Calculating Four Loop Corrections in QCD

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Motivation

QCD Lagrangian

- Classical part of QCD Lagrangian

$$\mathcal{L}_{\rm cl} \quad = \quad -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \sum_{\rm flavors} \bar{\psi}_i \left({\rm i} \not{D} - m_q \right)_{ij} \psi_j$$

- Matter fields $\psi_i, \bar{\psi}_j$ with $i, j = 1, \dots, 3$ (fundamental rep.)
  - covariant derivative $D_{\mu,ij} = \partial_\mu \delta_{ij} + {\rm i} g_s \left(t_a \right)_{ij} A^a_\mu$
- Field strength tensor $F^a_{\mu\nu}$ with $a = 1, \dots, 8$ (adjoint rep.)
  - covariant derivative $D_{\mu,ab} = \partial_\mu \delta_{ab} - g_s f_{abc} A^c_\mu$
  - $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$
- Formal parameters of the theory (no observables)
  - strong coupling $\alpha_s = g_s^2/(4\pi)$
  - quark masses m_q

Challenge

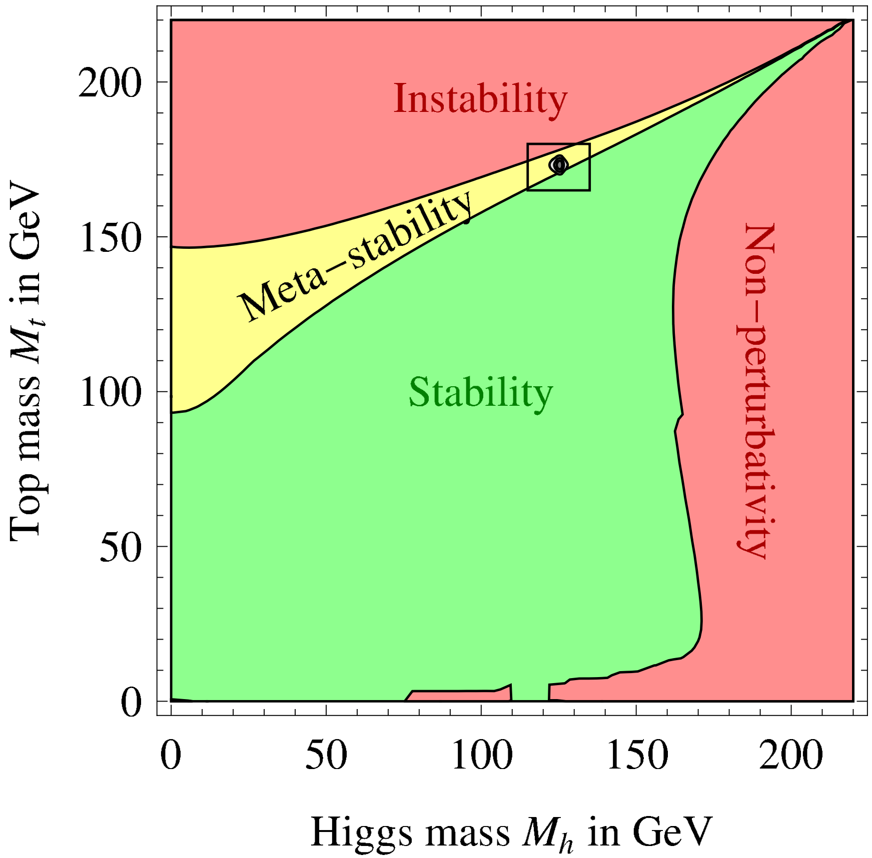
- Suitable observables for measurements of α_s, m_q, . . .
- Comparison of theory predictions and experimental data

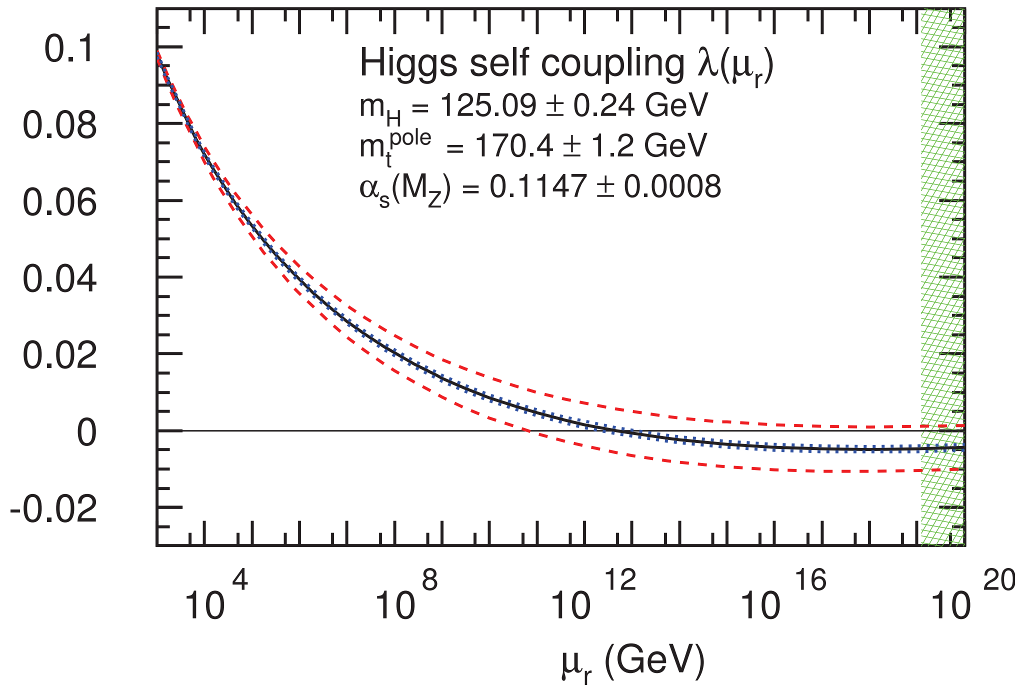
Strong coupling constant (2020)

BBG	$0.1134\ ^{+\,0.0019}_{-\,0.0021}$	valence analysis, NNLO	Blümlein, Böttcher, Guffanti '06
JR08	0.1128 ± 0.0010	dynamical approach	Jimenez-Delgado, Reya '08
	0.1162 ± 0.0006	including NLO jets	
ABKM09	0.1135 ± 0.0014	HQ: FFNS $n_f = 3$	Alekhin, Blümlein, Klein, S.M. '09
	0.1129 ± 0.0014	HQ: BSMN	
MSTW	0.1171 ± 0.0014		Martin, Stirling, Thorne, Watt '09
Thorne	0.1136	[DIS+DY, HT*] (2013)	Thorne '13
ABM11_J	$0.1134 \ldots 0.1149\ \pm 0.0012$	Tevatron jets (NLO) incl.	Alekhin, Blümlein, S.M. '11
NN21	0.1173 ± 0.0007	(+ heavy nucl.)	NNPDF '11
ABM12	0.1133 ± 0.0011		Alekhin, Blümlein, S.M. '13
	0.1132 ± 0.0011	(without jets)	
CT10	0.1140	(without jets)	Gao et al. '13
CT14	$0.1150\ ^{+\,0.0060}_{-\,0.0040}$	$\Delta\chi^2 > 1$ (+ heavy nucl.)	Dulat et al. '15
JR14	0.1136 ± 0.0004	dynamical approach	Jimenez-Delgado, Reya '14
	0.1162 ± 0.0006	standard approach	
MMHT	0.1172 ± 0.0013	(+ heavy nucl.)	Martin, Motylinski, Harland-Lang, Thorne '15
ABMP16	0.1147 ± 0.0008		Alekhin, Blümlein, S.M., Placakyte '17
NN31	0.1185 ± 0.0012	including NLO jets	NNPDF '18

- Measurements at NNLO (last ~ 10 years) from DIS data
- Large spread of fitted values at NNLO: $\alpha_s(M_Z) = 0.1128 \ldots 0.1185$
- [yellow box] 2019 PDG average: $\alpha_s(M_Z) = 0.1161 \pm 0.0018$

Electroweak vacuum stability





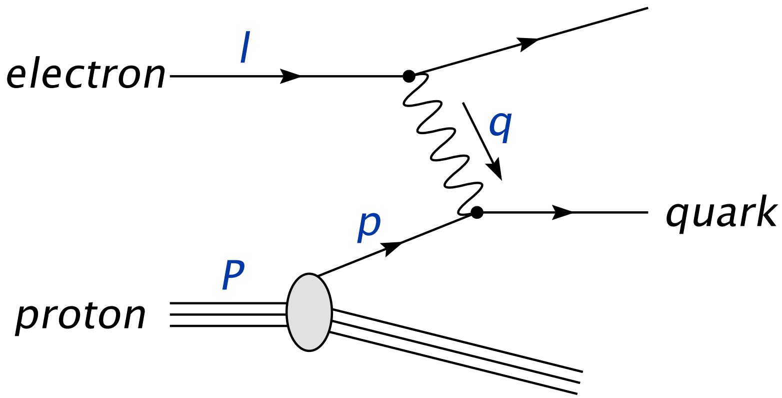
Bezrukov, Kalmykov, Kniehl, Shaposhnikov '12; Degrassi, Di Vita, Elias-Miro, Espinosa, Giudice et al. '12; Alekhin, Djouadi, S.M. '12; Masina '12; [many people]

- Condition of absolute stability of electroweak vacuum at Planck scale M_{Planck} requires Higgs self-coupling $\lambda(\mu_r) \geq 0$

$$m_H \geq 129.6 + 2.0 \times \left(m_t^{\text{pole}} - 173.34 \text{ GeV}\right) - 0.5 \times \left(\frac{\alpha_s(M_Z) - 0.1184}{0.0007}\right) \pm 0.3 \text{ GeV}$$

Deep-inelastic scattering

Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a\left(x,Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes PDF\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N^{3}LO known

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \ldots\right)$$

- Evolution equations up to N^{3}LO (work in progress)
  - non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

$$\frac{d}{d\ln\mu^2} PDF(x,\mu^2) = \left[P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)\right](x)$$

  - splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \ldots$

Evolution equations

- Parton distribution functions $q_i(x,\mu^2)$, $\bar{q}_i(x,\mu^2)$ and $g(x,\mu^2)$ for quarks, antiquarks of flavour i and gluons
- Flavor non-singlet combinations

$$q^{\pm}_{\mathrm{ns},ik} \;=\; (q_i \pm \bar{q}_i) - (q_k \pm \bar{q}_k) \text{ and } q^{\mathrm{v}}_{\mathrm{ns}} \;=\; \sum_{i=1}^{n_f} (q_i - \bar{q}_i)$$

  - splitting functions $P^{\pm}_{\mathrm{ns}}$ and $P^{\mathrm{v}}_{\mathrm{ns}} = P^{-}_{\mathrm{ns}} + P^{\mathrm{s}}_{\mathrm{ns}}$
- Flavor singlet evolution

$$\frac{d}{d\ln\mu^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} \quad \text{and} \quad q_{\mathrm{s}} \;=\; \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$$

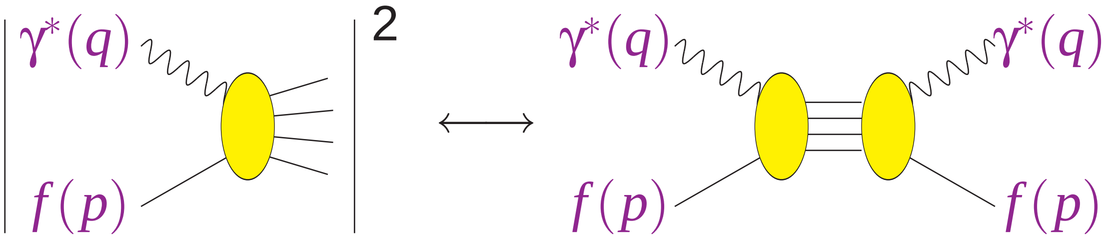
  - quark-quark splitting function $P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$
- Mellin transformation relates to anomalous dimensions $\gamma_{\mathrm{ik}}(N)$ of twist-two operators

$$\gamma^{(n)}_{\mathrm{ik}}(N,\alpha_s) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P^{(n)}_{\mathrm{ik}}(x,\alpha_s)$$

Operator product expansion (I)

Optical theorem

- Total cross section related to imaginary part of Compton amplitude
  - loop diagrams



- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x,Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x,Q^2)$$

- OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, x fixed
  Wilson '72; Christ, Hasslacher, Mueller '72

$$T_{\mu\nu} \quad = \quad \mathrm{i} \int d^4 z \, \mathrm{e}^{\mathrm{i}q \cdot z} \langle \mathrm{P} | \, T \left(j_\mu^\dagger(z) j_\nu(0) \right) | \mathrm{P} \rangle$$

Operator product expansion (II)

- Operator product expansion yields

$$T_{\mu\nu} \;=\; \sum_{N,j} \left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}\, C^N_{L,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right) + d_{\mu\nu}\, C^N_{2,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right) \right.$$
$$\left. + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p\cdot q} C^N_{3,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right] A^j_{\mathrm{P},N}\left(\mu^2\right) + \text{ higher twists}$$

