm{d}\eta$, scaling $x \to \eta x$ and integrating over η.

Let t be the small parameter, e.g. $-m^2/q^2$ for the Sudakov limit or $(p_1+p_3)^2/(p_1+p_2)^2$ for the Regge limit.

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Let P be a polynomial with positive coefficients,

$$P(x_1,\ldots,x_n,t) = \sum_{w\in S} c_w x_1^{w_1}\ldots x_n^{w_n} t^{w_{n+1}} \ ,$$

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The Newton polytope $\mathcal{N}_P$ of P is the convex hull of the set S in the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ equipped with the scalar product $v\cdot w = \sum_{i=1}^{n+1} v_i w_i$.

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A facet of P is a face of maximal dimension, i.e. n.

The main conjecture.

The asymptotic expansion of

$$G(t,\varepsilon) = \int_0^\infty \dots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \dots \mathrm{d}x_n \ ,$$

in the limit $t \to +0$ is given by

$$G(t,\varepsilon) \sim \sum_\gamma \int_0^\infty \dots \int_0^\infty \left[M_\gamma \left(P(x_1,\dots,x_n,t) \right)^{-\delta} \right] \mathrm{d}x_1 \dots \mathrm{d}x_n \ ,$$

where the sum runs over facets of the Newton polytope $\mathcal{N}_P$ of P, for which the normal vectors $r^\gamma = (r_1^\gamma,\dots,r_n^\gamma,r_{n+1}^\gamma)$, oriented inside the polytope have $r_{n+1}^\gamma > 0$.

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Let us normalize these vectors by $r_{n+1}^\gamma = 1$.

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For a given essential facet γ, let us define the polynomial

$$P^\gamma(x_1, \ldots, x_n, t) = P(t^{r_1^\gamma} x_1, \ldots, t^{r_n^\gamma} x_n, t) \equiv \sum_{w \in S} c_w x_1^{w_1} \ldots x_n^{w_n} t^{w \cdot r^\gamma}$$

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The scalar product $w \cdot r^\gamma$ is proportional to the projection of the point w on the vector r^γ. For $w \in S$, it takes a minimal value for all the points belonging to the considered facet $w \in S \cap \gamma$. Let us denote it by $L(\gamma)$.

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The polynomial P_0^γ is independent of t while P_1^γ can be represented as a linear combination of positive rational powers of t with coefficients which are polynomials of x.

For a given facet γ, the operator M_γ acts on the integrand as follows

$$
\begin{aligned}
& M_\gamma \left(P(x_1,\ldots,x_n,t)\right)^{-\delta} \\
= \; & t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \mathcal{T}_t \left(P_0^\gamma(x_1,\ldots,x_n) + P_1^\gamma(x_1,\ldots,x_n,t)\right)^{-\delta} \\
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In particular, the LO term of a given facet γ is

$$t^{-L(\gamma)\delta + \sum_{i=1}^n r_i^\gamma} \int_0^\infty \ldots \int_0^\infty \left(P_0^\gamma(x_1,\ldots,x_n)\right)^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \; .$$

An example:

$$G(t, \varepsilon) = \int_0^\infty (x^2 + x + t)^{\varepsilon - 1} \mathrm{d}x$$

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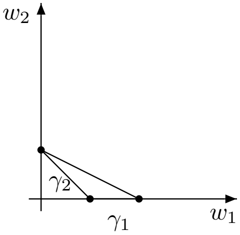
The Newton polytope (triangle)



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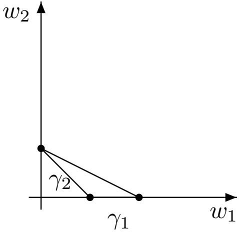

Two essential facets γ_1 and γ_2 with the corresponding normal vectors $r_1 = (0,1)$ and $r_2 = (1,1)$.

$\gamma_1 \rightarrow$ expanding the integrand in t. L0 is given by

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$\gamma_2 \to t$ times the integral of the integrand with $x \to tx$ expanded in powers of t. L0 is given by

$$t^{\varepsilon} \int_0^{\infty} (x + 1)^{\varepsilon - 1} \mathrm{d}x = -\frac{t^{\varepsilon}}{\varepsilon}$$

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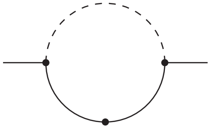
The sum of the contributions in the LO:

$$G(t,\varepsilon) \sim -\log t + O(\varepsilon)$$

Another example



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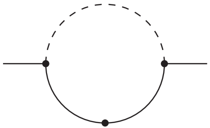

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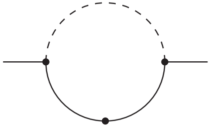

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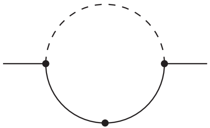

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The vertices A, B, C, D, E of the Newton polytope coincide with the set S

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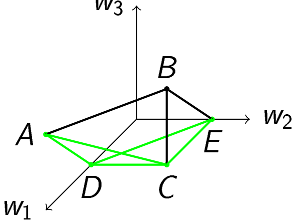
$$A(2,0,1), B(1,1,1), C(1,1,0), D(1,0,0), E(0,1,0).$$



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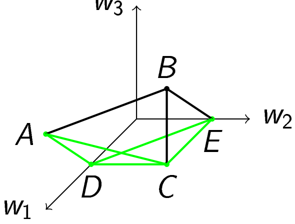

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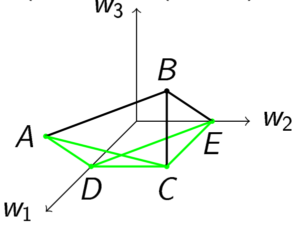

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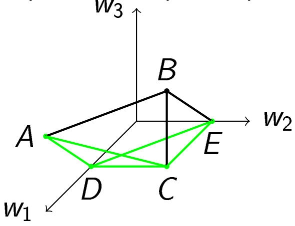

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$CDE \in$ the plane $w_3 = 0$, with the normal vector $(0,0,1) \rightarrow$ expansion in t.

$ACD \in$ the plane $w_1 - w_3 = 1$, with the normal vector $(-1,0,1)$

$\rightarrow t^{-2} \int_0^\infty x_1 \left[x_1/t + x_2 + (x_1/t)(t(x_1/t + x_2))\right]^{\varepsilon-2} = \ldots$

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The cancellation of these poles is a very natural check of the expansion procedure, i.e. the pole part of the sum of terms of the expansion should be equal to the pole part of the initial integral.

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One has to distinguish situations where contributions of individual facets are not regularized by the initial regularization parameter δ. A natural way to proceed is to introduce auxiliary analytic regularization by inserting powers $x_i^{\lambda_i}$.
For Feynman integrals at Euclidean external momenta, Speer proved that the corresponding dimensionally and analytically regularized parametric integral is convergent in a non-empty domain of parameters $(\varepsilon, \lambda_1, \ldots, \lambda_n)$.
A generalization of Speer's theorem to the case of LP representation [T. Semenova, A. Smirnov & V.S.'19].

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2. The new formulation has more chances to be proven. A proof in a special case [T. Semenova, A. Smirnov & V.S.'19].

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- *Divide et impera*