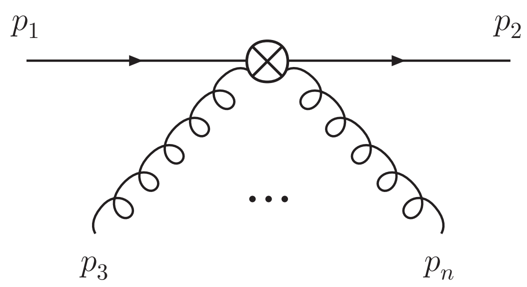
- Coefficient functions in Mellin space $C^N_{a,i}$
- Matrix elements $A^i_{\mathrm{P},N}$ of parton operators O^i at leading twist (twist = dimension minus spin)
  - e.g., quark operator $O^\psi_{\{\mu_1,\cdots,\mu_N\}} = \overline{\psi}\gamma_{\{\mu_1} D_{\mu_2} \cdots D_{\mu_N\}}\psi$
- Scale dependence of renormalized operators $O^{\text{bare}} \;=\; Z\, O^{\text{ren}}$
  $\longrightarrow$ anomalous dimensions $\gamma(\alpha_s, N)$

$$\frac{d}{d\ln\mu^2} O^{\text{ren}} = -\gamma\; O^{\text{ren}} \qquad\qquad \gamma \;=\; \left(\frac{d}{d\ln\mu^2} Z\right) Z^{-1}$$

Non-singlet

Calculation of anomlous dimensions

Operator matrix elements



- Quark operator of spin-N and twist two
  $O^{\psi}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi$
- N covariant derivatives $D_{\mu,ij} = \partial_\mu \delta_{ij} + \mathrm{i} g_s\, (t_a)_{ij}\,.$ between quark fields ψ, $\overline{\psi}$
- Feynman rules with new vertices for additional gluons coupling to operator
- Evaluation of operators in matrix elements $A^{\psi\psi}$ with external quark states
- Computation of quantum corrections up to four loops

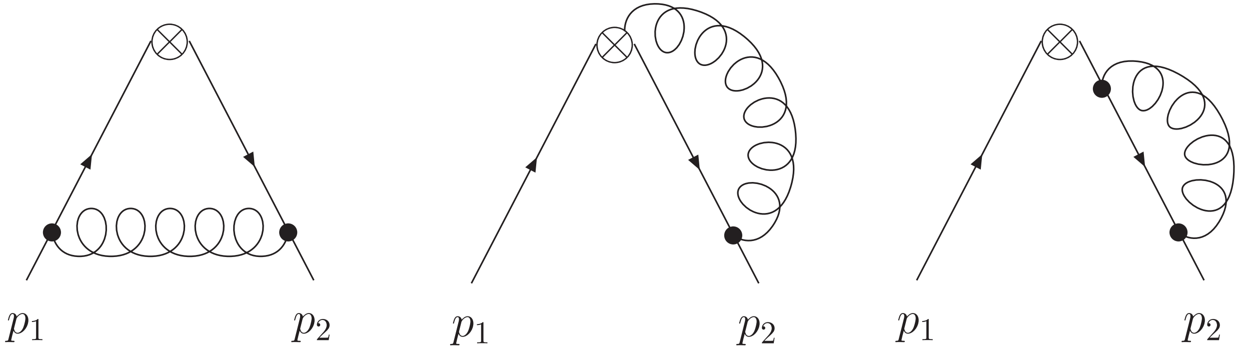
$$A^{\psi\psi}_{\{\mu_1,\dots,\mu_N\}} = \langle \psi(p_1) | O^{\psi}_{\{\mu_1,\dots,\mu_N\}}(-p_1 - p_2) | \overline{\psi}(p_2) \rangle$$

- Zero-momentum transfer through operator reduces problem to computation of propagator-type diagrams

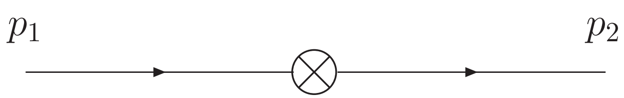
One-loop computation (I)

Quark anomalous dimension

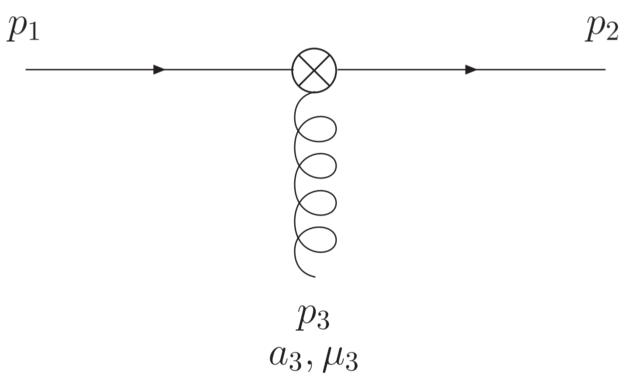
- Feynman diagrams



- New Feynman rules for vertices with light-like vector Δ, $\Delta^2 = 0$



$$\not{\Delta}\ (\Delta \cdot p_2)^{N-1}$$



$$-g t^{a_3} \not{\Delta} \Delta^{\mu_3} \sum_{j_1=0}^{N-2} (p_2 \cdot \Delta)^{N-2-j_1} (-p_1 \cdot \Delta)^{j_1}$$

One-loop computation (II)

One-loop result

- Computation of loop integral in $D = 4 - 2\epsilon$ dimensions and expansion in ϵ

$$\Delta^{\mu_1} \ldots \Delta^{\mu_N} \langle \psi(p_1) | O^{\psi}_{\{\mu_1,\ldots,\mu_N\}}(0) | \overline{\psi}(-p_1) \rangle \; =$$

$$= \quad 1 + \frac{\alpha_s}{4\pi} C_F \frac{1}{\epsilon} \left\{ 4S_1(N) + \frac{2}{N+1} - \frac{2}{N} - 3 \right\} + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2)$$

- One-loop result contains harmonic sum $S_1(N)$

$$S_1(N) \quad = \quad \sum_{i=1}^{N} \frac{1}{i} \; = \; 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{N}$$

$$S_1(N+1) - S_1(N) \quad = \quad \frac{1}{N+1}$$

- Details in chapt. 4.6 of

  The Theory of Quark and Gluon Interactions

  F.J. Yndurain

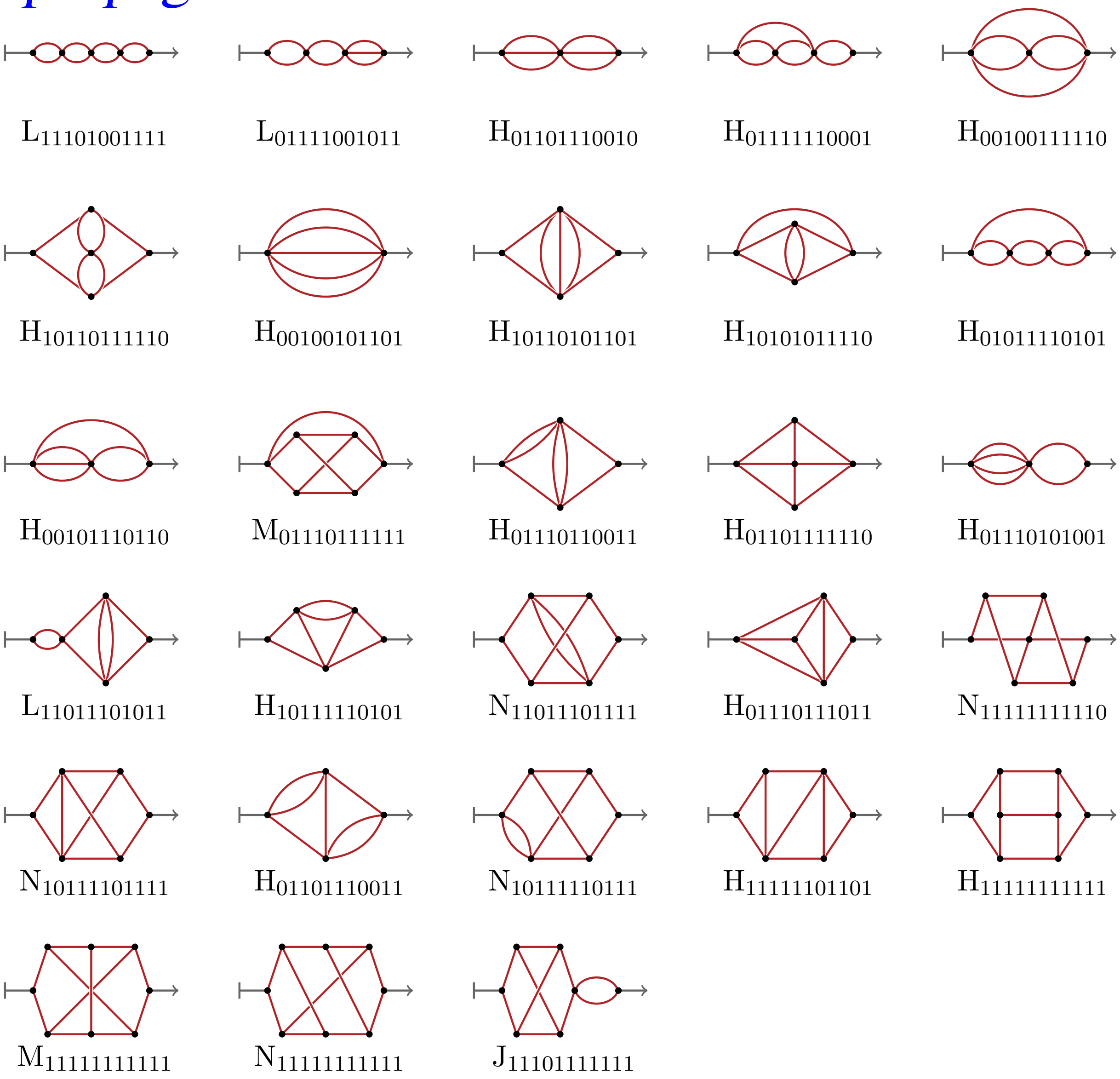
Four-loop computation (I)

Work flow

- Anomalous dimensions $\gamma(N)$ from ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
    - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma_{\rm ns}^{\pm}$
    - 1 three- and 29 four-loop meta diagrams for $\gamma_{\rm ns}^{\rm s}$

Four-loop computation (II)

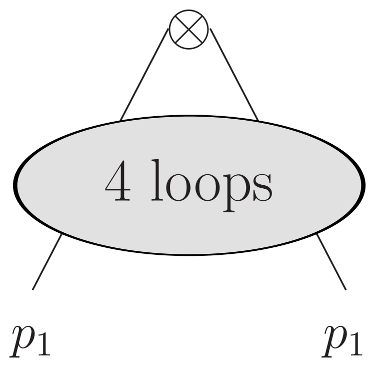
Four-loop propagators



Four-loop computation (III)

Fixed Mellin moments

- Computation of anomalous dimensions $\gamma(N)$ for Mellin moments mostly up to $N = 18$
  - sometimes higher for complicated topologies ($N = 19$, $N = 20$, . . .)
  - much higher for "easy" topologies, e.g., large-n_c ($N \simeq 40$, . . .)



- Hardware
  - multi-core Linux servers with 1 TByte RAM memory and several TByte of disk space

- Resources
  - CPU time for $N = 18$ added up to 6403 days on a single core (200 days real time with **TForm**)
  - $\mathcal{O}(15)$ TByte of disk space at intermediate stages of computation for moments $N = 20$

Reconstruction the anomalous dimensions

- Anomalous dimensions $\gamma(N)$ of leading twist non-singlet local operators
  - expressible in harmonic sums up to weight 7

$$S_{\pm m_1,\dots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\dots,m_k}(i)$$

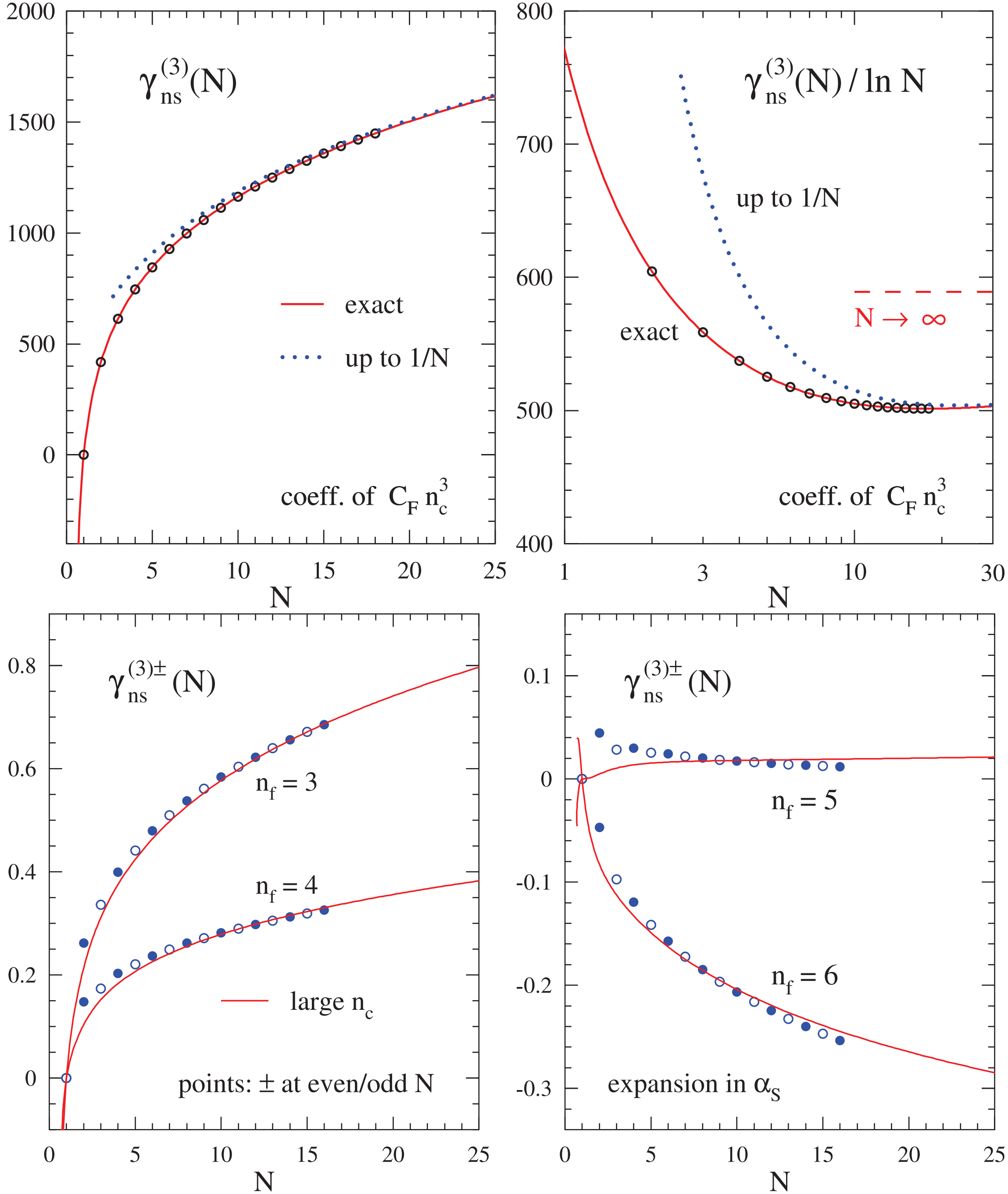
  - $2 \cdot 3^{w-1}$ sums at weight w
- Reciprocity relation $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(a)\right)$ reduces number of 2^{w-1} sums at weight w for $\gamma_{\rm u}$
  - additional denominators with powers $1/(N+1)$ give $2^{w+1} - 1$ objects (255 at weight 7)
- Constraints at large-x/small-x ($N \to \infty$/$N \to 0$) give additional 46 conditions

Upshot

- Computation of Mellin moments up to $N = 18$ for anomalous dimensions feasible
- Reconstruction of analytic all-N expressions in large-n_c limit from solution of Diophantine equations

Four-loop Mellin moments

- Top:
  n_f^0 part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$ in large-n_c limit and large-N expansion

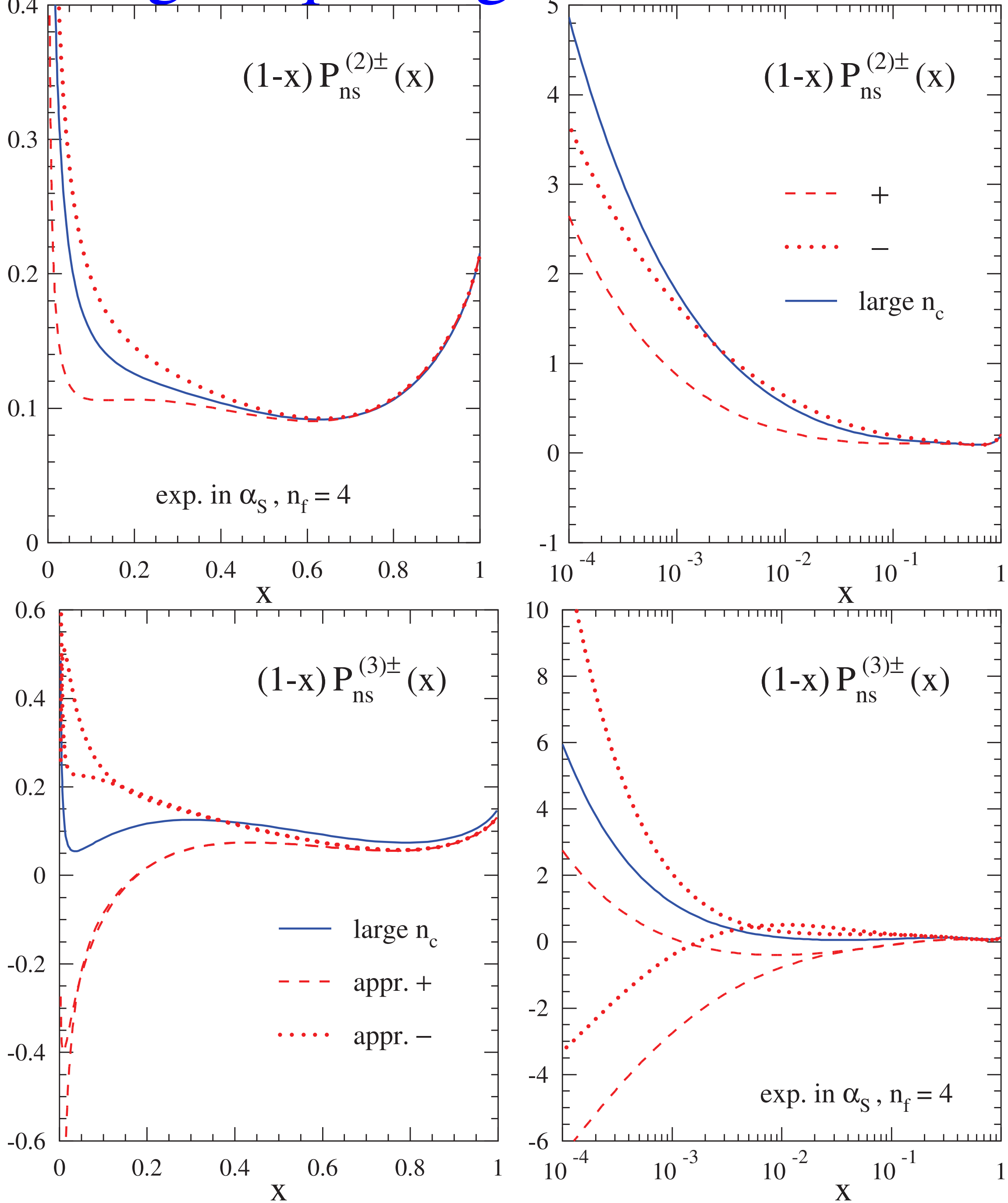


- Bottom: results for even-N ($\gamma_{\rm ns}^{(3)+}(N)$) and odd-N ($\gamma_{\rm ns}^{(3)-}(N)$) in large-n_c limit for $n_f = 3, \ldots 6$

Four-loop non-singlet splitting functions

- Top:
  three-loop $P_{\rm ns}^{(2)\pm}(x)$
  and large-n_c limit
  with $n_f = 4$

- Bottom:
  four-loop $P_{\rm ns}^{(3)\pm}(x)$
  and uncertainty bands
  beyond large-n_c limit
  with $n_f = 4$

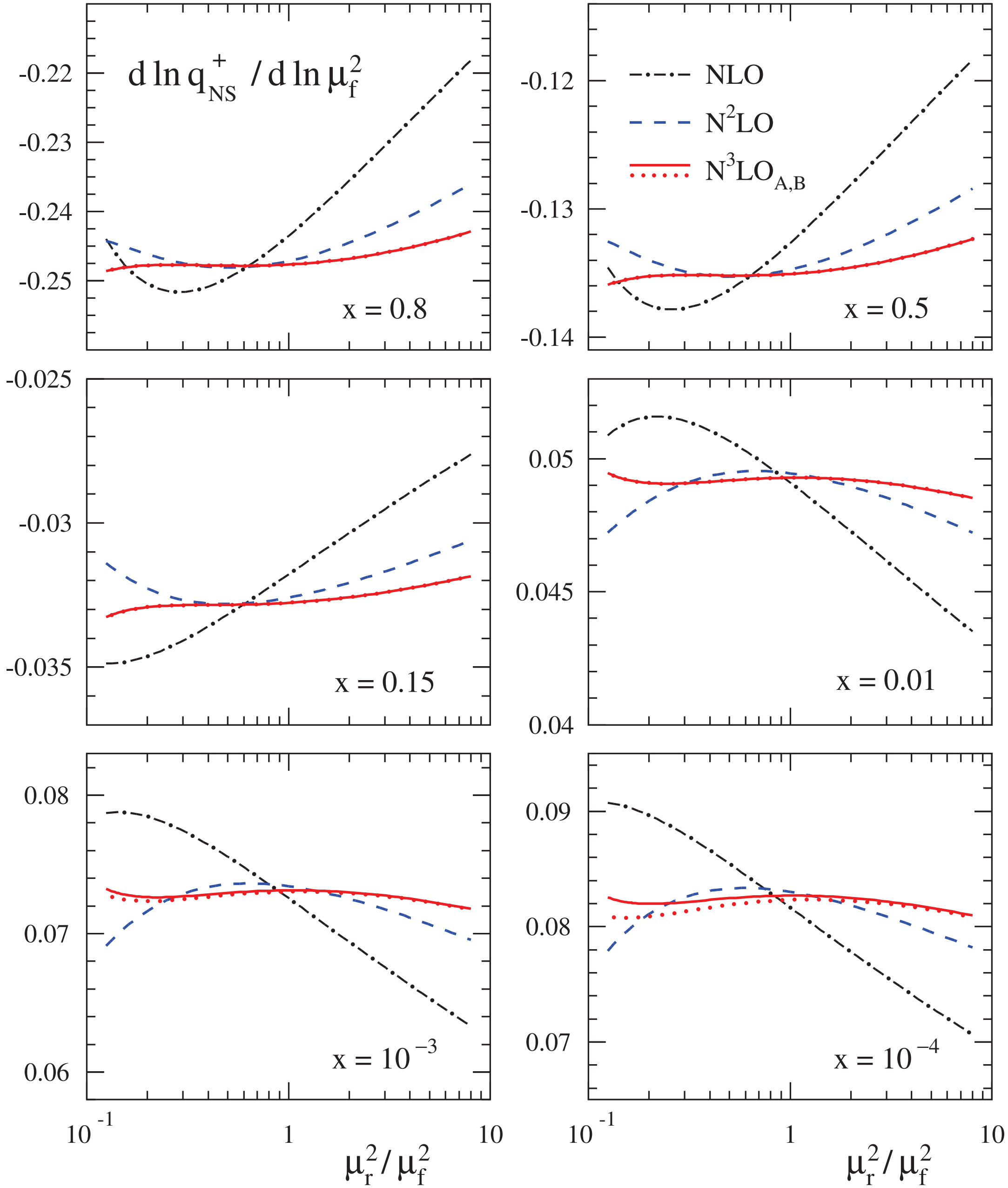


Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

$$x q_{\rm ns}^{+}(x, \mu_0^2) \; = \; x^{0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



Five-loop Mellin moments

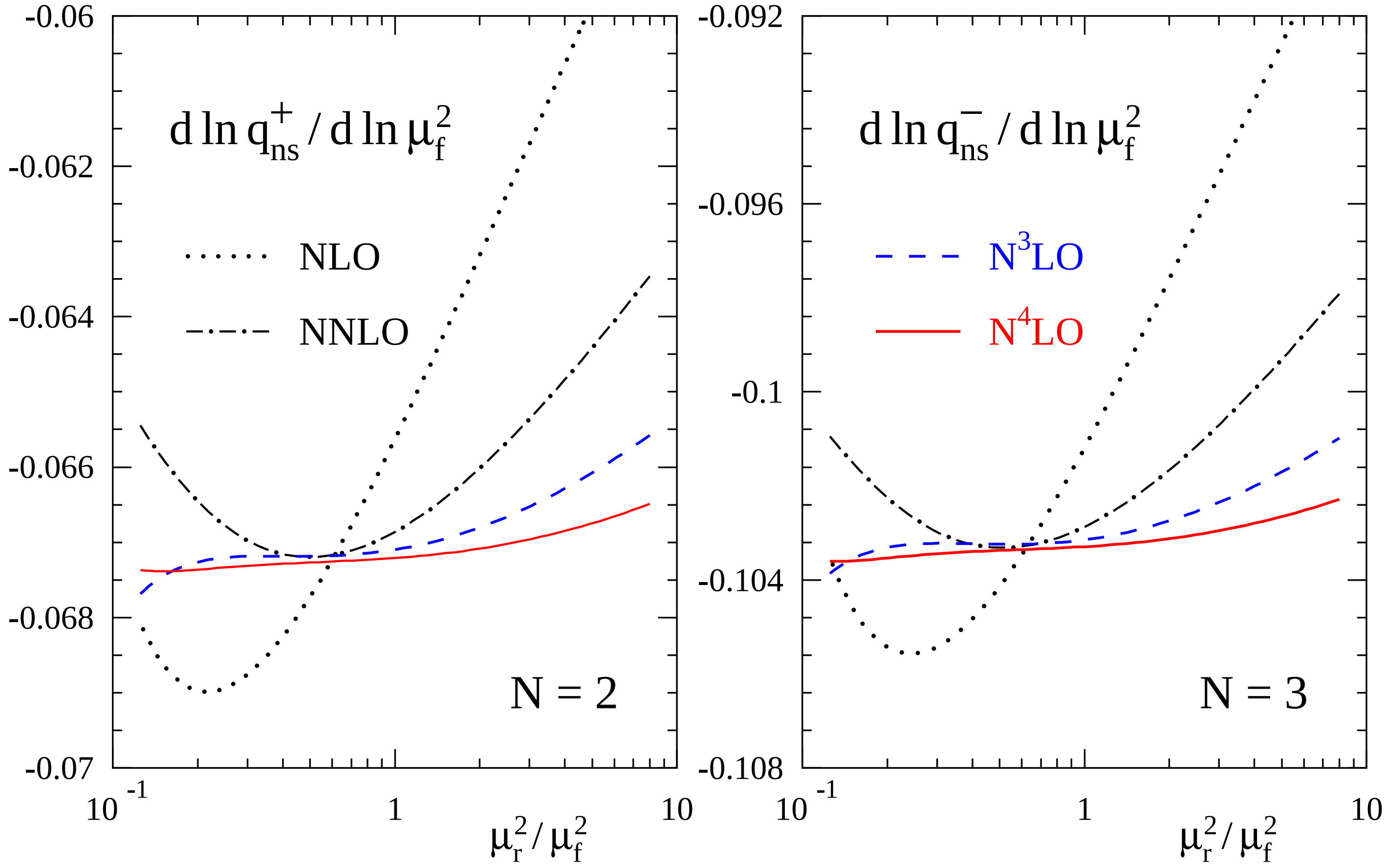
- Moments $N = 2$ and $N = 3$ for nonsinglet anomalous dimensions $\gamma_{\rm ns}^{\pm}$
  - implementation by Herzog, Ruijl '17 of local R^* operation Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84 for reduction of five-loop self-energy diagrams to four-loop ones computed with **Forcer** Ruijl, Ueda, Vermaseren '17

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)+}(N=2) \;=\;& \\
& C_F^5 \left[\frac{9306376}{19683} - \frac{802784}{729}\zeta_3 - \frac{557440}{81}\zeta_5 + \frac{12544}{9}\zeta_3^2 + 8512\zeta_7 \right] \\
& - C_A C_F^4 \left[\frac{81862744}{19683} - \frac{1600592}{243}\zeta_3 + \frac{59840}{81}\zeta_4 - \frac{142240}{27}\zeta_5 + 3072\zeta_3^2 - \frac{35200}{9}\zeta_6 + 19936\zeta_7 \right] \\
& + C_A^2 C_F^3 \left[\frac{63340406}{6561} - \frac{1003192}{243}\zeta_3 - \frac{229472}{81}\zeta_4 + \frac{61696}{27}\zeta_5 + \frac{30976}{9}\zeta_3^2 - \frac{35200}{9}\zeta_6 + 15680\zeta_7 \right] \\
& - C_A^3 C_F^2 \left[\frac{220224724}{19683} + \frac{4115536}{729}\zeta_3 - \frac{170968}{27}\zeta_4 - \frac{3640624}{243}\zeta_5 + \frac{70400}{27}\zeta_3^2 + \frac{123200}{27}\zeta_6 + \frac{331856}{27}\zeta_7 \right] \\
& + C_A^4 C_F \left[\frac{266532611}{39366} + \frac{2588144}{729}\zeta_3 - \frac{221920}{81}\zeta_4 - \frac{3102208}{243}\zeta_5 + \frac{74912}{81}\zeta_3^2 + \frac{334400}{81}\zeta_6 + \frac{178976}{27}\zeta_7 \right] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \left[\frac{15344}{81} - \frac{12064}{27}\zeta_3 - \frac{704}{3}\zeta_4 + \frac{58400}{27}\zeta_5 - \frac{6016}{3}\zeta_3^2 - \frac{19040}{9}\zeta_7 \right] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_F \left[\frac{23968}{81} - \frac{733504}{81}\zeta_3 + \frac{176320}{81}\zeta_5 + \frac{6400}{3}\zeta_3^2 + \frac{77056}{9}\zeta_7 \right] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_A \left[\frac{82768}{81} - \frac{555520}{81}\zeta_3 + \frac{10912}{9}\zeta_4 - \frac{1292960}{81}\zeta_5 + \frac{84352}{27}\zeta_3^2 + \frac{140800}{27}\zeta_6 + 12768\zeta_7 \right] \\
& + n_f C_F^4 \left[\frac{1824964}{19683} - \frac{463520}{243}\zeta_3 + \frac{21248}{81}\zeta_4 - \frac{16480}{81}\zeta_5 + \frac{6656}{9}\zeta_3^2 - \frac{6400}{9}\zeta_6 + \frac{8960}{3}\zeta_7 \right] \\
& - n_f C_A C_F^3 \left[\frac{3375082}{6561} - \frac{420068}{243}\zeta_3 - \frac{48256}{81}\zeta_4 + \frac{458032}{81}\zeta_5 + \frac{3968}{3}\zeta_3^2 - \frac{8000}{3}\zeta_6 + \frac{4480}{3}\zeta_7 \right] \\
& + n_f C_A^2 C_F^2 \left[\frac{15291499}{13122} + \frac{1561600}{243}\zeta_3 - \frac{114536}{27}\zeta_4 - \frac{252544}{243}\zeta_5 + \frac{24896}{27}\zeta_3^2 + \frac{13600}{27}\zeta_6 + \frac{11200}{27}\zeta_7 \right] \\
& - n_f C_A^3 C_F \left[\frac{48846580}{19683} + \frac{4314308}{729}\zeta_3 - \frac{274768}{81}\zeta_4 - \frac{1389080}{243}\zeta_5 + \frac{27808}{81}\zeta_3^2 + \frac{184000}{81}\zeta_6 + \frac{39088}{27}\zeta_7 \right] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \left[\frac{22096}{27} + \frac{43712}{81}\zeta_3 - \frac{512}{9}\zeta_4 - \frac{217280}{81}\zeta_5 - \frac{25088}{27}\zeta_3^2 + \frac{25600}{27}\zeta_6 - 2464\zeta_7 \right] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \left[\frac{170752}{81} - \frac{328832}{81}\zeta_3 + \frac{650240}{81}\zeta_5 - \frac{8192}{9}\zeta_3^2 - \frac{35840}{9}\zeta_7 \right] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \left[\frac{207824}{81} + \frac{251392}{81}\zeta_3 - \frac{5632}{9}\zeta_4 - \frac{522880}{81}\zeta_5 + \frac{15872}{27}\zeta_3^2 + \frac{70400}{27}\zeta_6 - \frac{29120}{9}\zeta_7 \right] \\
& + n_f^2 C_F^3 \left[\frac{1082297}{6561} - \frac{145792}{243}\zeta_3 + \frac{1072}{81}\zeta_4 + \frac{55552}{81}\zeta_5 + \frac{1792}{9}\zeta_3^2 - \frac{3200}{9}\zeta_6 \right] \\
& + n_f^2 C_A C_F^2 \left[\frac{332254}{2187} - \frac{85016}{243}\zeta_3 + \frac{20752}{27}\zeta_4 - \frac{28544}{81}\zeta_5 - \frac{13952}{27}\zeta_3^2 + \frac{1600}{27}\zeta_6 \right] \\
& + n_f^2 C_A^2 C_F \left[\frac{631400}{6561} + \frac{214268}{243}\zeta_3 - 784\zeta_4 - \frac{53344}{243}\zeta_5 + \frac{25472}{81}\zeta_3^2 + \frac{22400}{81}\zeta_6 \right] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[\frac{43744}{81} - \frac{35648}{81}\zeta_3 - \frac{1792}{9}\zeta_4 - \frac{52480}{81}\zeta_5 + \frac{2048}{27}\zeta_3^2 + \frac{12800}{27}\zeta_6 \right] \\
& + n_f^3 C_F^2 \left[\frac{265510}{19683} + \frac{11872}{729}\zeta_3 - \frac{128}{3}\zeta_4 + \frac{512}{27}\zeta_5 \right] \\
& + n_f^3 C_A C_F \left[\frac{168677}{19683} + \frac{11872}{729}\zeta_3 + \frac{2752}{81}\zeta_4 - \frac{4096}{81}\zeta_5 \right] - n_f^4 C_F \left[\frac{5504}{19683} + \frac{1024}{729}\zeta_3 - \frac{128}{81}\zeta_4 \right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)-}(N=3) \;=\;& \\
& C_F^5 \left[\frac{81472935625}{80621568} + \frac{99382175}{23328}\zeta_3 - \frac{3395975}{162}\zeta_5 - \frac{9650}{9}\zeta_3^2 + \frac{34685}{2}\zeta_7 \right] \\
& - C_A C_F^4 \left[\frac{286028134219}{80621568} - \frac{23916529}{7776}\zeta_3 - 4490\zeta_3^2 + \frac{134090}{81}\zeta_4 - \frac{2468075}{108}\zeta_5 - \frac{55000}{9}\zeta_6 + \frac{155155}{4}\zeta_7 \right] \\
& + C_A^2 C_F^3 \left[\frac{20173099267}{3359232} - \frac{15401281}{864}\zeta_3 + \frac{732787}{1296}\zeta_4 + \frac{1972075}{216}\zeta_5 - \frac{63830}{9}\zeta_3^2 - \frac{79750}{9}\zeta_6 + \frac{139895}{4}\zeta_7 \right] \\
& - C_A^3 C_F^2 \left[\frac{166662991819}{20155392} - \frac{36397493}{2916}\zeta_3 - \frac{103763}{54}\zeta_4 + \frac{30994565}{3888}\zeta_5 - \frac{133990}{27}\zeta_3^2 - \frac{72875}{54}\zeta_6 + \frac{2127335}{108}\zeta_7 \right] \\
& + C_A^4 C_F \left[\frac{75932079965}{10077696} - \frac{27693563}{23328}\zeta_3 - \frac{1791229}{1296}\zeta_4 - \frac{9417425}{1944}\zeta_5 - \frac{96700}{81}\zeta_3^2 + \frac{163625}{81}\zeta_6 + \frac{199640}{27}\zeta_7 \right] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \left[\frac{81725}{162} - \frac{33505}{18}\zeta_3 - \frac{1100}{3}\zeta_4 + \frac{52025}{18}\zeta_5 - \frac{7000}{3}\zeta_3^2 - \frac{48125}{36}\zeta_7 \right] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_F \left[\frac{231575}{36} + \frac{6351445}{324}\zeta_3 - \frac{2927225}{162}\zeta_5 + \frac{23210}{3}\zeta_3^2 - \frac{200410}{9}\zeta_7 \right] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_A \left[\frac{165871}{54} + \frac{1816625}{162}\zeta_3 - \frac{41800}{9}\zeta_4 - \frac{4456145}{162}\zeta_5 + \frac{196880}{27}\zeta_3^2 + \frac{200750}{27}\zeta_6 - \frac{7525}{4}\zeta_7 \right] \\
& + n_f C_F^4 \left[\frac{1776521549}{40310784} - \frac{1332919}{486}\zeta_3 + \frac{5000}{9}\zeta_3^2 + \frac{33290}{81}\zeta_4 - \frac{30325}{81}\zeta_5 - \frac{10000}{9}\zeta_6 + \frac{14000}{3}\zeta_7 \right] \\
& - n_f C_A C_F^3 \left[\frac{3737356319}{3359232} - \frac{2327111}{432}\zeta_3 + \frac{1280}{3}\zeta_3^2 + \frac{262069}{648}\zeta_4 + \frac{1693715}{162}\zeta_5 - \frac{14000}{3}\zeta_6 + \frac{7000}{3}\zeta_7 \right] \\
& + n_f C_A^2 C_F^2 \left[\frac{5637513931}{3359232} + \frac{2711207}{486}\zeta_3 - \frac{5020}{27}\zeta_3^2 - \frac{457499}{108}\zeta_4 + \frac{508820}{243}\zeta_5 - \frac{20375}{27}\zeta_6 + \frac{50155}{108}\zeta_7 \right] \\
& - n_f C_A^3 C_F \left[\frac{8766012215}{2519424} + \frac{45697231}{5832}\zeta_3 + \frac{1195}{81}\zeta_3^2 - \frac{2848403}{648}\zeta_4 - \frac{1808870}{243}\zeta_5 + \frac{222250}{81}\zeta_6 + \frac{250915}{108}\zeta_7 \right] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \left[\frac{24385}{27} - \frac{334010}{81}\zeta_3 - \frac{8480}{9}\zeta_3^2 + \frac{1622600}{81}\zeta_5 - \frac{135380}{9}\zeta_7 \right] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \left[\frac{297889}{162} + \frac{154970}{81}\zeta_3 - \frac{62600}{27}\zeta_3^2 + \frac{3700}{9}\zeta_4 - \frac{122780}{81}\zeta_5 - \frac{36500}{27}\zeta_6 - 910\zeta_7 \right] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \left[\frac{241835}{162} + \frac{333487}{81}\zeta_3 + \frac{30560}{27}\zeta_3^2 - 10780/9\,\zeta_4 - \frac{316900}{81}\zeta_5 + \frac{110000}{27}\zeta_6 - \frac{71960}{9}\zeta_7 \right] \\
& + n_f^2 C_F^3 \left[\frac{512848319}{1679616} - \frac{57109}{54}\zeta_3 + \frac{2800}{9}\zeta_3^2 + \frac{9118}{81}\zeta_4 + \frac{86440}{81}\zeta_5 - \frac{5000}{9}\zeta_6 \right] \\
& + n_f^2 C_A C_F^2 \left[\frac{1080083}{5832} - \frac{296729}{972}\zeta_3 - \frac{21800}{27}\zeta_3^2 + \frac{56327}{54}\zeta_4 - \frac{42860}{81}\zeta_5 + \frac{2500}{27}\zeta_6 \right] \\
& + n_f^2 C_A^2 C_F \left[\frac{61747877}{419904} + \frac{2496811}{1944}\zeta_3 + \frac{39800}{81}\zeta_3^2 - \frac{3503}{3}\zeta_4 - \frac{88990}{243}\zeta_5 + \frac{35000}{81}\zeta_6 \right] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[\frac{19435}{27} - \frac{53366}{81}\zeta_3 + \frac{3200}{27}\zeta_3^2 - \frac{3160}{9}\zeta_4 - \frac{70000}{81}\zeta_5 + \frac{20000}{27}\zeta_6 \right] \\
& + n_f^3 C_F^2 \left[\frac{28758139}{1259712} + \frac{21673}{729}\zeta_3 - \frac{610}{9}\zeta_4 + \frac{800}{27}\zeta_5 \right] \\
& + n_f^3 C_A C_F \left[\frac{13729181}{1259712} + \frac{14947}{729}\zeta_3 + \frac{4390}{81}\zeta_4 - \frac{6400}{81}\zeta_5 \right] - n_f^4 C_F \left[\frac{259993}{629856} + \frac{1660}{729}\zeta_3 - \frac{200}{81}\zeta_4 \right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)\,\rm v}(N=3) \;=\;& \gamma_{\rm ns}^{(4)-}(N=3) \\
& + n_f \frac{d_{abc}d^{abc}}{N_F} \Bigg\{ C_F^2 \left[\frac{79906955}{46656} + \frac{246955}{54}\zeta_3 - \frac{504550}{81}\zeta_5 \right] \\
& \quad - C_A C_F \left[\frac{9797321}{3888} - \frac{475655}{54}\zeta_3 + \frac{17600}{9}\zeta_4 + \frac{516950}{81}\zeta_5 - \frac{500}{9}\zeta_3^2 + \frac{2800}{9}\zeta_7 \right] \\
& \quad + C_A^2 \left[\frac{166535}{486} - \frac{1783913}{324}\zeta_3 + \frac{5555}{9}\zeta_4 + \frac{507515}{81}\zeta_5 - \frac{2035}{27}\zeta_3^2 - \frac{5500}{27}\zeta_6 - \frac{2765}{18}\zeta_7 \right] \\
& \quad + n_f C_A \left[\frac{285985}{3888} + \frac{41954}{81}\zeta_3 + \frac{160}{27}\zeta_3^2 - \frac{1010}{9}\zeta_4 - \frac{56480}{81}\zeta_5 + \frac{1000}{27}\zeta_6 \right] \\
& \quad + n_f C_F \left[\frac{1098323}{3888} - \frac{49720}{81}\zeta_3 + \frac{3200}{9}\zeta_4 \right] - n_f^2 \left[\frac{21823}{1944} \right] \Bigg\}
\end{aligned}
$$

Scale stability of evolution



- Renormalization-scale dependence of $d \ln q_{\rm ns}^{\pm} / d \ln \mu_f^2$ at $N = 2$ and $N = 3$ using NLO, NNLO, N^{3}LO and N^{4}LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

Singlet

Color factors of $SU(n_c)$

- Quadratic Casimir factors $C_r\delta^{ab} \equiv \mathrm{Tr}\,(\,T_r^a\,T_r^b\,)$
  - fundamental representation $C_F = (n_c^2 - 1)/(2n_c)$;
    adjoint representation $C_A = n_c$
- Quartic Casimir invariants occur for the first time at four loops
  - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ for representations labels x, y with generators T_r^a

$$d_r^{abcd} = \frac{1}{6}\,\mathrm{Tr}\,(\,T_r^a\,T_r^b\,T_r^c\,T_r^d + \text{ five } bcd \text{ permutations}\,)$$

- SU(n_c) with fermions in fundamental representation

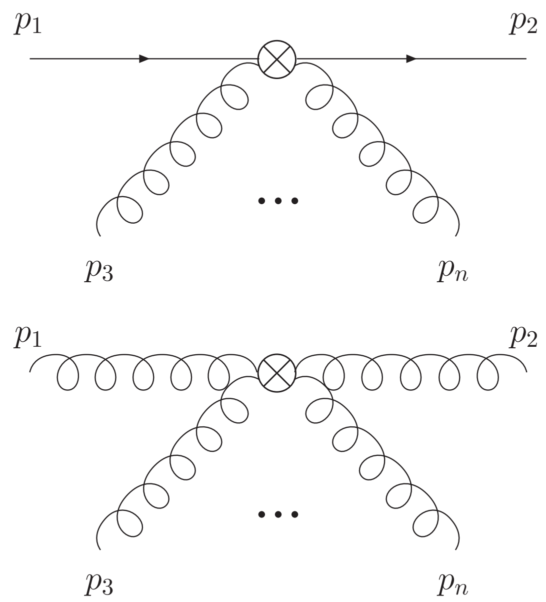
$$d_{AA}^{(4)}/n_A = \frac{1}{24}\,n_c^2(n_c^2 + 36)\;,$$

$$d_{FA}^{(4)}/n_A = \frac{1}{48}\,n_c(n_c^2 + 6)\;,$$

$$d_{FF}^{(4)}/n_A = \frac{1}{96}\,(n_c^2 - 6 + 18n_c^{-2})$$

  - trace-normalized with $T_F = \frac{1}{2}$
  - dimensions of representations $n_F = n_c$ and $n_A = (n_c^2 - 1)$

Operator matrix elements



- Singlet operators of spin-N and twist two

$$O^{\rm q}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi \;,$$
$$O^{\rm g}_{\{\mu_1,\dots,\mu_N\}} = F_{\nu\{\mu_1} D_{\mu_2} \cdots D_{\mu_{N-1}} F_{\mu_N\}}{}^{\nu}$$

- Quartic Casimir terms at four loops are effectively ‘leading-order’
  - anomalous dimensions fulfil relation for $\mathcal{N}=1$ supersymmetry

$$\gamma^{(3)}_{\rm qq}(N) + \gamma^{(3)}_{\rm gq}(N) - \gamma^{(3)}_{\rm qg}(N) - \gamma^{(3)}_{\rm gg}(N) \overset{Q}{=} 0$$

- color-factor substitutions for $n_f=1$ Majorana fermions (factor $2n_f$)

$$(2n_f)^2\,\frac{d^{(4)}_{FF}}{n_A} = 2n_f\,\frac{d^{(4)}_{FA}}{n_A} = 2n_f\,\frac{d^{(4)}_{FF}}{n_F} = \frac{d^{(4)}_{FA}}{n_F} = \frac{d^{(4)}_{AA}}{n_A}$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
  - quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer ‘99

$$\gamma^{(0)}_{\rm qg}(N)\,\gamma^{(3)}_{\rm gq}(N) \overset{Q}{=} \gamma^{(0)}_{\rm gq}(N)\,\gamma^{(3)}_{\rm qg}(N)$$

Calculation and results

- Splitting functions (diagonal) in the large-x limit

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_{n,\mathrm{i}}}{(1-x)_+} + B_{n,\mathrm{i}}\,\delta(1-x) + C_{n,\mathrm{i}}\,\ln(1-x) + D_{n,\mathrm{i}}$$

- Cusp anomalous dimensions related by Casimir scaling up to three loops

$A_{n,\mathrm{g}} = \frac{C_A}{C_F} A_{n,\mathrm{q}}$ for $n \leq 3$

  - at four loops Casimir scaling holds in large n_c-limit — Dixon '17

- Generalized Casimir scaling at four loops for new color factors

S. M., Ruijl, Ueda, Vermaseren, Vogt '18

  - $A_{4,\mathrm{g}}\Big|_{\frac{d_A^{abcd}d_A^{abcd}}{n_A}} = A_{4,\mathrm{q}}\Big|_{\frac{d_F^{abcd}d_A^{abcd}}{n_F}}$
  - $A_{4,\mathrm{g}}\Big|_{\frac{d_F^{abcd}d_A^{abcd}}{n_A}} = A_{4,\mathrm{q}}\Big|_{\frac{d_F^{abcd}d_F^{abcd}}{n_F}}$
  - $A_{4,\mathrm{g}}\Big|_{\frac{d_F^{abcd}d_F^{abcd}}{n_A}} = 0$

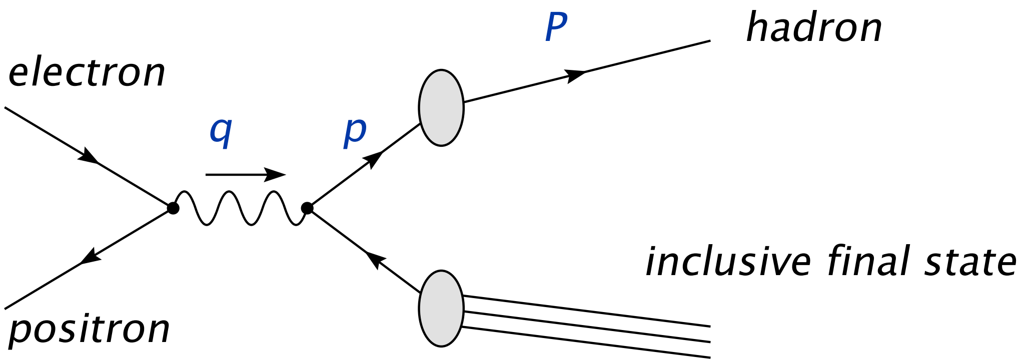
Quark cusp anomalous dimensions

$$
\begin{aligned}
A_{4,\mathrm{q}} \;=\; & C_F C_A^3 \left(\frac{84278}{81} - \frac{88400}{81}\zeta_2 + \frac{20944}{27}\zeta_3 + 1804\zeta_4 - \frac{352}{3}\zeta_2\zeta_3 - \frac{3608}{9}\zeta_5 \right. \\
& \left. - 16\zeta_3^2 - \frac{2504}{3}\zeta_6 \right) + \frac{d_{FA}^{(4)}}{n_c} \left(-128\zeta_2 + \frac{128}{3}\zeta_3 + \frac{3520}{3}\zeta_5 - 384\zeta_3^2 - 992\zeta_6 \right) \\
& + C_F^3 n_f \left(\frac{572}{9} + \frac{592}{3}\zeta_3 - 320\zeta_5 \right) \\
& + C_F^2 C_A n_f \left(-\frac{34066}{81} + \frac{440}{3}\zeta_2 + \frac{3712}{9}\zeta_3 - 176\zeta_4 - 128\zeta_2\zeta_3 + 160\zeta_5 \right) \\
& + C_F C_A^2 n_f \left(-\frac{24137}{81} + \frac{20320}{81}\zeta_2 - \frac{23104}{27}\zeta_3 - \frac{176}{3}\zeta_4 + \frac{448}{3}\zeta_2\zeta_3 + \frac{2096}{9}\zeta_5 \right) \\
& + n_f \frac{d_{FF}^{(4)}}{n_c} \left(256\zeta_2 - \frac{256}{3}\zeta_3 - \frac{1280}{3}\zeta_5 \right) + C_F^2 n_f^2 \left(\frac{2392}{81} - \frac{640}{9}\zeta_3 + 32\zeta_4 \right) \\
& + C_F C_A n_f^2 \left(\frac{923}{81} - \frac{608}{81}\zeta_2 + \frac{2240}{27}\zeta_3 - \frac{112}{3}\zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27}\zeta_3 \right)
\end{aligned}
$$

Large-n_c (Henn, Lee, Smirnov, Smirnov, Steinhauser '16; S. M., Ruijl, Ueda, Vermaseren, Vogt '17); n_f terms (Grozin '18; Henn, Peraro, Stahlhofen, Wasser '19); n_f^2 terms (Davies, Ruijl, Ueda, Vermaseren, Vogt '16; Lee, Smirnov, Smirnov, Steinhauser '17); n_f^3 terms (Gracey '94; Beneke, Braun, '95); quartic colour factors (Lee, Smirnov, Smirnov, Steinhauser '19; Henn, Korchemsky, Mistlberger '19)

e^+e^- *annihilation*

e^+e^- *annihilation*



Kinematic variables

- momentum transfer $Q^2 = +q^2$ (time-like)
- scaling variable $x = (2p \cdot q)/Q^2$

- One-particle inclusive cross section (parametrized by functions F_a)

$$F_a\left(x, Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes FF\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N^{3}LO (work in progress)

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \ldots\right)$$

- Evolution equations for fragmentation functions FF up to N^{2}LO known
  - non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

$$\frac{d}{d \ln \mu^2} FF(x, \mu^2) \quad = \quad \left[P^T(\alpha_s(\mu^2)) \otimes FF(\mu^2)\right](x)$$

  - (time-like) splitting fcts. $P_{ji}^T = \alpha_s P_{ji}^{T,(0)} + \alpha_s^2 P_{ji}^{T,(1)} + \alpha_s^3 P_{ji}^{T,(2)} + \ldots$

Relating space- and time-like kinematics

Crewther relation

- From conformal and chiral invariance of leading singularity of short distance OPE simple relation between Crewther '72
  - amplitude $\pi^0 \to \gamma\gamma$
  - polarized Bjorken sum rule $\int_0^1 dx\, g_1^{ep-en}(x, Q^2)$
  - Adler function D_V (derivative of correlator $Q^2 \frac{\partial}{\partial Q^2} \Pi_V$)

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- Higher order radiative QCD corrections exhibit relations between
  - polarized Gross-Llewellyn Smith sum-rule C_{GLS} at $\mathcal{O}(\alpha_s^4)$ (first Mellin moment of $F_3^{\bar{\nu}p+\nu p}$) Baikov, Chetyrkin, Kühn '10
  - Adler function D_V at $\mathcal{O}(\alpha_s^4)$ Baikov, Chetyrkin, Kühn '10
- $C_{\rm GLS}$ and D_V related by running coupling (β-function) through $\mathcal{O}(\alpha_s^4)$ Broadhurst, Kataev '96; Maxwell, Broadhurst, Kataev '06; Baikov, Chetyrkin, Kühn '10; Baikov, Chetyrkin, Kühn, Rittinger '12

$$C_{\rm GLS}(\alpha_s)\, D_V(\alpha_s) \quad = \quad d_R \left(1 + \frac{\beta(\alpha_s)}{\alpha_s} K(\alpha_s)\right)$$

Drell-Yan-Levy relation

- Analytic continuation in energy $-q^2 \to +q^2$ (exploit analyticity properties)
  Curci, Furmanski, Petronzio '80; Floratos, Kounnas, Lacaze '81; Stratmann, Vogelsang '96; Blümlein, Ravindran, van Neerven '00; ...
- Relation between DIS structure function $F_1^{\rm s-like}$ and fragmentation function $F_T^{\rm t-like}$

$$F_T^{\rm t-like}(x) \quad = \quad -xF_1^{\rm s-like}\left(\frac{1}{x}\right)$$

- Leading order splitting function $P^{(0)}_{\rm qq}$
  - respects "naive" Drell-Yan-Levy relation (with $\delta(1-x) \to \delta(1-x)$)

$$P^{(0)}_{qq}(x) \quad = \quad 2C_F\left(\frac{2}{1-x} - 1 - x\right) + 3C_F\delta(1-x)$$

- Beyond leading order naive version of Drell-Yan-Levy relation not valid

Gribov-Lipatov reciprocity

- Leading order diagonal splitting functions identical for space- and time-like kinematics $P^{\rm t-like}(x) = P^{\rm s-like}(x) = -xP^{\rm s-like}\left(\frac{1}{x}\right)$
- reciprocity relation implied (realized in $N=4$ SYM theory)

Mapping DIS to e^+e^- annihilation

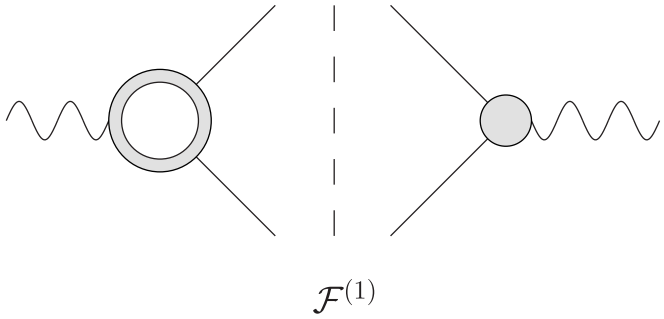
- DIS theory results recycled for time-like evolution
- Use of
  - factorization in dimensional regularization ($D = 4 - 2\epsilon$)
  - infrared safety (Kinoshita – Lee-Nauenberg theorem) of observables
  - generalized reciprocity of space-/time-like anomalous dimension with reciprocity respecting function $f(N)$ Dokshitzer, Marchesini, Salam '05; Basso, Korchemsky '06
    $\gamma^{T/S}(N) = f(N \mp \gamma^{T/S}(N)/2 - \beta(\alpha_s)/2)$
  - scheme invariance of physical kernels
- Theory predictions for time-like singlet splitting functions at NNLO (using analyticity)
  - complete: $P_{qq}^{(2)T}$, $P_{gg}^{(2)T}$, $P_{gq}^{(2)T}$
    Mitov, S.M., Vogt '06; S.M., Vogt '07; Almasy, S.M., Vogt '11
  - almost complete: $P_{qg}^{(2)T}$ Almasy, S.M., Vogt '11
  - complete $P_{qg}^{(2)T}$ (with additional soft-collinear factorization)
    Hao Chen, Yang, Zhu, Zhu '20

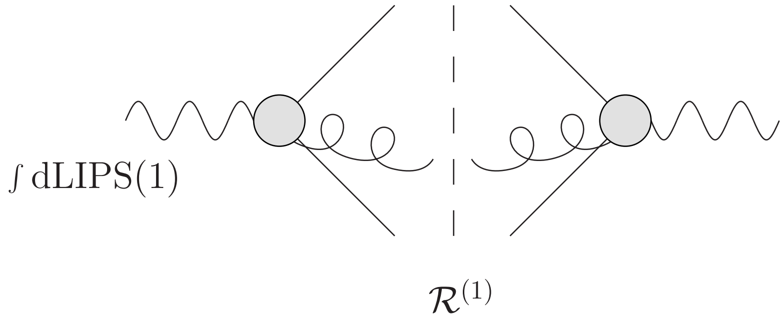
Direct computation of e^+e^- annihilation

Task

- Direct computation of e^+e^- annihilation to check analyticity
- Access to coefficient functions at N^{3}LO Magerya, PhD thesis, to appear

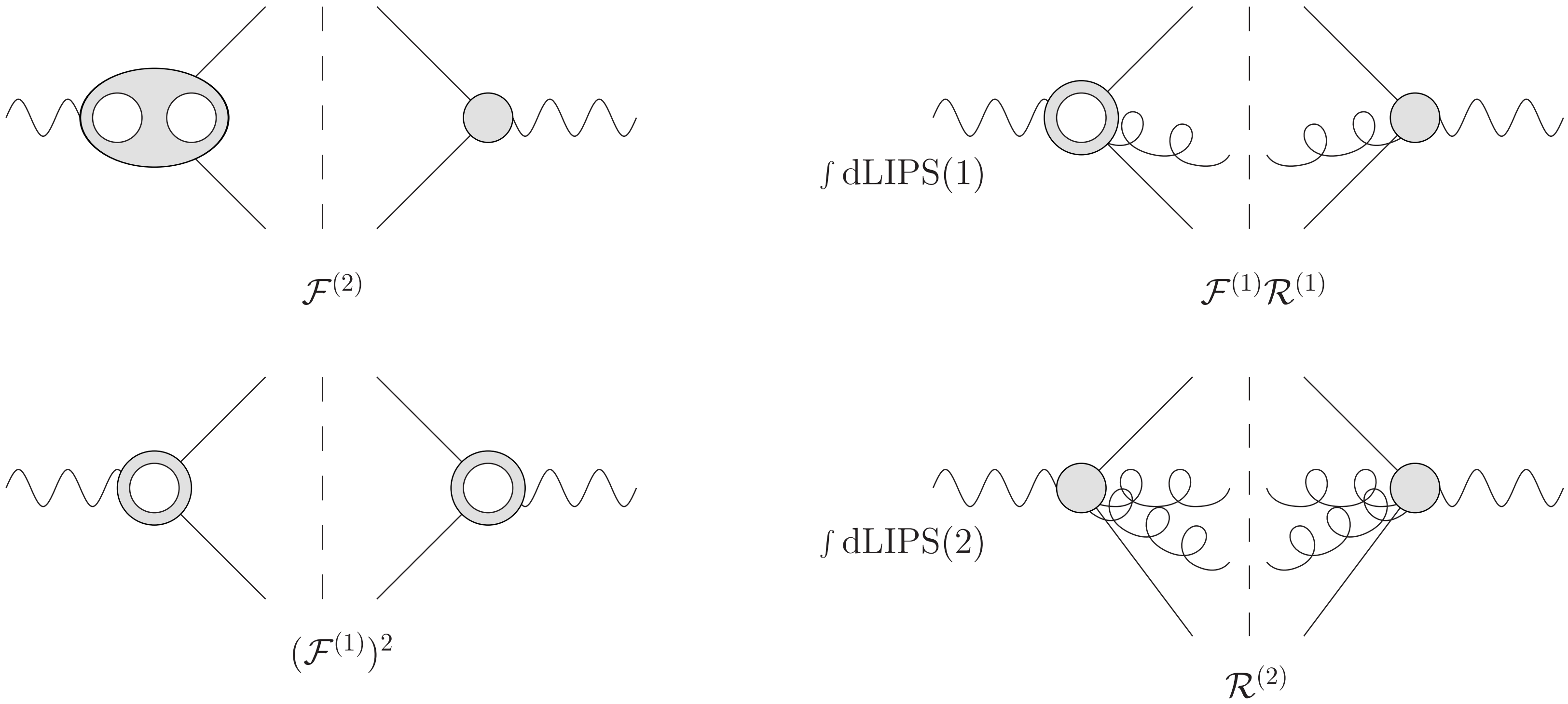
e^+e^- *annihilation at one loop*





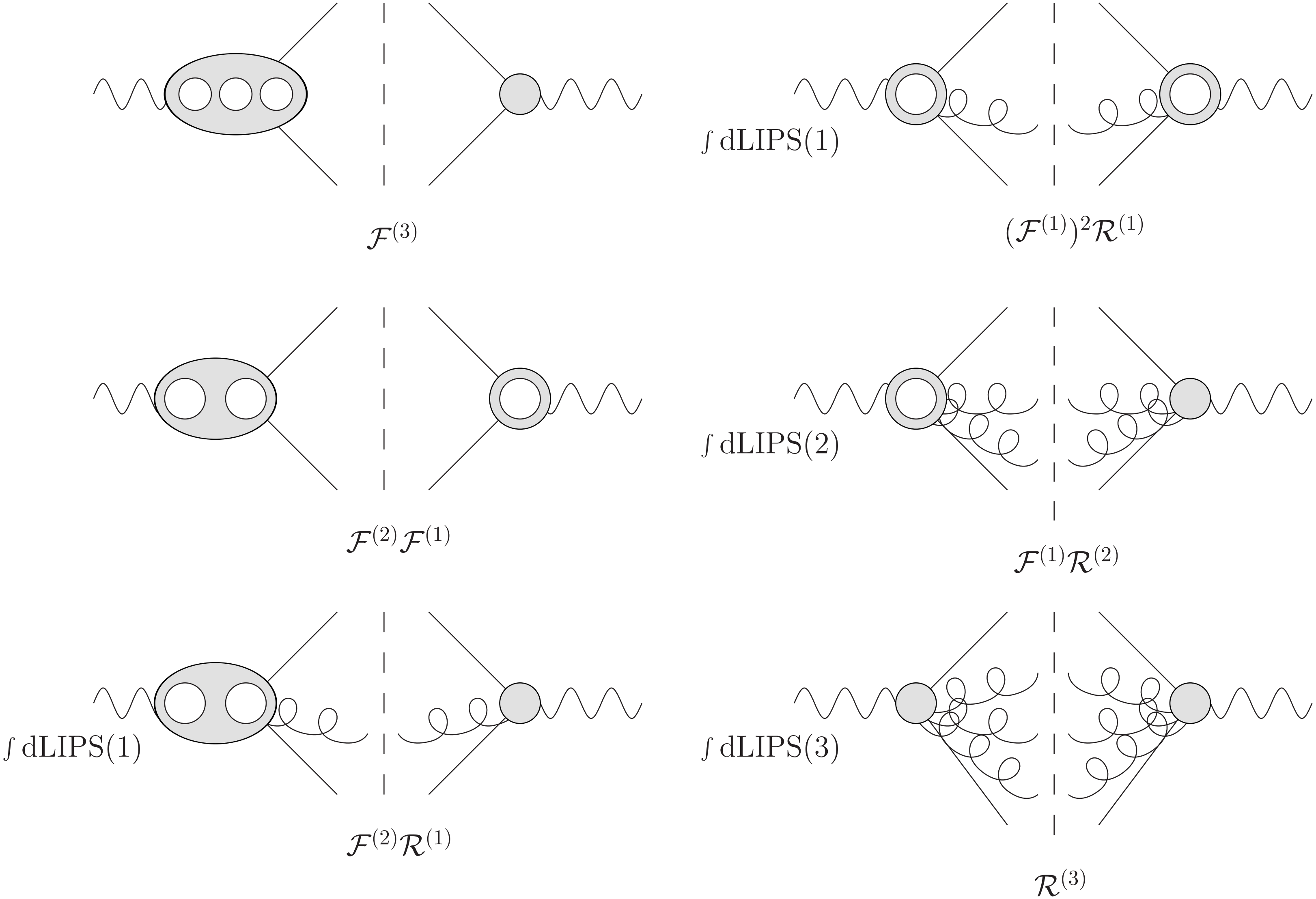
$$\mathcal{T}_1^{\mathrm{b}} \quad = \quad 2\,\mathrm{Re}\,\mathcal{F}_1\,\delta(1-x) + \mathcal{R}_1$$

e^+e^- *annihilation at two loops*



$$\mathcal{T}_2^{\rm b} \quad = \quad (2\,\mathsf{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\delta(1-x) + 2\,\mathsf{Re}\,\mathcal{F}_1\mathcal{R}_1 + \mathcal{R}_2$$

e^+e^- annihilation at three loops



$$\mathcal{T}_3^{\rm b} \quad = \quad (2\,\text{Re}\,\mathcal{F}_3 + 2\,|\mathcal{F}_1\mathcal{F}_2|)\,\delta(1-x) + (2\,\text{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\mathcal{R}_1 + 2\,\text{Re}\,\mathcal{F}_1\mathcal{R}_2 + \mathcal{R}_3$$

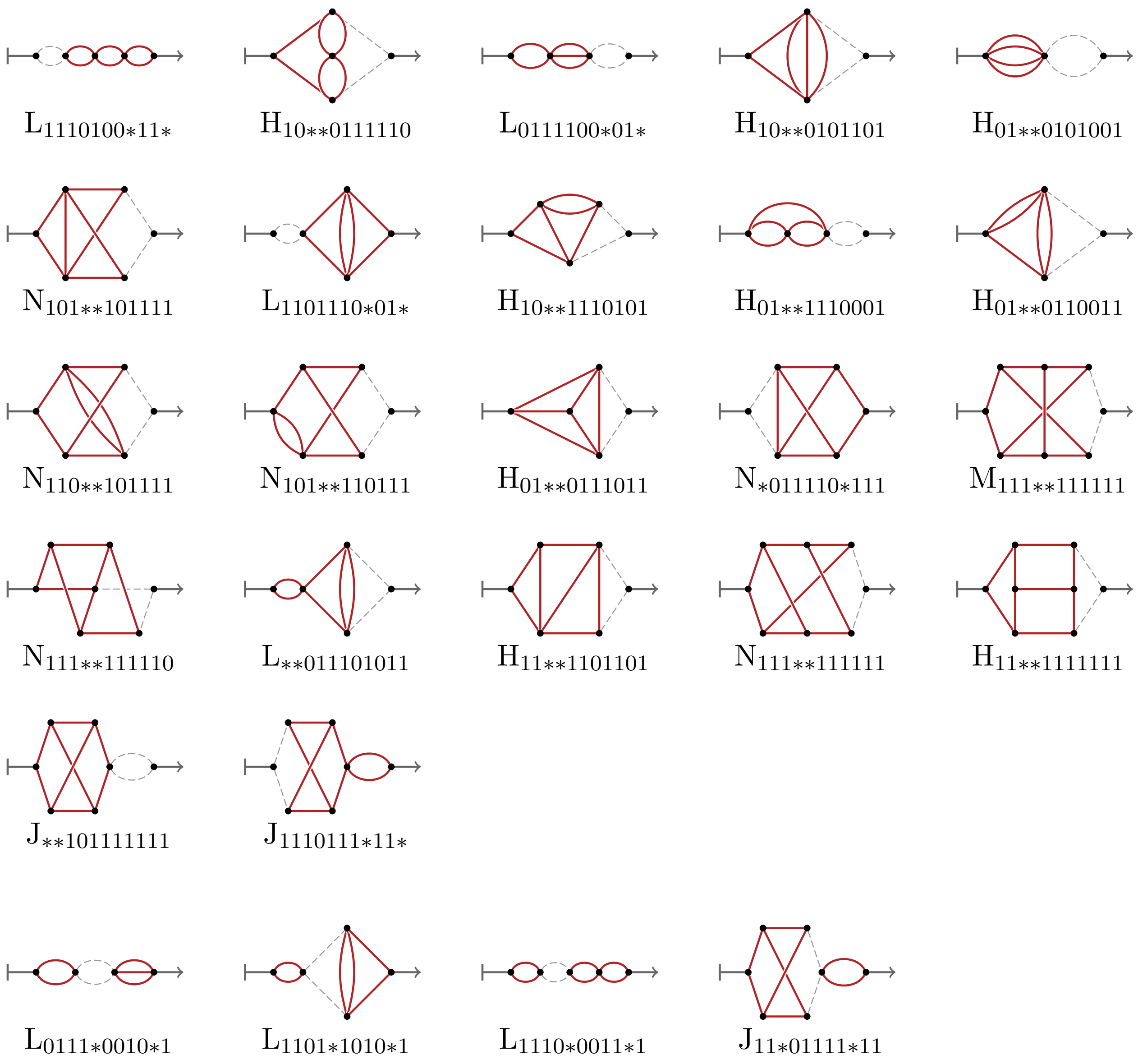
Four-loop computation

Work flow

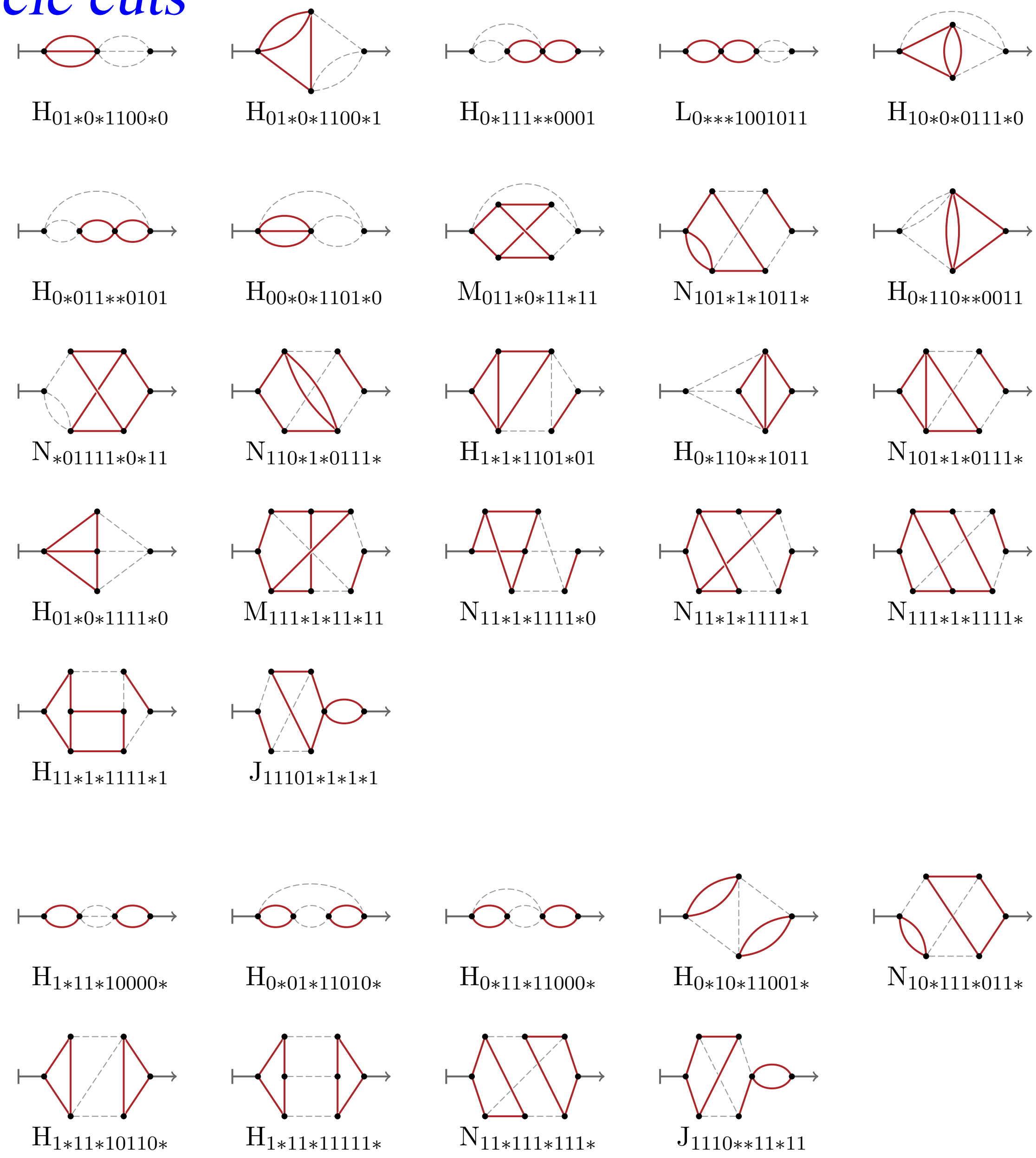
- Application of “reverse unitarity” requires to consider four-loop massless propagator diagrams with cuts
- Parametric reduction of all physical cuts of four-loop massless propagator
  Gituliar, Magerya, Pikelner ‘18; Magerya, Pikelner ‘19
  - two-particle cuts
  - three-particle cuts
  - four-particle cuts
  - five-particle cuts
- Solution of integration-by-parts reductions for inclusive integrals
  - dimensional recurrence relations
- Semi-inclusive integrals $F_{semi-incl}(x)$ from solution of differential equations in x
  - boundary conditions given by inclusive integrals
    $F_{incl} = \int_0^1 dx F_{semi-incl}(x)$
- Checks with help of Cutkosky rules for each diagram F

$$\mathrm{Im}F = -\sum_i \mathrm{cuts}_i F$$

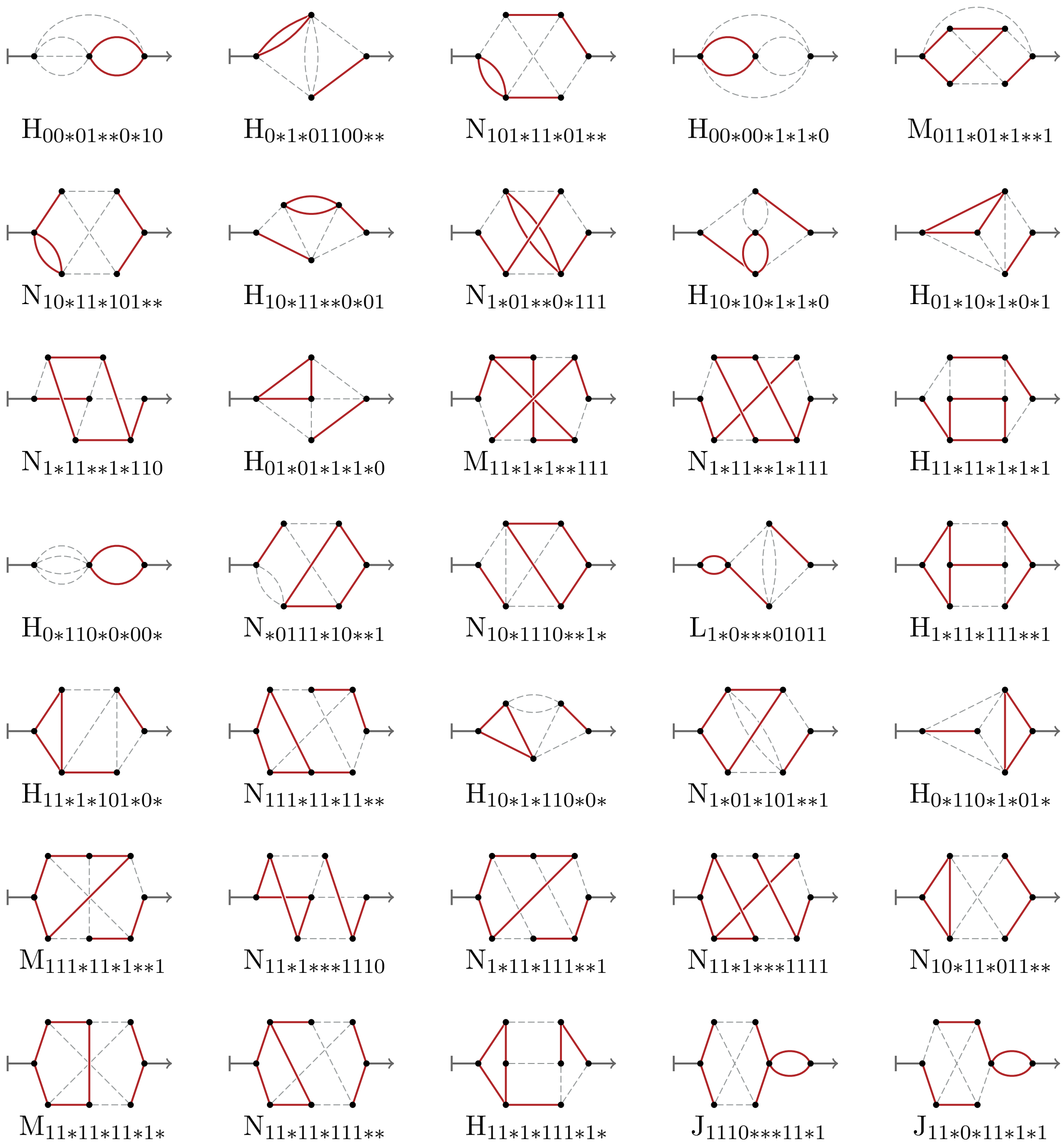
Two-particle cuts



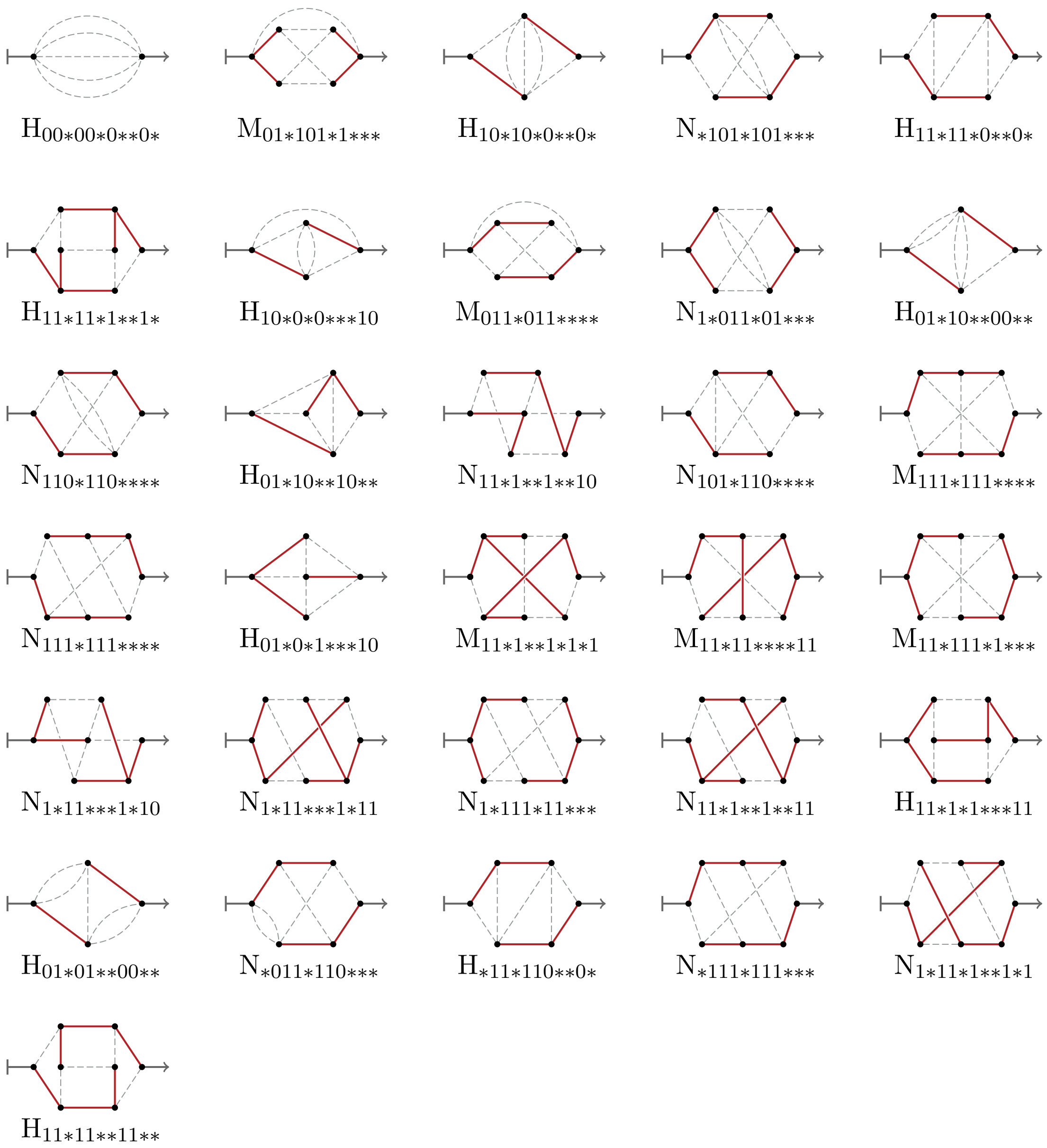
Three-particle cuts



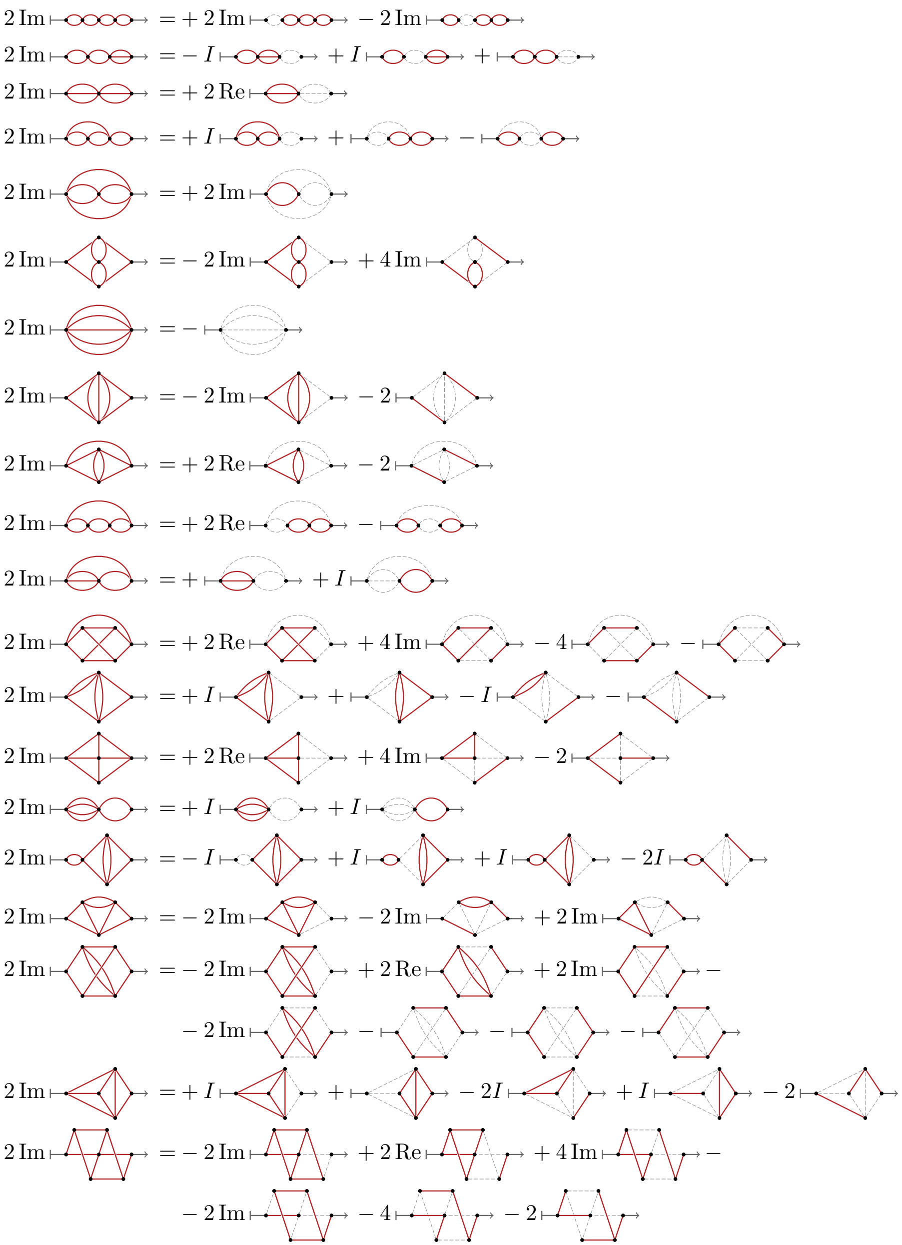
Four-particle cuts

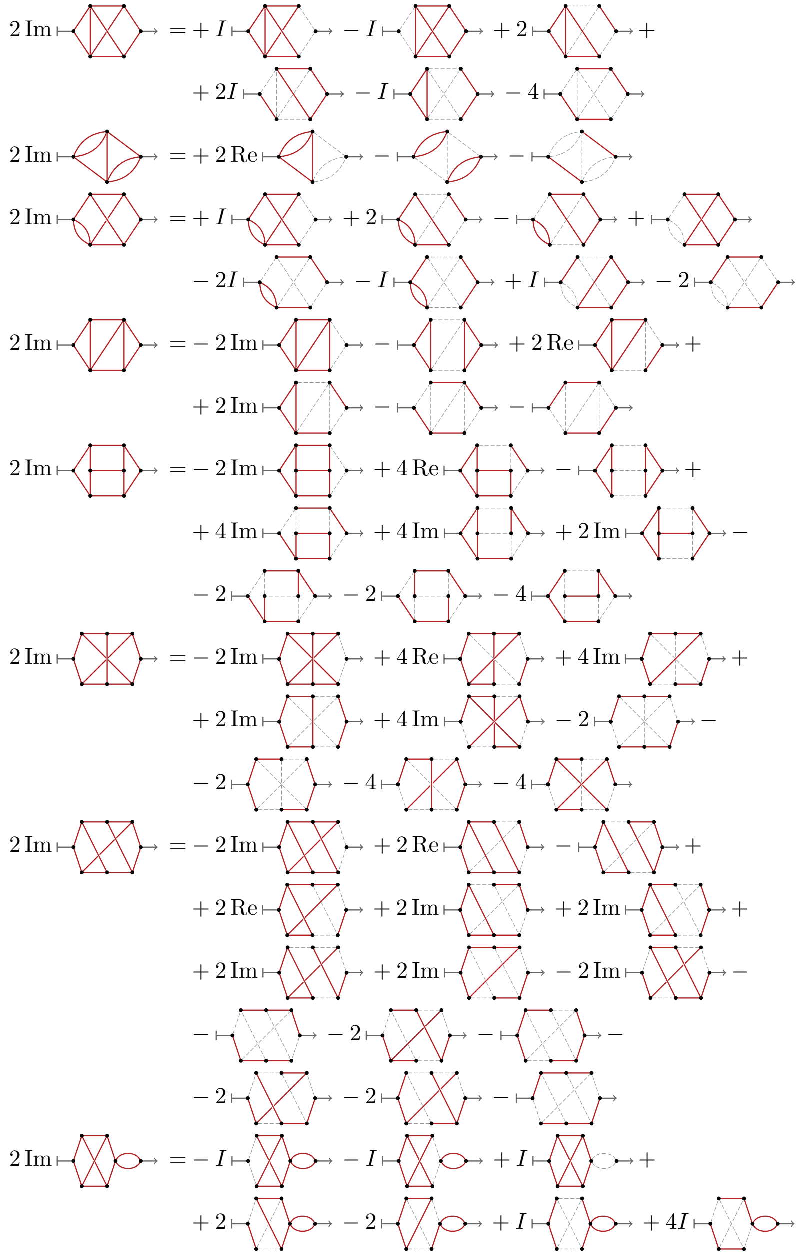


Five-particle cuts



Cutkosky rules





Summary

- Determination of strong coupling α_s at 1% precision requires QCD radiative corrections to evolution equations at N^{3}LO

Deep-inelastic scattering

- Coefficient functions at N^{3}LO known
- Non-singlet anomalous dimensions $\gamma_{\rm ns}^{(3),\pm,{\rm v}}(N)$ (fixed Mellin moments and exact results for large-n_c) at N^{3}LO
- Quartic Casimir contributions to singlet anomalous dimension $\gamma_{ij}^{(3)}(N)$ at N^{3}LO

e^+e^- *annihilation*

- Coefficient functions at N^{2}LO known
- Splitting functions at N^{2}LO known
- Four-loop massless propagator diagrams with cuts
  - independent direct computation of splitting functions at N^{2}LO
  - coefficient functions at N^{3}LO within reach